J. A. Stott
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# A TREATISE

OF

# MECHANICS,

BY


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MEMBER OF THE INSTITUTE, ETC.


TRANSLATED FROM THE FRENCH, AND ELUCIDATED WITH EXPLANATORY NOTES,

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IN TWO VOLUMES.

VOL. II.


LONDON: LONGMAN AND CO.

DUBLIN: A. MILLIKEN; AND HODGES AND SMITH.

MDCCCXLII.



IIA Lib.,

*00714*

DUBLIN:
PRINTED AT THE UNIVERSITY PRESS,
BY GRAISBERRY AND GILL.

# CONTENTS OF VOLUME II.

## BOOK THE FOURTH.

### DYNAMICS.

SECOND PART.

CHAPTER I.—GENERAL PRINCIPLE OF DYNAMICS.—page 1.

CHAPTER II.—DETERMINATION OF MOMENTS OF INERTIÆ AND OF PRINCIPAL AXES.—p. 33.

## CHAPTER III.—OF THE MOTION OF A SOLID BODY ABOUT A FIXED AXIS.—p. 62.

### I. *Uniform Motion of Rotation.*—Ibid.



### II. *Variable Motion of Rotation.*—p. 74.

CHAPTER VI.—OF THE MOTION OF A HEAVY SOLID BODY ON A GIVEN PLANE.—p. 168.

I. *Case in which Friction is not taken into account.*—Ibid.



II. *Case in which Friction is taken into account.*—p. 186.

CHAPTER VII.—OF THE IMPACT OF TWO BODIES OF ANY FORM WHATEVER.—p. 206.

III. *Longitudinal Impact of elastic Rods.*—p. 269.



IV. *Digression on the Integrals of Equations of partial Differences.* —p. 282.

IV. *Principles of living Forces and of least Action.*—p. 383.

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# BOOK THE FIFTH.

## HYDROSTATICS.

### Chapter I.—Preliminary Notions.—p. 405.

CHAPTER II.—GENERAL EQUATIONS OF THE EQUILIBRIUM OF FLUIDS. —p. 416. Nos.

CHAPTER III.—OF THE EQUILIBRIUM OF HEAVY FLUIDS.—p. 445.



CHAPTER IV.—OF THE EQUILIBRIUM AND MOTION OF FLOATING BODIES.—p. 465.

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# BOOK THE SIXTH.

## HYDRODYNAMICS.

### CHAPTER I.—GENERAL EQUATIONS OF THE MOTION OF FLUIDS.— p. 531.

## ADDITION.

RELATIVE TO THE APPLICATION OF THE PRINCIPLE OF LIVING FORCES IN THE CALCULATION OF MACHINES IN MOTION.—p. 618.

# ERRATA.

Page 12, line 14, for $\frac{dx^2}{dt^2}$ read $\frac{d^2x}{dt^2}$.
—— 37, —— 4, for $dx$ read $dz$.
—— 54, —— 25, *for* $\sin\psi = \frac{1}{\sqrt{1+u^2}}$ *read* $\cos\psi = \frac{1}{\sqrt{1+u^2}}$; and last line for

$$(f'u - g'\sqrt{1+u^2} \text{ read } (f'u - g')\sqrt{1+u^2}.$$

—— 85, —— 14, *for* compound *read* simple.
—— 89, —— 6 from bottom, *for* sum of the forces, *read* sum of the moments of the forces.
—— 105, —— 8, *for* $= qz$, *read* $- qx_{,}$.
—— 107, —— 3 from bottom, *for* motion *read* moment; line 6 from bottom, *for* o*x* o*y* *read* o$x_{,}$ o$y_{,}$.
—— 176, —— 19, *after* equation *add* (2).
—— 180, —— 1, for $\theta$ read $d\theta$.
—— 189, last line, for $\frac{d^2x'}{dt^2}$ $\frac{d^2x}{dt^2}$ read $\frac{dx'^2}{dt^2}$ $\frac{dx^2}{dt^2}$.
—— 240, line 4 from bottom, *for* (1) *read* (2).
—— 244, —— 11, *for* form *read* from.
—— 272, —— 3, for $a$ read $a^2$.
—— 273, —— 15, *for* $\sin\frac{i\pi c}{l}\cos\frac{i\pi at}{l}$ *read* $\sin\frac{i\pi c}{l}\sin\frac{i\pi x}{l}\cos\frac{i\pi at}{l}$.
—— 274, —— 10 from bottom, *for* $\Sigma\cos\frac{i\pi}{l}$ *read* $\Sigma\frac{1}{i}\cos\frac{i\pi}{l}$.
—— 308, —— 4 *for* $\sin m^2bt$ *read* $\sin m^2bt$; and line 16, for

$$\left(\frac{\int_0^l X\phi x dx}{\int_0^l X^2\phi' x dx}\right. \text{ read } \left(\frac{\int_0^l X\phi x dx}{\int_0^l X^2\phi x dx}\right..$$

—— 310, last line, *for* $\frac{1}{2}l$ *read* $\frac{1}{3}l$.
—— 314, line 19, for $i = 0$ read $i = 1$.
—— 348, —— 7, *for* the $n-1$ *read* $n-1$ of the.
—— 352, last line, *for* number *read* member.
—— 437, line 10, *for* $\gamma$ *read* $\gamma^2$.
—— 468, —— 7 from bottom, *after* value *read* of $y$; and for $y$ read $h$.
—— 480, —— 14, before $g\rho\gamma a\theta^2$ place $\frac{1}{2}$.
—— 483, —— 6, for $x$ read $x_{,}$.
—— 637, —— 9, *for* (o) *read* ($l$).
—— 687, —— 9 from bottom, *after* portion *read* between.
—— 688, —— 6 from bottom, *after* term *read* of the

# BOOK THE FOURTH.

## DYNAMICS.

## SECOND PART.

### CHAPTER I.

#### GENERAL PRINCIPLE OF DYNAMICS.

350. When material points, subjected to the action of given forces, are connected together in any manner whatever, they acquire, during each instant, infinitely small velocities, different from those which these forces would impress on them, if they were free. When these forces are known, these last velocities will be so likewise; and the general problem of dynamics consists in deducing from them, in magnitude and direction, the increments of the velocities which are actually produced. Its solution depends on an extremely simple principle, for which we are indebted to D'Alembert; by means of it all questions relative to motion are reduced to simple questions of equilibrium, that can be always resolved by the rules explained in the preceding book.

In order to express this principle in an accurate manner, let $m$ be the mass of one of the material points in question, and $u\tau$ the velocity which the force that solicits it would impress upon it, in an infinitely small time $\tau$, if it was free.

Let $q\tau$ denote the increment of velocity which actually obtains during this same instant, the direction of which will, in general, be different from that of the given velocity $u\tau$; and let $u\tau$ be decomposed by the rule of the parallellogram of forces, which is equally applicable to velocities, (No. 145), into two other velocities, of which let one be $q\tau$, and the other $p\tau$. The measure of the motive force applied to the moveable will be the product $mu$; $mq$ and $mp$ will be the values of those which are capable of producing the velocities $q\tau$ and $p\tau$; and the given force $mu$ may be regarded as the resultant of the force $mq$, to which the increment of the velocity which actually obtains, is due, and of the force $mp$, the effect of which is destroyed by the connexion of the points of the system. We shall term this last, the *force lost*.

If the quantities analogous to $m$, $u$, $q$, which refer to the other points of the system, be denoted by the same letters with accents, namely, by $m'$, $u'$, $q'$, $p'$, $m''$, $u''$, $q''$, $p''$, &c., it is evident, that whatever be their number and mutual connexion, $mp$, $m'p'$, $m''p''$, &c., the forces lost should constitute an equilibrium, for if this equilibrium did not take place, these forces would produce certain infinitely small velocities during the the instant $\tau$, and, consequently, $q\tau$, $q'\tau'$, $q''\tau''$, &c., would no longer be the increments of velocity that actually have place, which is contrary to hypothesis. It is in this that the principle of D'Alembert consists. Instead of the forces $mp$, $m'p'$, $m''p''$, &c., we may substitute, in the equations of equilibrium of the system that is considered, the quantities of motion $mp\tau$, $m'p'\tau'$, $m''p''\tau''$, &c., which are proportional to them, and then the statement will be, that there is an equilibrium between the infinitely small quantities of motion, lost during each instant by all the points of the system, in consequence of their mutual connexion.

351. The preceding general statement of this principle may be changed into another, which will be frequently more convenient.

For this purpose, it may be observed, that $mu$ being the resultant of $mq$ and $mp$, each of these components, the second for example, is likewise the resultant of $mu$ and of the other component, taken in an opposite direction from that in which it acts; by replacing in this manner each of the forces lost, i. e. $mp$, $m'p'$, $m''p''$, &c., by the two forces of which it is the resultant, it is evident the principle of D'Alembert implies, that there is constantly an equilibrium between the given forces, which act on all the points of a system of material points in motion, and the forces which produce the infinitely small increments of the velocity which have place at each instant, these last forces being taken in an opposite direction from that in which they act. We may, if we please, replace the first forces by the quantities of motion $mu\tau$, $m'u'\tau'$, $m''u''\tau''$, &c., and the last by $mq\tau$, $m'q'\tau'$, $m''q''\tau''$, &c., by assigning to each of the velocities $q$, $q'$, $q''$, &c., a direction contrary to that which it really has, and by supposing the directions of $u$, $u'$, $u''$, &c., to be those which they actually have.

This second manner of stating this principle has the advantage of leading directly to equations between the unknown quantities $q$, $q'$, $q''$, &c., and the data of the problem, which are evidently the velocities $u$, $u'$, $u''$, &c. These equations will result, both from the conditions of the equilibrium, and also from the manner in which the different points of the system are, in each case, connected together; the number of them will be always the same as that of the coordinates of all those points (No. 342), and, consequently, the same as that of the components of the velocities $q$, $q'$, $q''$, &c., parallel to the axes of these coordinates, so that they will make known, both in magnitude and direction, the increments of velocity of all these material points at each instant, which is, as has been stated, the general solution of the problem of Dynamics. It is the province of the integral calculus to enable us to deduce from these infinitely small increments, the velocities themselves and the coordinates of each material point, in functions of the time.

352. When the forces $mq$, $m'q'$, $mq''$, &c., shall have been determined, if they are laid off in a direction opposite to that in which they act, and if they be then compounded with the given forces $mu$, $m'u'$, $m''u''$, &c., $mp$, $m'p'$, $m''p''$, &c., the forces lost will be obtained. It is from these last forces that the tensions of threads, of elastic rods, and of all the physical modes of connexion that can exist between the different points of the system, as also the pressures exerted on the surfaces and given curves, which the material points are constrained to describe, arise; and, according to the first method of stating the principle of D'Alembert, these pressures or tensions may be determined in the state of motion of the system, by the rules of statics applied to the forces lost, (No. 343).

Therefore, during the motion, a part of the given force, which acts on each moveable, is employed in making the velocity to vary, and does not influence the pressures or tensions in question; and the other part, which is considered to be destroyed or lost, produces these pressures or tensions, and has no influence whatever on the velocity. When the system has attained to a permanent state, in which all the points that compose it move uniformly, the first part of each force is cipher, and the entire force is destroyed, that is to say, employed in producing pressures against fixed obstacles, and the tensions of physical strings, &c., just as if this system was in equilibrio.

Hence, if a cord be supposed to move in the direction of its length, and if given forces act at its two extremities in the directions of its productions, (when this motion continues to be uniform,) the two forces will be equal, and their common value will express the tension of the cord; if, on the contrary, the two forces are unequal, the excess of the greatest above the least will be employed in accelerating or retarding the motion of the cord, and the measure of its tension will be the part of the greatest force which is destroyed by the least, or equal and contrary to this. For example, when a horse draws a

load on a road, and the motion of the system continues uniform, the effort of the horse parallel to the road is equal to the weight of the load resolved in this direction, plus the friction of the load against the road; it is constant when the state of the road and its inclination do not vary; if it be supposed to be transmitted to the load by means of cords parallel to each other and to the road, the entire effort will be equal to the sum of the tensions of all these cords; and in practice, the effort exerted in the direction of each cord, is measured by the extension of a spring interposed in the direction of its length. The inclination and state of the roads remaining the same, if the efforts of the animal increase or diminish, the motion of the system will be accelerated or retarded, while the tensions do not undergo any variation. When the road is horizontal, the friction insensible, and the motion uniform, the horse has no other force to develope but that which is necessary for his own advance; he exerts no effort in the direction of the cords attached to the load, and their tensions are constantly cipher.

353. The principle of D'Alembert obtains also in the case of the finite quantities of motion, which are lost by bodies connected together in any manner whatever, and on which simultaneous percussions are made, which percussions are, in fact, motive forces, acting on the material points with great intensities, and during very short intervals of time (No. 126).

Thus, let us suppose that a force of this nature acts on the point, whose mass is $m$, during a time which is finite, but so short, that the point $m$ and all the other points of the system do not sensibly change their position in this interval. Let us denote it by $\varepsilon$, and by $\text{U}$ the velocity of finite magnitude which this force would impress on the point $m$, if it was entirely free; likewise let $\text{Q}$ be the velocity which is actually impressed upon it, so that at the end of the time $\varepsilon$, $m$ is actuated by the velocity which it had previously to the impact, by the velocity $\text{Q}$, and by the velocity which is communicated to it, during the

same time, by the motive forces which may act on the system, independently of the percussions. Let the velocity U be resolved into two others, one equal to Q, and the other denoted by P. Let similar suppositions be made with respect to $m'$, $m''$, &c., the other points of the system, and if, relatively to these points, the quantities corresponding to U, P, Q, be denoted by U′, P′, Q′, U″, P″, Q″, &c.; an equilibrium will exist in the system between $m$P, $m'$P′, $m''$P″, &c., the quantities of motion lost, whether they be considered at the commencement or end of the time $\epsilon$.

In fact, let $\epsilon$, the duration of the percussions, be decomposed into an infinite number of infinitely small instants, and let $\tau$ be one of these instants, $m\omega\tau$, $m'\omega'\tau'$, $m''\omega''\tau''$, &c., the infinitely small parts of $m$P, $m'$P′, $m''$P″, &c., lost during this instant, and, as before, $mp\tau$, $m'p'\tau'$, $m''p''\tau$, &c., the infinitely small quantities of motion arising from the motive forces, and that are also lost during this same instant. By what has been stated in No. 350, there will be an equilibrium in the system between these two groups of quantities of motion; each of the equations relative to this equilibrium will be of the form,

$$\text{A}m\omega\tau + \text{A}'m'\omega'\tau + \text{A}''m''\omega''\tau + \&\text{c}.$$
$$+ \text{B}mp\tau + \text{B}'m'p\tau + \text{B}''m''p''\tau + \&\text{c}. = 0.;$$

in which A, A′, A″, &c., B, B′ B″, &c., denote coefficients depending an the positions of $m$, $m'$, $m''$, &c.; and this equation will subsist during $\epsilon$, the whole duration of the percussions. Therefore the sum of the values of its first member, which correspond to all instants of this duration, will be equal to the cipher; but, in this sum, the coefficients may be considered as invariable, since, by hypothesis, the positions of the points $m$, $m'$, $m''$, &c., do not sensibly change during the entire continuance of the percussions; moreover, the sum of the values of $m\omega\tau$, $m'\omega'\tau$, $m''\omega''\tau$, &c., will be the quantities of motion $m$P, $m$P′, $m''$P″, &c.; those of $mp\tau$, $m'p\tau$, $m''p''\tau$, &c., may

be neglected relatively to the first, since the effects of motive forces, such as weights and attractions directed to fixed or moveable centres, during the percussions, are generally insensible with respect to the effects of these other forces; consequently, we shall have

$$Amp + A'm'p' + A''m''p'' + \&c. = 0.$$

The same will be the case with respect to all the equations of equilibrium of the system, which will subsist between $mp$, $m'p'$, $m''p''$, &c., the quantities of motion lost; which was required to be demonstrated.

In consequence of the invariability of the coefficients $A$, $A'$, $A''$, &c., during the continuance of the percussions, these equations refer indifferently to the commencement or end of the time $\varepsilon$. For greater convenience, it is assumed that this duration is the same for all the percussions, which is evidently allowable, provided that $\varepsilon$ is the longest duration of the percussions that are considered at the same time.

These percussions, in general, arise from the impacts of the moveables, either against one another, or against fixed obstacles. It may happen that during the time $\varepsilon$, these points slide ever so little, either against one another, or against these obstacles; the frictions which they will by this means experience, will abstract from them a certain quantity of motion. Now, these quantities cannot be neglected, like those which arise from gravity and attractions; for the friction is a force proportional to the pressure; that is to say, a force which abstracts from $m$, $m'$, $m''$, &c., in each instant, infinitely small quantities of motion proportional to that which the pressure would impress on them in the same instant; hence it follows, that the effects of frictions during the time $\varepsilon$, may be comparable to those of the percussions; consequently, when the moveables $m$, $m'$, $m''$, &c., slide one against another, during the percussions, it is necessary to establish the equilibrium between the quantities of motion lost by

friction, and those represented by $mP$, $m'P'$, $m''P''$, &c. The velocities P, P′, P″, &c., may, if we please, be replaced by their components, that is to say, by velocities respectively equal and contrary to Q, Q′, Q″, &c., and by the velocities U, U′, U″, &c., taken in their proper directions.

This extension of the general principle of Dynamics to quantities of motion which have a finite magnitude, will enable us to determine the velocities of the bodies of a system at the commencement of the motion, and also during its continuance, when they strike against each other, or impinge against fixed obstacles, and generally, when the velocities of the moveables experience what are termed *sudden changes.*

354. In the different applications of the general principle of dynamics, which we propose to make in the subsequent part of this treatise, the moveables between which any mode of physical connexion whatever exists, may, moreover, act on each other, either by means of attractions or repulsions at a distance, and also may experience percussions at particular instants. But before we proceed farther, we propose to give in this chapter a simple example of each of these three circumstances, which will be of service in the general developments that we propose hereafter to detail.

Let us, in the first place, consider, as in the fourth case of No. 329, two heavy bodies attached to the extremities of an inextensible thread, and placed on two inclined planes which rest against each other. Let $h$ be the common height of those two planes, $l$ the length of one of them, $l'$ that of the other, $m$ the mass of the body placed upon the first, $m'$ that of the body placed upon the second, and $g$ the gravity. If the friction is not taken into account, the accelerating force of the first body will be the component of the weight resolved in the direction of the first plane, which is equal to $\frac{gh}{l}$; and in like manner the accelerating force of the second will be equal to $\frac{gh}{l'}$. At the end of the time $t$, let $v$ denote the velocity com-

mon to all the points of $m$, and $v'$ that of all the points of $m'$; these are supposed to be positive or negative, according as $m$ and $m'$ descend or ascend. During the instant $dt$, $v$ and $v'$ will be increased by $dv$ and $dv'$; but, during this same instant, the accelerating forces would, if they were free, impress on $m$ and $m'$ the positive velocities $\frac{gh}{l}dt$ and $\frac{gh}{l'}dt$; therefore the velocities which they lose during the same instant $dt$, in consequence of the connexion which exists between the two bodies, are $\frac{gh}{l}dt - dv$ and $\frac{gh}{l'}dt - dv'$. Now, in order that the two corresponding quantities of motion may be in equilibrio (No. 350), it is evidently necessary that they should be equal; consequently, we shall have

$$m\left(\frac{gh}{l}dt - dv\right) = m'\left(\frac{gh}{l'} - dv'\right). \qquad (1)$$

Moreover, the two velocities $v$ and $v'$ are equal, and affected with contrary signs; for, in the motion in question, one of the two masses descends, and the other ascends, describing equal spaces on the respective inclined planes; hence we have

$$v' = -v, \quad dv' = -dv.$$

If this value of $dv'$ be substituted in equation (1), we can deduce

$$dv = \frac{(ml' - m'l)h}{(m + m')ll'} gdt;$$

and, by integrating,

$$v = \frac{(ml' - m'l)h}{(m + m')ll'} . gt + c;$$

$c$ being the arbitrary constant.

If this integral be multiplied by $dt$, and integrated a second time, the space described by $m$ on its inclined plane, will be obtained; and from the preceding value of $v$, it is evident that its motion is uniformly accelerated or retarded, according

as $ml' > m'l$ or $ml' \angle m'l$. In virtue of the equation $v' = -v$, the contrary obtains with respect to $m'$.

Let T denote the tension of the thread to which $m$ and $m'$ are attached, this tension arises, as we know, from the force lost during each instant by these two bodies respectively. The value of this motive force will be one of the quantities of motion which constitute the two members of equation (1), divided by $dt$; consequently we have

$$\text{T} = m\left(\frac{gh}{l} - \frac{dv}{dt}\right);$$

which, by substituting for $dv$ its preceding value, becomes($a$)

$$\text{T} = \frac{(l+l')\,mm'hg}{(m+m')\,ll'};$$

This value becomes $\frac{mhg}{l}$, as we know it should, in the case of $ml' = m'l$, which is that of equilibrium. With respect to the pressure exerted on each inclined plane, it is equal to the weight of the body which it sustains resolved perpendicularly to this plane, and is the same as in the state of equilibrium.

335. The constant $c$ is the initial velocity of $m$; if both bodies set out from a state of rest, $c = 0$; but if one or both of them experience a percussion at the commencement of the motion, their initial velocities should be deduced from it.

Let us suppose, therefore, that at the commencement of the motion, $m$ and $m'$ experience percussions, which if the bodies were entirely free, would impress, in the direction of the productions of the thread to which they are attached, a velocity $a$ on all the points of $m$, and a velocity $a'$ on all those of $m'$. As their initial velocities are $c$ and $-c$, it follows that at the commencement, the quantities of motion lost have been, in magnitude and direction, equal to $m(a-c)$ and $m'(a'+c)$ respectively; and, in order that they may constitute an equilibrium, it is necessary that, agreeably to No. 353, they should be equal; hence we shall have

$$m(a-c) = m'(a'+c);$$

from which we can deduce

$$c = \frac{ma - m'a'}{m+m'}.$$

The percussion which the thread experiences at this instant, in the direction of each of its productions, is due to one or other of these lost quantities of motion, the common value of which is(b)

$$\frac{mm'(a+a')}{m+m'};$$

so that the initial percussion of the thread is the same, as if it was suspended vertically at a fixed point, and a body attached to its inferior extremity, was struck in the direction of the weight, by a second body actuated by this quantity of motion, and which is then united with the first.

356. In place of two heavy bodies, we can consider three or a greater number, placed on a series of inclined planes, each of them being connected with the following, by means of an inextensible string; the motion of this system will be of the same nature, and will be determined in the same manner, as in the case already considered.

We may likewise substitute for the two bodies which have been considered, a heavy chain laid on the two inclined planes. If it be homogeneous, and of a constant thickness, and if at the end of the time $t$, the lengths of its two parts be denoted by $x$ and $x'$, their masses will be to each other as these quantities, so that in equation (1), $m$ and $m'$ should be replaced by $x$ and $x'$. Moreover, during the instant $dt$, the first of these two parts is increased by the element $dx$, which is actuated by the velocity $v$, common to all its points; on this account the quantity of motion lost will be diminished by a positive or negative quantity equal to $vdx$. For a like reason, the quantity of motion lost by the second part of the chain during the same instant, should be diminished by a quantity equal to

$v'dx'$; we should therefore take $vdx$ and $v'dx'$ from the first and second members of this equation (1), which will thus become

$$x\left(\frac{gh}{l}dt - dv\right) - vdx = x'\left(\frac{gh}{l'}dt - dv'\right) - v'dx'.$$

Denoting the constant length of the chain by $\lambda$, we shall have

$$x + x' = \lambda, \quad dx + dx' = 0;$$

moreover, $v$ and $v'$ the velocities of its two parts will be respectively equal to

$$v = \frac{dx}{dt}, \quad v' = \frac{dx'}{dt};$$

hence there results $dx' = -dx$, and $vdx = v'dx'$; and, by eliminating $x'$ and $dv'$ from the equation of the motion, we obtain(*c*)

$$\frac{d^2x}{dt^2} - a^2x + \beta = 0,$$

in which, in order to abridge, we make

$$\frac{gh(l+l')}{\lambda ll'} = a^2, \quad \frac{gh}{l'} = \beta;$$

the complete integral of this linear equation is

$$x = ae^{at} + be^{-at} + \frac{\lambda l}{l+l'},$$

$e$ being the base of the Naperian system of logarithms, and $a$ and $b$ two arbitrary constants, the values of which can be determined by means of those of $x$ and $\frac{dx}{dt}$, when $t = 0$. If the entire chain exists on the same inclined plane, that is to say, if the difference $x - x'$ becomes equal to $\pm\lambda$, the nature of the motion will be changed, and it will become uniformly accelerated.

In order that the chain may remain at rest, it is necessary that we should have $a = 0$ and $b = 0$; hence we infer

$$x = \frac{\lambda l}{l + l'}, \quad x' = \frac{\lambda l'}{l + l'};$$

from which it appears, that in the state of equilibrium, $x$ and $x'$, the two parts of the chain, are to each other as $l$ and $l'$, the lengths of the inclined planes on which they are laid; so that its two extremities exist on the same horizontal(d) line. Conversely, if this condition is satisfied at any given instant, and if at this instant, the points comprising the chain are not actuated by any velocity, the equilibrium will obtain; for as the proportion

$$x : \lambda - x :: l : l',$$

gives $\frac{\lambda l}{l + l'}$ for the value of $x$, we shall have, at the instant in question,

$$ae^{at} + be^{-at} = 0;$$

and as the velocity is supposed to be equal to cipher, we shall, at the same time, have

$$\frac{dx}{dt} = aae^{at} - bae^{-at} = 0;$$

from which there results $a = 0$ and $b = 0$.

357. For a second example of the application of the general principle of dynamics, (let the motion of two bodies subjected to their mutual repulsion be considered); and, in order to reduce the question to the simplest possible case, we shall suppose that no initial velocity is impressed on them, (perpendicularly to the line drawn from the one to the other) so that both of them must move on the same right line, given in position.

Let their masses be $m$ and $m'$, $x$ and $x'$ their distances, at the end of the time $t$, from a fixed point taken on this line, $v$ and $v'$ their velocities, so that at this instant, we have

$$v = \frac{dx}{dt}, \quad v' = \frac{dx'}{dt}.$$

At the same time, let R be the repulsive force acting in

opposite directions on $m$ and $m'$, and which, for greater clearness, we shall suppose to increase the distance $x'$, and to diminish the distance $x$. During the instant $dt$, this motive force would impress a velocity $\frac{\text{R}dt}{m'}$ on the mass $m'$; and, as the increase of velocity of $m'$ is really $dv'$, it follows that its velocity, and quantity of motion, which are lost during this instant, will be $\frac{\text{R}dt}{m'} - dv'$ and $\text{R}dt - m'dv'$. Likewise the quantity of motion lost by $m$, in the same direction and during the same instant, will be $-\text{R}dt - mdv$. Now, as these two material points are entirely free, it is necessary, in order that these quantities of motion may be in equilibrio, that they be separately equal to cipher; consequently, we shall have

$$mdv + \text{R}dt = 0, \quad m'dv' - \text{R}dt = 0.$$

Let $r$ be the distance btween the two material points $m$ and $m'$, so that we may have

$$x' - x = r, \quad dx' - dx = dr.$$

Then since $dx = vdt$ and $dx' = v'dt$, we shall obtain from the preceding equations

$$mdv + m'dv' = 0, \quad 2mvdv + 2m'v'dv' = 2\text{R}dr,$$

and by integrating, we shall have(*c*)

$$mv + m'v' = c, \quad mv^2 + m'v'^2 = 2\int \text{R}dr + c'.$$

The force R will be a given function of $r$; we can therefore obtain the integral $\int \text{R}dr$ either exactly or by approximation; and if we suppose that this integral is cipher when $r = a$, ($a$ being the value of $r$ at the commencement of the motion), its value at any instant whatever will be a function of $r$ and $a$, which we can denote $f(r, a)$. Likewise, let $\alpha$ and $\alpha'$ be the initial velocities of $m$ and $m'$; we shall have, at the same time,

$$r = a, \quad f(r, a) = 0, \quad v = \alpha, \quad v' = \alpha',$$

and, consequently,

$$c = ma + m'a', \quad c' = ma^2 + m'a'^2;$$

hence there will result at any instant whatever,

$$\left.\begin{aligned} mv + m'v' &= ma + m'a', \\ mv^2 + m'v'^2 &= 2f(r, a) + ma^2 + m'a'^2. \end{aligned}\right\} \quad (1)$$

By means of these last equations, the velocities of the two moveables may be known in functions of their mutual distance; and it follows from them, that whenever the value of this distance becomes the same as it was at any previous time, the squares $v^2$ and $v'^2$ will also have the same values, and that each moveable will resume an equal velocity, in the same direction or in an opposite one.

$v$ and $v'$ being known in functions of $r$; by means of the equation

$$dt = \frac{dr}{v' - v},$$

the value of $t$ can be determined in a function of $r$, by a second integration, and conversely.

Moreover, if the first of equations (1) be multiplied by $dt$, there results, by integrating,

$$mx + m'x' = (ma + m'a')\,t + b,$$

$b$ being an arbitrary constant, which will be known by means of the initial positions of the two moveables; and this equation, (together with the equation $x' - x = r$,) will make known their positions at any instant whatever, that is to say, the values of $x$ and $x'$ in the functions of $r$ or of $t$; which is the complete solution of the problem.

If the mutual action of the two moveables was attractive, the sign of R, and, consequently, that of $f(r, a)$, should be changed in the preceding formulæ. If this force was repulsive at certain distances, and attractive at others, R should be such a function of $r$ as would change its sign within the

limits of the values of the variable. It results from the preceding equation, that in no case is the motion of the centre of gravity altered by the mutual action of the two moveables; for its first member divided by $m + m'$, expresses the distance of the centre of gravity from the fixed origin of the axes of $x$ and $x'$; so that the motion of the centre of gravity is uniform and independent of the force R.

358. Equations (1) obtain also in the case of the motion of two solid bodies, $m$ and $m'$, of any magnitude whatever, and subjected to the action of the force R, provided that the velocities of all the points of these two bodies are constantly parallel to a given line. This force R, whether attractive or repulsive, may then be produced by a spring which is dilated or contracted between the two bodies against which its two extremities are pressed; or we may even suppose that the force R arises from an elastic fluid which is developed between these two bodies, and repels them in opposite directions, from one another. This last case is that of the motion of a bullet and gun, during the time that the first traverses the barrel of the piece. Let $m$ denote the mass of the bullet, and $m'$ that of the gun, then in order to be able to apply the preceding formulæ, we must suppose that all the powder is converted into gas at the very commencement of the motion. The length of the charge will be equal to $a$ the initial distance of the two bodies, and when this distance becomes $r$, the force R will express the pressure which the gas, thus dilated, exerts on each of these two bodies. It is necessary, moreover, to make some hypothesis on the value of R as a function of $r$. Now, if the temperature of the gas remains constant during its dilatation, the force R, by the law of Mariotte, will vary in the inverse ratio of the spaces which it occupies in the interior of the barrel. Let, therefore, $k$ be the pressure on the unit of the surface, exerted by the gas, the instant that the powder is ignited, and when it still occupies the same space as the charge. If $\omega$ denotes the section of the charge perpendicular to its length, which is like-

wise the interior section of the piece; $k\omega$ will be the value of R at the commencement of the motion; and, if the temperature be constant, we shall have ($f$)

$$R = \frac{k\omega a}{r},$$

at the instant, when the distance of the two moveables is equal to $r$; for, at these two epochs, the spaces occupied by the gas are to each other as the lengths $a$ and $r$.

This expression for R has been generally adopted, though it is founded on two inaccurate hypotheses; for, 1st, the entire of the charge is not converted into gas before the bullet begins to move; and 2ndly, during its dilatation in the barrel of the piece, the gas which is generated must experience very great diminutions of temperature. But these two causes have opposite effects on the decrease of the value of R: the second tends evidently to render this decrease more rapid, while the effect of the first must be to retard it, in consequence of the new quantities of gas which are successively added to the initial quantity. If these two opposite causes be supposed very nearly to compensate each other, their influence on the expression of R, as a function of $r$, need not be taken into account. This being agreed on, in consequence of the value of R given above, we shall have ($g$)

$$f(r, a) = k\omega a \log\frac{a}{r},$$

the integral $f(r, a)$ being supposed to be cipher when $r = a$. The initial velocities of the bullet and barrel are supposed to be cipher($h$); hence if in equations (1), we make $\alpha = 0$, and $\alpha' = 0$, and if the preceding value of $f(r, a)$ be substituted, we shall have

$$mv + m'v' = 0, \quad mv^2 + m'v'^2 = 2k\omega a \log\frac{a}{r}.$$

Let $l$ be the length of the barrel, v the velocity of the bullet

at the mouth of the barrel, v′ the corresponding velocity o
*recoil*, we shall have, at the same time,

$$r = l, \quad v = \mathrm{v}, \quad v' = \mathrm{v}';$$

and by means of the preceding equations we shall obtain

$$\mathrm{v}^2 = \frac{2m'k\omega a}{m(m + m')} \log \frac{l}{a};$$

from which v, the velocity of projection, will be obtai
and that of the recoil will be, abstracting from the sign, e
to this velocity v multiplied by $\frac{m}{m'}$.

If the differential of $\mathrm{v}^2$ relatively to $a$, be put equ
cipher, the length of the charge, which, every thing
being the same, renders the velocity of projection a *maxim*
will be determined.

By this means we obtain

$$\log \frac{l}{a} = 1;$$

and as this logarithm is taken in the Naperian system, it
lows, that if, as usual, $e$ denotes the base of this system,
shall have $l = ea$; so that the value of $a$ in question, wi
a little greater than the third of $l$ the length of the barrel

359. As the mass $m'$ consists of that of the barrel
stock or frame of the gun, it is always very great relat
to that of the bullet; consequently, in the value of $\mathrm{v}^2$,
divisor $m + m'$ may be reduced to $m'$, by which means it
become

$$\mathrm{v}^2 = \frac{2k\omega a}{m} \log \frac{l}{a}.$$

In order to apply this formula, the constant $k$, whicl
notes the elastic force of the powder, when it is converte
gas, should be known. For this purpose, let D be the de
of the powder in its natural state; then the mass of the cl

will be $D\omega a$; and, its weight being supposed equal to one-third of the weight of the bullet, we shall have

$$m = 3D\omega a;$$

hence, by means of equation (2), we can deduce

$$k = \frac{3DV^2}{2\log\frac{l}{a}}.$$

This quantity $k$ will be the maximum pressure of the gas produced by the ignition of the powder, referred to the unit of surface; in order to compare it with the atmospheric pressure, if $p$ be this other pressure, $h$ the height of the mercury in the barometer, $g$ the gravity, and $\mu$ the density of the mercury; we shall have

$$p = g\mu h.$$

Likewise, if $M$ be the modulus of the tables in the vulgar system of logarithms, and $\lambda$ the logarithm of $\frac{l}{a}$ taken in these tables, so that we may have

$$\lambda = M\log\frac{l}{a};$$

we shall obtain from these values

$$\frac{k}{p} = \frac{3MDV^2}{2\lambda\mu g h}.$$

Since the densities of the powder and mercury at the mean temperature, i. e. at 18° of the centigrade thermometer, are

$$D = 0,8335, \quad \mu = 13,548;$$

likewise, as

$$g = 9^{m},80896, \quad h = 0^{m},76; \quad M = 0,4342945,$$

the preceding formula, will become

$$\frac{k}{p} = (0,0053761)\,\frac{V^2}{\lambda}.$$

In the case of a piece of 24, the charge of which is a third of the weight of the bullet, we have (No. 216).

$$v = 463^m, \quad \frac{l}{a} = \frac{1368}{134};$$

hence we obtain

$$k = 1142 . p.$$

Relatively to a piece of 12, we have, in the same manner,

$$v = 493^m, \frac{l}{a} = \frac{1248}{99};$$

which gives

$$k = 1187 . p.$$

These two values of $k$ should be equal, if the theory was rigorously true, and the data exact; as it is, if their mean value be taken, we shall have

$$k = 1165 . p.$$

which is the value of $k$ that should be employed in formula (2); but this expression of $v^2$ can only be regarded as an empirical formula, on account of the hypothesis on which it is founded, and also because in the direct computation of the motion of the bullet in the barrel of the piece, the mass of the powder converted into gas, should be taken into account. At the same time that this fluid urges the bullet and the piece in opposite directions, a part of the force which it developes is employed in moving its own mass, which should not be neglected with respect to that of the projectile; and we may conceive that in this case, the velocity of projection must be less, than if, the elastic force of the powder remaining the same, its mass was insensible, as is assumed in the preceding analysis. This remark, for which we are indebted to Lagrange, shews the necessity of considering at the same time the motions of the powder and of the two masses $m$ and $m'$, while the bullet is within the barrel; but then the question becomes so complicated, and, in consequence, the difficulty of the computation

so great, that it is impossible to obtain any useful practical result. Therefore, it is preferable to have recourse to experiment in order to determine the velocities of projection of bodies shot from the mouth of pieces of artillery. In (No. 216), a method of obtaining these velocities from the consideration of the ranges of the projectile was explained; but besides this, there is also another method, which will be detailed in one of the subsequent chapters, (Nos. 402, 403, 404).

360. Let the principle of D'Alembert be now applied to the simplest case of the impact of bodies, and let us suppose that the bodies are two homogeneous spheres, whose centres move in the same right line, and that all their points describe parallels to this line.

Let $m$ and $m'$ be the masses of these two bodies, $v$ and $v'$ their velocities at the very commencement of their contact, that is to say, at the first instant of the impact; the signs of $v$ and $v'$ will be same or opposite, according as the two moveables proceed in the same or contrary directions. In both cases, we shall consider the velocity $v$ as positive, and, after the impact, the velocity of each of the moveables will be regarded as positive or negative, according as it has the same or an opposite direction from the velocity of $m$ before the impact.

Whatever be the degree of hardness of the two moveables, they are always more or less compressible; therefore, because the bodies move with different velocities $v$ and $v'$, one of them must impinge on, and consequently compress the other, and, during this compression, the velocity of one of the bodies, of $m$, for example, will diminish by infinitely small degrees, while that of $m'$ will increase in the same manner, until these two velocities become equal. Now, setting out from this instant, two distinct cases present themselves to be considered.

1. If the two spheres are altogether destitute of elasticity, they will cease to act on each other, from the instant that their velocities are thus reduced to an equality, and they will continue to move with a common velocity, remaining in juxta-

position with each other, and retaining the forms which in consequence of their compression they have acquired.

2. If, on the contrary, the two spheres are elastic, they will tend to resume their natural form. While they revert towards it, and thus continually press the one against the other, the velocity of $m$ will gradually decrease, and that of $m'$ gradually increase; until the instant in which these two bodies separate, and this will be the termination of the impact. Now, in the case of perfect elasticity, we suppose that the second part of the impact is altogether similar to the first; that at the end of the impact, the two bodies have exactly resumed their spherical form and a velocity common to all the points of each of them; and that, during its second part, they lose or gain quantities of motion equal to those which have been already lost or gained during the first.

The problem of the impact of two spheres would present no new difficulty, and would be a case of that discussed in No. 357, if their radii were infinitely small.

In order to resolve it completely, when the radii are of a finite magnitude, (we should take into account the propagation of motion in their masses, and determine the state of the two bodies at any instant during the continuance of the phenomenon; which, in the actual state of the science, we may regard as impossible.) We shall therefore admit the suppositions, which have been now explained, as being the data of the question in which we are engaged; and by combining these data with the principle of D'Alembert, applied to quantities of motion of a finite magnitude, it will be only necessary to determine the velocities of the two spheres at the end of the shock, by means of their primitive masses and velocities, both when these two bodies are entirely devoid of elasticity, and also when they are perfectly elastic. *Soft* bodies are the only ones which have no sensible elasticity; the greater part of *hard* bodies revert to their primitive form, when they are not broken by the impact.

361. In the case of soft bodies, if $u$ be the velocity after the impact, which is common to the two spheres, the velocity lost by $m$ will be $v-u$, and that gained by $m'$ will be $u-v'$. If, therefore, these two bodies move, the one before the other, with these velocities, $v-u$, $u-v'$, by the principle established in No. 353, they must constitute an equilibrium; this requires (No. 127) that the quantities of motion corresponding to these velocities should be equal. Consequently we shall have

$$m(v-u)=m'(u-v');$$

from which there results

$$u=\frac{mv+m'v'}{m+m'}, \tag{a}$$

which is the value of $u$ required to be obtained.

If $m'$ be at rest before the shock, and if by reason of its density, this mass is extremely great, so that it may be considered as infinite relatively to $m$, we shall have $u=0$, q. p.($l$). The mass $m'$ will in this case represent a fixed obstacle; and the body, when devoid of elasticity, will be reduced to a state of rest, when it impinges against this obstacle.

By the *living force*, or the *vis viva* of a material point, or more generally of a body, all the points of which are endowed with the same velocity, is meant the product of its mass by the square of this velocity. Therefore, the sum of the living forces of $m$ and $m'$ before the shock, is $mv^2+mv'^2$, and $mu^2+m'u^2$, after the shock. Now, it appears from formula (a), that the second sum is always less than the first; for if the quantity

$$-2u(mv+m'v'-mu-m'u),$$

which, in virtue of equation (a), is cipher, be taken from

$$mv^2+m'v'^2-mu^2-m'u^2,$$

the difference, when concinnated, becomes

$$m(v-u)^2+m'(u-v')^2,$$

which is a positive quantity.

Therefore, in the impact of two spheres, which are destitute of elasticity, there is a loss of living force; and this loss is equal, as appears from what precedes, to the sum of the living forces due to the velocities $v-u$, $u-v'$, which have been respectively lost and gained by these two bodies. This result is a particular case of a general theorem, for which we are indebted to Carnot, and which will be demonstrated in the sequel.

362. In the first part of the impact, that is to say, until the instant of the greatest compression, the two spheres are always thus affected, whatever be the degree of their elasticity; so that the velocity $u$, the expression for which has been determined above, is always that which is common to them at this instant. Therefore, during this first part, the diminution of the velocity of $m$, and the increase of that of $m'$, will be $v-u$ and $u-v'$. Now, if these two spheres are perfectly elastic, $m$ will experience, in the second part of the impact, a second diminution of velocity equal to the first, and consequently its velocity at the end of the impact will be $v-2(v-u) = 2u-v$. At the same time, $m'$ will experience a second increase of velocity equal to $u-v'$, and its final velocity will be $v'+2(u-v')$ or $2u-v'$. If, therefore, $\mathrm{v}$ and $\mathrm{v}'$ denote the velocities of $m$ and $m'$ after the impact, we shall have

$$\mathrm{v} = 2u - v, \quad \mathrm{v}' = 2u - v';$$

the value of $u$ being, as stated above, always furnished by formula (a).

If these two velocities be respectively taken the one from the other, we shall have

$$\mathrm{v} - \mathrm{v}' = v' - v;$$

from which it appears, that in this impact the relative velocity of the two moveables changes its sign, but preserves the same magnitude.

If the density of the mass $m'$ be such, that relatively to

the mass $m$, it may be considered as infinite, and if $v' = 0$, we shall have $u = 0$, and, consequently, $\mathrm{v} = -v$; hence it follows, that when a perfectly elastic sphere strikes a fixed obstacle, it is reflected back with a velocity equal and contrary to that which it had before the impact. Therefore, in the case of a heavy sphere, which falls in a vacuo, on a horizontal immoveable plane, it should be reflected back to the height from which it fell.

The sum of the living forces, before and after the impact will be the same; or, in other words, we shall have

$$mv^2 + m'v'^2 = m(2u - v)^2 + m'(2u - v')^2;$$

which may be reduced to

$$4u(mu + m'u - mv - m'v') = 0,$$

an equation which by virtue of formula (a) is evidently identical.

363. If we suppose $m = m'$, we shall have

$$2u = v + v', \quad \mathrm{v} = v', \quad \mathrm{v}' = v.$$

Therefore, in the impact of two perfectly elastic spheres, the masses of which are equal, there is an interchange of velocity; and if one of the two is at rest before the impact, the other will remain at rest after the impact, and the first will be actuated by the primitive velocity of the second.

It follows from this, that if there be a series of balls equal in mass, whose centres exist all in the same right line, and if the first be the only one of them in motion, and actuated by a velocity equal to $v$, in the direction of this line, and moving towards the other balls, this first ball will be reduced to a state of repose by impinging on the second, which will be actuated by the velocity $v$, with which it will impinge on the third, and will afterwards be reduced to a state of rest, then the third will be actuated by the velocity $v$, which it will lose by impinging on the fourth, and so on until the last, which will retain the velocity $v$. Therefore, after this series of impacts, all

the balls will remain at rest except the last, which will move off with the velocity with which the first was primitively actuated; and as this result is independent of the magnitude of the intervals existing between the consecutive balls, it follows that it will likewise obtain when these intervals disappear, and when, consequently, all the balls after the first are in contact. Thus, when a series of any number of perfectly elastic balls, of equal mass, the centres of which are in the same right line at rest, and successively in contact with each other, is struck by another elastic ball equal to each of them, and moving in the line of the centres, this last will be united to the series, all the balls composing which will remain at rest, with the exception of the ball placed at the other extremity, which will be detached from the others, and move off with the velocity of the striking ball; this is, in fact, what may be frequently observed in the case of billiard balls.

In general, the laws of the impact of spherical bodies, both hard and soft, which are consequences of the hypotheses of No. 360, have been confirmed by numerous experiments made on equal and unequal balls, consisting of the same or of different materials, and actuated by velocities which were in any ratio whatever to each other.

364. The motion of the centre of gravity of a system of bodies is never altered by the impact or mutual action of the moveables. This important proposition, the simplest case of which was already taken notice of in No. 357, and which can also be verified in the impact of soft or perfectly elastic bodies, will be demonstrated in all its generality in a subsequent part of this treatise. In order to prove that it obtains in the impact of bodies, whether soft or elastic, let $x$ and $x'$ be the distances at the end of the time $t$, of the centres of $m$ and $m'$ from a fixed point on the line along which they move. Likewise, let $x_1$ be the distance at this instant of the centre of gravity of $m$ and $m'$ from the same point; we shall have (No. 65)

$$(m + m')x_1 = mx + m'x'.$$

By differentiating, we obtain,

$$(m+m')\frac{dx_1}{dt}=m\frac{dx}{dt}+m'\frac{dx'}{dt}; \qquad \text{(b)}$$

by means of this equation, $\frac{dx_1}{dt}$, the velocity of the centre of gravity corresponding to the velocities of the two spheres, can be determined. Now, before the impact, we have

$$\frac{dx}{dt}=v, \quad \frac{dx'}{dt}=v';$$

and, consequently,

$$\frac{dx_1}{dt}=\frac{mv+m'v'}{m+m'}.$$

After the impact, we have

$$\frac{dx}{dt}=\frac{dx'}{dt}=u,$$

in the case of soft bodies, and

$$\frac{dx}{dt}=2u-v, \quad \frac{dx'}{dt}=2u-v',$$

in the case of perfectly elastic bodies. By substituting these values successively in equation (b), we shall obtain, by means of equation (a), $\frac{dx_1}{dt}=u$ in both cases, which, in virtue of this same equation (a), is the value of $\frac{dx_1}{dt}$ before the impact. Consequently, in the impact of the two spheres, the motion of their centre of gravity is not affected.

As the velocity of this point is always the sum of the quantities of motion of the bodies divided by the sum of their masses, it is evident that in the impact of two spherical bodies, either soft or elastic, the sum of the quantities of motion is not changed, regard being had, in this sum, to the signs of the velocities.

If $v'$, the velocity of $m'$, be cipher, and if this mass is very small relatively to $m$, the quantity of motion impressed on $m'$, and taken from $m$, will be, very nearly, $m'v$ or $2m'v$, according as these bodies are devoid of elasticity or perfectly elastic($l$).

365. Heretofore the resistance of fluids has been assimilated to a series of impacts of the moveable against the molecules of the medium which it traverses; and although, according to M. Poisson, the theory of resistance founded on this hypothesis should be abandoned, he deems it expedient, notwithstanding, briefly to explain it here.

Let us suppose that the moveable is a right cylinder which moves in the direction of its length. Let $\omega$ be the area of its base, perpendicular to this dimension and to the direction of its motion; likewise let $m$ be the mass of the moveable, and $\rho$ the density of the liquid or aeriform fluid in which it moves. At the end of the time $t$, let $v$ denote its velocity, and $x$ the distance of its anterior base from a fixed point, taken on the perpendicular to this plane, so that we may have $dx = vdt$. In the instant $dt$, this base will traverse the space $dx$, the body will therefore strike all the material points of the fluid comprised in a section, of which the base is $\omega$, the height $dx$, and the mass $\rho\omega dx$. Now, all these points are considered as detached from each other, and not acting at all on the surrounding fluid; and, in this hypothesis, the diminution of the quantity of motion experienced by the body during the instant $dt$, will be the product of its velocity $v$, and of $\rho\omega dx$, the mass struck, or the double of this product, according as this impact is compared to that of bodies destitute of all elasticity, or to the impact of perfectly elastic bodies. The value on the first hypothesis, namely, $v\rho\omega dx$, is that which deviates least from experiment; therefore by adopting it, and observing that the variation of the quantity of motion of $m$ during the instant $dt$, is $mdv$, we shall have

$$mdv = -v\rho\omega dx;$$

and if for $dx$ its value $vdt$ be substituted, there results, by dividing by $dt$,

$$m\frac{dv}{dt} = -\rho\omega v^2,$$

which expresses the motive force, arising from the resistance exerted on a plane surface, perpendicular to the direction of the motion.

It appears from inspection of the above expression, that this resistance is proportional to the density of the fluid, to the surface on which it acts, and to the square of the velocity of the body. Denoting the height through which the body should fall freely to acquire this velocity by $h$, and the gravity by $g$, that is to say, making $v^2 = 2gh$, its value will become $2g\rho\omega h$, so that it is equal to the weight of a cylinder of the fluid, the base of which is the surface perpendicular to the direction of the velocity, and the height is equal to twice that through which a heavy body should fall in a vacuo, to acquire this same velocity.

If the direction of the motion is not perpendicular to the plane surface which experiences the resistance, the velocity of the body should be resolved into two others, the one perpendicular, the other parallel to this plane; we suppose that the velocity parallel to the plane gives rise only to a friction, which we will not now take into account, so that the resistance properly so called, is the same as if the normal velocity solely existed; this is the reason why this component is substituted for the velocity $v$ in the preceding value of the resistance, which then becomes $\rho\omega v^2\cos^2 i$, $i$ being the angle which the normal to the surface $\omega$ makes with the direction of the velocity $v$.

366. This result being admitted, and extended to the infinitely small elements of curved surfaces, the resistance experienced by a solid body of any form whatever, may be obtained in the following manner:

Let us suppose, for greater simplicity, that the question related to a solid of revolution, all the points of which de-

scribed lines parallel to the axis of its figure, with a velocity $v$. Let this axis, represented by AB (fig. 1), be that of the abscissæ, and AMB its generating curve; and let $x$ and $y$ denote the abscissa CP and ordinate PM of M any point of this curve. Let us suppose that the greatest section of the solid, perpendicular to the axis of the figure, is that which belongs to the point C, the origin of the coordinates, and that, consequently, CD is the greatest ordinate of the curve MAB. As the motion is directed from B to A, the portion of the surface which experiences the resistance of the medium will be that which belongs to the part DMA of this curve; $ds$ being the differential element of this curve at any point whatever M, we shall have

$$\cos i = \frac{dy}{ds},$$

which is the value of the cosine of the angle that the normal at this point, makes with the axis of $x$, that is to say, with the direction of the motion; and this angle will be the same in the entire extent of the zone generated by the revolution of $ds$ about AB, the surface of which zone is equal, as we know, to $2\pi y ds$. Therefore each of the plane elements of this zone will experience a normal resistance equal to the product of this element multiplied by $\rho v^2 \cos^2 i$. If this force be resolved into two others, one perpendicular, and the other parallel to AB, it is evident that the components perpendicular to AB will destroy each other's effects, two by two; and the value of each component parallel to AB will be equal to the resistance which is perpendicular to the zone, multiplied by $\cos i$; consequently the sum of these components for the entire zone will be equal to the product of the surface $2\pi y ds$ by $\rho v^2 \cos^2 i$, and by $\cos i$, which, by substituting for $\cos i$ its value, is equal to $2\pi\rho v^2 y \dfrac{dy^3}{ds^2}$.

Hence, if we denote the entire resistance experienced by the solid in a direction contrary to that of its motion, by R, and make CA $= a$, we shall have

$$\text{R} = 2\pi\rho v^2 \int_0^a y \frac{dy^3}{ds^2}, \qquad \text{(c)}$$

for the value of this motive force.

If the body be a sphere, its centre will be at the point C, and its radius will be equal to $r$. Denoting the angle MCA by $\theta$, we shall have

$$y = a \sin\theta, \quad dy = a \cos\theta d\theta, \quad ds = a d\theta;$$

from which there results ($m$)

$$\text{R} = 2\pi\rho v^2 a^2 \int_0^{\frac{1}{2}\pi} \cos^3\theta \sin\theta . d\theta = \tfrac{1}{2}\pi\rho a^2 v^2;$$

which shews that the resistance experienced by a sphere is the half of that which the circumscribed cylinder, whose base $\pi a^2$ is perpendicular to the direction of the motion, experiences.

367. It is Newton to whom we are indebted for this first essay on the resistance of fluids; he it was also who first determined the motion of bodies subject to the action of a force depending on their velocity. From a comparison of the result of his computation, with the observed time of descent of a sphere, which falls in the air, from a great height, he observed that it was necessary, in order to make them agree together, to reduce the preceding value of R to the one-half.

From other experiments instituted by Borda, it would appear that this value should be only reduced to three-fifths, which gives

$$\text{R} = \tfrac{3}{10}\pi\rho a^2 v^2.$$

D denoting the density of the sphere, its mass will be $\dfrac{4\pi \text{D} a^3}{3}$; and if R be divided by this mass, and if the accelerating force resulting from this division be denoted by $\phi$, we shall obtain

$$\phi = \frac{9\rho v^2}{40\text{D}a},$$

which is, in fact, the expression for the resistance most generally adopted by authors who have written on the Ballistic pendulum, and which has been already cited in No. 216.

In virtue of formula (c), the determination of the solid of revolution which experiences the least resistance in its passage through a fluid, consists in finding the generating curve of this solid, for which the integral $\int_0^a y \frac{dy^3}{ds^2}$ is a *minimum;* this problem can be easily solved by the rules of the calculus of variations. Newton gave the solution long before other geometers occupied themselves with this description of questions, however he did not point out the method that he followed in arriving at this result ($n$).

The preceding theory of resistance is founded, as we have seen, on a vague comparison of the action of a fluid with the impact of bodies, and on the inadmissible supposition that in this impact the molecules of the fluid act solely on the body, and not at all on each other. It is contradicted by observation, inasmuch as the absolute magnitude assigned by the calculus is double of that which results from experiment; it is also at variance with observation as to the law of the resistance in a function of the velocity, which, according to this theory, should be always proportional to the square of the velocity, while it results from the observed decrease of amplitudes in very small oscillations of the pendulum (No. 187), that this force is only proportional to the first power of the very small velocities. The resistance which a liquid or aeriform fluid opposes to the motion of a solid body, is made up of a friction against the surface, and of the resultant of the pressures which this fluid exerts against the entire of this surface. In order to determine this second part, which is the resistance properly so called, we should consider at the same time the motions of the body and of the fluid, as M. Poisson did in the memoir already cited (No. 191). This force may be different in oscillatory and progressive motion, in liquids and in gases; and in these last, it may depend on their temperature, and not on their density solely, a point which it would be of consequence to verify by experience.

## CHAPTER II.

### DETERMINATION OF MOMENTS OF INERTIA AND OF PRINCIPAL AXES.

368. In the subsequent chapters of this book, the different cases of the motion of a solid body will be considered. In order to obtain more readily the equations of its motion, we shall suppose each body to be divided into parts, which though insensible, are nevertheless of a finite magnitude, comprising immense numbers of molecules. (Even though this body should consist of detached molecules, the sums relative to its insensible parts may still, without appreciable error, be changed, as in No. 98, into definite integrals;) so that in the following discussions, the parts in question may be treated as infinitely small.

The equations of motion of a solid body will contain nine definite integrals, namely,

$$\begin{array}{lll} \int xdm, & \int ydm, & \int zdm, \\ \int xydm, & \int xzdm, & \int yzdm, \\ \int z^2dm, & \int y^2dm, & \int x^2dm; \end{array}$$

$dm$ denotes the differential element of that point of the mass, whose coordinates are $x, y, z$, and the integrals are supposed to extend to the entire mass of the body, which we shall designate by M. The three first depend on the position of the centre of gravity; and if its three coordinates be $x_1, y_1, z_1$, we shall have (No. 91)

$$\int xdm = \text{M}x_1, \quad \int ydm = \text{M}y_1, \quad \int zdm = \text{M}z_1;$$

so that when this point is taken for the origin of the coordinates, each of these three integrals will be cipher.

It will be proved farther on, that wherever this origin may be, we can always determine the *direction* of the three axes in such a manner, that we may have

$$\int xydm = 0, \quad \int xzdm = 0, \quad \int yzdm = 0.$$

The three rectangular axes relatively to which these three integrals vanish, are termed *principal axes.*

With respect to the three last of the nine integrals, we shall express them by means of three others denoted by A, B, C, which will be respectively

$$A = \int(y^2 + z^2)\, dm, \quad B = \int(z^2 + x^2)\, dm, \quad C = \int(x^2 + y^2)\, dm;$$

from which we can obtain

$$2\int z^2dm = A + B - C,$$
$$2\int y^2dm = C + A - B,$$
$$2\int x^2dm = B + C - A.$$

In general, the sum of the elements of the mass of a body, multiplied by the squares of their respective distances from any line, is termed the moment of inertia of the body with respect to this line. Thus, A, B, C, will be the moments of inertia of the body, with respect to the axes of $x, y, z$; because, for example, $y^2 + z^2$ is the square of the distance of $dm$ from the axis of $x$. When these lines are principal axes, A, B, C, are termed the *principal* moments of inertia.

By placing the origin of the coordinates at the centre of gravity, and by making the principal axes, the axes of the coordinates, the equations of motion are simplified, because some of their terms by this means disappear; this point and these axes possess, moreover, other important properties in dynamics, which we shall advert to in the sequel.

369. The determination of the moments of inertia is a problem of the integral calculus, which may be always solved either exactly, or by the method of quadratures.

The simplest example is the calculation of the moment of inertia of a homogeneous rectangular parallelopiped, with respect to one of its sides. Let its three adjacent sides be taken for the axes of $x, y, z$, and let their lengths be denoted by $a, b, c$; then if each of these three lines be divided into an infinite number of infinitely small parts, by drawing through all the points of division, planes parallel to the faces of the parallellopiped, these three series of planes will divide it into elements which are infinitely small in their three dimensions. The volume of the element whose three coordinates are $x, y, z$, will be evidently $dx\,dy\,dz$; therefore, its mass will be

$$dm = \rho dx\,dy\,dz;$$

$\rho$ being the density of the parallellopiped, which is supposed to be constant. Consequently, C the moment of inertia with respect to the side which is taken for the axis of $z$, and whose length is $c$, will be

$$\text{C} = \rho \iiint (x^2 + y^2)\, dx\,dy\,dz.$$

This triple integral should be extended to all the elements of the given parallellopiped, and then integrated, in any order we please, from $x = 0$, $y = 0$, $z = 0$, to $x = a$, $y = b$, $z = c$; from this there results, without any difficulty,

$$\text{C} = \rho \left(\frac{a^3bc}{3} + \frac{ab^3c}{3}\right),$$

or, what comes to the same thing($a$),

$$\text{C} = \tfrac{1}{3}\,\text{M}\,(a^2 + b^2);$$

M being the mass of the body, and, consequently, equal to $\rho\,(abc)$. In the same manner, it might be shewn that the moments of inertia of the same body, with respect to the sides whose lengths are $b$ and $a$, will be respectively

$$\text{B} = \tfrac{1}{3}\text{M}\,(c^2 + a^2), \quad \text{A} = \tfrac{1}{3}\text{M}\,(b^2 + c^2).$$

370. For a second example, let it be proposed to determine the moment of inertia of a homogeneous ellipsoid with respect to one of its three axes of figure.

Denoting the lengths of its three principal diameters by $2a$, $2b$, $2c$, if the directions of the axes of the coordinates $x$, $y$, $z$, coincide with these diameters, the equation of the surface of the ellipsoid will be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \tag{a}$$

Its moment of inertia, with respect to the axis of $z$, will be expressed by the same triple integral as in the preceding problem; the constant $\rho$ denoting always the density of the body. In order to obtain this triple integral, which should be extended to the entire mass of the spheroid, we shall integrate first with respect to $z$, $x$ and $y$ being considered as constant; then, with respect to $y$, $x$ being still considered as constant, and finally, with respect to $x$ itself. We may follow any order we please in these three successive integrations; in them the ellipsoid is supposed to be divided into an infinite number of elliptic slices parallel to the plane of the axes of $y$ and $z$; and each slice to be divided in the same manner into an infinite number of parallellopipeds parallel to the axes of $z$, and terminated at the surface, and each parallellopiped into elements infinitely small in their three dimensions. The limits of the integral with respect to $z$ will be the two values of this variable which are furnished by equation (a); this definite integral will express, in a function of $x$ and $y$, the moment of inertia of any one of the parallellopipeds. The limits of the integral relative to $y$ will be the two values of this variable, which belong to the same value of $x$, in the equation of the section of the ellipsoid made by the plane of $x$ and $y$; it will express the moment of inertia of the section parallel to the plane of the axes of $y$ and $z$, whose distance from this plane is equal to $x$. Finally, if the integral relative to $x$ be taken from

$x = -a$ to $x = a$, it will express the moment of inertia of the entire ellipsoid.

By integrating with respect to $z$, there results

$$(x^2 + y^2)\, z\, dx\, dy + \text{constant}.$$

The two limits furnished by equation (a) being

$$z = \pm c \sqrt{1 - \frac{y^2}{b^2} - \frac{x^2}{a^2}};$$

the definite integral will consequently be

$$2\rho c x^2 \sqrt{1 - \frac{y^2}{b^2} - \frac{x^2}{a^2}}\, dx\, dy + 2\rho c y^2 \sqrt{1 - \frac{y^2}{b^2} - \frac{x^2}{a^2}}\, dx\, dy.$$

If, in order to abridge, we make,

$$b^2 - \frac{b^2 x^2}{a^2} = r^2,$$

the integral relative to $y$ of the first part of the preceding formula, will become

$$\frac{2\rho c x^2 dx}{b} \int \sqrt{r^2 - y^2}\, dy.$$

From the equation of the section of the ellipsoid by the plane of the axes of $x$ and $y$, namely,

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1,$$

it appears that $y = \pm r$ are the two limits of the integral relative to $y$; and as $(b)$

$$\int_{-r}^{r} \sqrt{r^2 - y^2}\, dy = \tfrac{1}{2} \pi r^2,$$

it follows, by substituting for $r^2$ its value, that

$$2\rho c x^2 dx \int \sqrt{1 - \frac{y^2}{b^2} - \frac{x^2}{a^2}}\, dy = \frac{\pi\rho bc}{a^2}(a^2x^2 - x^4)\, dx.$$

Therefore, by integrating with respect to $x$ from $x = -a$ to $x = a$, we shall obtain

$$2\rho c \iint x^2 \sqrt{1 - \frac{y^2}{b^2} - \frac{x^2}{a^2}}\, dx\, dy = \frac{4\pi\rho a^3 bc}{15};$$

by merely changing the letters, we shall have, without entering on any new calculations,

$$2\rho c \iint y^2 \sqrt{1 - \frac{y^2}{b^2} - \frac{x^2}{a^2}}\, dx\, dy = \frac{4\pi\rho b^3 ac}{15};$$

consequently, the value of C the moment of inertia with respect to the axis of $z$, which is the sum of these two last integrals, will be

$$C = \frac{4\pi\rho abc}{15}(a^2 + b^2).$$

In the same manner, B and A, the moments of inertia with respect to the axes of $x$ and $y$, may be obtained. The mass of the ellipsoid being denoted by M, we shall have, in consequence of the value of its volume given in No. 89,

$$M = \frac{4\pi\rho abc}{3};$$

hence it appears, that the three moments of inertia with respect to the diameters $2a$, $2b$, $2c$, will be

$$A = \tfrac{1}{5}M(b^2 + c^2), \quad B = \tfrac{1}{5}M(c^2 + a^2), \quad C = \tfrac{1}{5}M(a^2 + b^2).$$

These diameters are the three *principal* axes of the body which intersect at its centre of gravity; for if they be taken as the axes of the coordinates of $x, y, z$, the three integrals $\int xydm$, $\int zxdm$, $\int yzdm$, extended to the entire ellipsoid are cipher, since each of them consists of elements, which, taken two by two, are equal and affected with opposite signs.

It appears from the preceding values of A, B, C, that the

greatest and least of them are those which belong to the least and greatest of the three diameters, which indeed is also evident, from the definition of moments of inertia.

371. Since in the case of a sphere, $a = b = c$, the three moments of inertia become equal to each other, and their common value is $\frac{8\pi}{15}\rho a^5$. If the radius $a$ be increased by an infinitely small quantity, and becomes equal to $a + da$, the corresponding increment of this moment of inertia of the sphere, namely, $\frac{8\pi}{3}\rho a^4 da$, will express the moment of inertia of a spherical stratum, whose interior and exterior radii are respectively $a$ and $a + da$ (c). Now, if the sphere is not homogeneous, but only consists of concentrical homogeneous strata, so that $r$ denoting the radius of any stratum whatever, the density $\rho$ may be a given function of $r$, in order to obtain the moment of inertia of the entire sphere in this case, that of any stratum whatever, namely, $\frac{8\pi}{3}\rho r^4 dr$ should be integrated with respect to $r$, and the integral then extended to the entire radius of the sphere; therefore, denoting this radius by $c$, we shall obtain

$$\frac{8\pi}{3}\int_0^c \rho r^4 dr$$

for the required moment of inertia.

It evidently appears from a comparison of this last expression, with that of the moment of inertia of a homogeneous sphere of the same radius, and whose density is equal to the mean density of the sphere which we have been just considering, that it will be greater or less than in the case of the homogeneous sphere, according as the density $\rho$ increases or decreases from the centre to the surface, this is also evident from the definition of moments of inertia.

372. In the case of a homogeneous body terminated by a surface of revolution, the determination of its moment of inertia with respect to the axis of figure, is reduced to one inte-

gration depending on the generating curve. In this case, the solid should be decomposed into circular rings, of an infinitely small thickness and breadth, each of them having its centre in the axis, and being bounded on one part by two planes perpendicular to the axis; and on the other part, by two cylindrical surfaces, of which this line is the common axis. Denoting the radius of the interior surface by $r$, that of the exterior surface by $r + dr$, and the distance of the two planes by $dx$, the volume of the ring will be $\pi(r + dr)^2 dx - \pi r^2 dx$, or $2\pi r dr dx$, neglecting an infinitely small term of the third order($d$). Consequently, if $\rho$ be the density of the body, its mass will be $2\pi\rho r dr dx$, and as all the points of this ring are at the same distance $r$ from the axis of the figure, $2\pi\rho r^3 dr dx$, the product of this mass and of $r^2$, will express its moment of inertia with respect to this axis. Therefore, if the lines AB and AMB (fig. 1) represent this axis and the generating curve, and if we make

$$\text{AP} = x, \quad \text{PM} = y,$$

the moment of inertia of the infinitely slender slice of the solid of revolution, perpendicular to AB, and corresponding to the point P, will be obtained by integrating $2\pi\rho r^3 dr dx$ from $r = 0$ to $r = y$, the result of which integration is $\frac{1}{2}\pi\rho y^4 dx$. Hence, therefore, if we denote the length of the axis AB by $l$, and the moment of inertia of the entire solid by $\mu$, the value of $\mu$ will be obtained by integrating this differential $\frac{1}{2}\pi\rho y^4 dx$, from $x = 0$ to $x = l$, and it will give

$$\mu = \tfrac{1}{2}\pi\rho \int_0^l y^4 dx. \qquad \text{(b)}$$

If $\alpha$ and $\beta$ denote given values of $x$, such that $\alpha \angle \beta$ and $\beta \angle l$, it will be sufficient to integrate from $x = \alpha$ to $x = \beta$, in order to obtain the moment of inertia of the slice of the solid comprised between two planes perpendicular to the axis, the distances of these planes from the point A, being $\alpha$ and $\beta$. If this body is a solid hollowed out, and comprised be-

tween two surfaces of revolution, which have the same axis AB, its moment of inertia will be obtained, by considering this body as the difference of two solids of revolution, and by taking one of their moments of inertia, with respect to the common axis, from the other. Finally, if the moment of inertia of a portion of the solid of revolution, comprised between two planes drawn through the axis of the figure, be required; it is evident, that this moment with respect to this axis, will be to that of the entire solid as the angle between the two planes is to four right angles(*e*).

373. If the generating curve AMB be the circumference of a circle whose radius is $a$, the value of $y^2$ which should be substituted in formula (b), is equal to $2ax - x^2$; and if the moment of inertia of the spherical segment, of which the sagitta is $\alpha$, with respect to the diameter perpendicular to its base, be required, we should integrate, in this formula, from $x = 0$ to $x = \alpha$, from which there results(*f*)

$$\mu = \tfrac{1}{2}\pi\rho\alpha^3\left(\frac{4a^2}{3} - a\alpha + \frac{\alpha^2}{5}\right).$$

In the case of the entire sphere, $\alpha = 2a$, which gives $\mu = \dfrac{8\pi\rho a^5}{15}$, as before.

If AMB be a right line passing through the point A, and making with the axis AB an angle, of which the tangent is $\theta$, we shall have

$$y = \theta x.$$

If this value be substituted in equation (b), there results, by integrating it from $x = \alpha$ to $x = \beta$,

$$\mu = \tfrac{1}{10}\pi\rho\theta^4(\beta^5 - \alpha^5).$$

This expression will be the value of the moment of inertia of a truncated cone, with respect to its axis of figure. If $a$ and $b$ denote the radii of its two bases, and $h$ its altitude, we shall have

$$\theta\alpha = a, \quad \theta\beta = b, \quad \beta - \alpha = h;$$

and by means of these expressions, the preceding value of $\mu$ may be made to assume the form($g$)

$$\mu = \tfrac{1}{10}\pi\rho h\,(a^4 + a^3b + a^2b^2 + ab^3 + b^4).$$

In the case of an entire cone, $b = 0$; consequently,

$$\mu = \tfrac{1}{10}\pi\rho h a^4,$$

and as M the mass of the cone is equal to $\frac{1}{3}\pi\rho h a^2$, we shall have

$$\mu = \tfrac{3}{10}\,\mathrm{M}a^2.$$

When $b = a$, the truncated cone will become a cylinder, therefore, we shall have

$$\mu = \tfrac{1}{2}\pi\rho h a^4,$$

and as M the mass is in this case equal to $\pi\rho h a^2$, there will result,

$$\mu = \tfrac{1}{2}\,\mathrm{M}a^2.$$

374. If the moment of inertia of a body with respect to an axis passing through the centre of gravity be known, the moment of inertia of the same body, relatively to any other axis parallel to the first may be easily obtained.

In fact, if the origin of the coordinates be at the centre of gravity, and the first axis be that of the coordinates of $z$, $\alpha$ and $\beta$ the coordinates of the point where the second axis intersects the plane of the axes of $x$ and $y$, to which plane it is also perpendicular, and if $a$ be the distance of the centre of gravity from the second axis, $r$ the distance of $dm$ any element of the body from the first axis, $r'$ the distance of the same material point from the second axis; the moment of inertia $\int r^2dm$ will be known, from which it is proposed to obtain that of $\int r'^2dm$, these integrals being supposed to be respectively extended to the entire mass of the solid. Now, as

$$r'^2 = (x-\alpha)^2 + (y-\beta)^2 = x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2;$$

by multiplying by $dm$, integrating and observing that

$$x^2+y^2=r^2, \quad \alpha^2+\beta^2=a^2,$$

we shall have

$$\int r'^2 dm = \int r^2 dm - 2\alpha\int x dm - 2\beta\int y dm + a^2\int dm;$$

but since the centre of gravity exists on the axis of the ordinates $z$, the integrals $\int x dm$, $\int y dm$ are cipher (No. 361); moreover, $\int dm$ is the mass of the entire body denoted by M; consequently, the preceding equation will be reduced to

$$\int r'^2 dm = \int r^2 dm + \text{M}a^2.$$

Thus, the required moment of inertia will be obtained, by adding to the given moment of inertia the mass of the body, multiplied by the square of the distance of the centre of gravity from the new axis.

By means of this rule, the moment of inertia of a homogeneous sphere, or of one composed of concentrical strata, can be immediately obtained with respect to any axis whatever, since this moment is known for all axes passing through the centre of figure, which is also the centre of gravity.

In any body whatever, the moment of inertia, with respect to an axis passing through its centre of gravity, is less than with respect to any other axis parallel to this last. The moments of inertia of the same body, with respect to all axes parallel to each other, and equally distant from the centre of gravity, are equal to each other; and their common value will increase according as their distance from this point becomes greater.

375. The moment of inertia not only varies with the absolute position of the axis to which it is referred, it also changes with the direction of this line. In order to shew how this direction influences the magnitude of the moment of inertia of any body whatever, we will investigate an expression for that of the mass M with respect to an axis drawn through

the origin of the coordinates, and which makes with the axes of $x, y, z$, the three given angles $\alpha, \beta, \gamma$.

Let $p$ be the perpendicular let fall from the element $dm$ on the new axis, D the distance of this material point from the origin of the coordinates, $\delta$ the angle contained between the line D and the new axis. The coordinates of $dm$ being $x, y, z$, the cosines of the angles which the direction of its radius vector D makes with the axes of these coordinates, will be $\frac{x}{\text{D}}, \frac{y}{\text{D}}, \frac{z}{\text{D}}$; consequently, we shall have (No. 9)

$$\cos\delta = \frac{x}{\text{D}}\cos\alpha + \frac{y}{\text{D}}\cos\beta + \frac{z}{\text{D}}\cos\gamma.$$

Moreover, we have

$$p = \text{D}\sin\delta;\quad p^2 = \text{D}^2 - (\text{D}\cos\delta)^2;$$

hence there will result, by substituting for $\text{D}\cos\delta$, the preceding value of $\cos\delta$ multiplied by D, and putting $x^2+y^2+z^2$ for $\text{D}^2$ ($h$),

$$p^2 = x^2\sin^2\alpha + y^2\sin^2\beta + z^2\sin^2\gamma$$
$$- 2xy\cos\alpha.\cos\beta - 2xz\cos\alpha\cos\gamma - 2yz\cos\beta\cos\gamma;$$

from which we can obtain

$$\int p^2dm = \sin^2\alpha\int x^2dm + \sin^2\beta.\int y^2dm + \sin^2\gamma\int z^2dm$$
$$- 2\cos\alpha.\cos\beta.\int xydm - 2\cos\alpha\cos\gamma\int xzdm$$
$$- 2\cos\beta.\cos\gamma\int yzdm.$$

By means of this formula, the moment of inertia $\int p^2dm$, relative to an axis of a given direction, and passing through the origin of the coordinates will be obtained, when the six integrals $\int x^2dm$, $\int y^2dm$, $\int z^2dm$, $\int xydm$, $\int xzdm$, $\int yzdm$, relative to the axes of the coordinates, and which extend to the entire mass of the body, are known. If these three lines are principal axes, the three last integrals are cipher (No. 368), and the preceding formula will be reduced to

$$\int p^2dm = \sin^2\alpha\int x^2dm + \sin^2\beta\int y^2dm + \sin^2\gamma\int z^2dm.$$

But, in consequence of the equation

$$\left|\begin{array}{l}\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1;\\ \sin^2\alpha = \cos^2\beta + \cos^2\gamma,\\ \sin^2\beta = \cos^2\gamma + \cos^2\alpha,\\ \sin^2\gamma = \cos^2\alpha + \cos^2\beta;\end{array}\right.$$

by means of which the value of $\int p^2dm$ may be changed into the following,

$$\begin{aligned}\int p^2dm = & (\int y^2dm + \int z^2dm)\cos^2\alpha,\\ & + (\int z^2dm + \int x^2dm)\cos^2\beta,\\ & + (\int x^2dm + \int y^2dm)\cos^2\gamma;\end{aligned}$$

hence by combining each couple of integrals into one, and denoting the moments of inertia with respect to the axes of $x$, $y$, $z$ by A, B, C, as in No. 368, we shall finally have

$$\int p^2dm = \text{A}\cos^2\alpha + \text{B}\cos^2\beta + \text{C}\cos^2\gamma. \qquad (c)$$

Consequently, the moment of inertia corresponding to any axis whatever, passing through a given point, will be known immediately, if the three moments of inertia relative to the three principal axes which intersect in this point are given; and, by combining this result with that of the preceding number, it appears that the determination of all moments of inertia of the same body, will depend on the three principal moments of inertia of its centre of gravity. Having determined, for example, (No. 370), the values of these three moments of inertia, in the case of a homogeneous ellipsoid, the moment of inertia of this body with respect to any axis whatever, may be considered as known.

376. The greatest and least of the three principal moments of inertia A, B, C, which occur in formula (c), are also the greatest and least of all moments of inertia of the same body, with respect to axes passing through the origin of the coordinates. In fact, if A be the greatest of the three quan-

tities A, B, C, by substituting $1 - \cos^2\beta - \cos^2\gamma$ for $\cos^2\alpha$ in equation (c) we shall have

$$\int p^2 dm = A - (A - B)\cos^2\beta - (A - C)\cos^2\gamma;$$

hence it follows, that whatever be the values of the angles $\beta$ and $\gamma$, $\int p^2 dm$ is less than A. In like manner, C being the least of the three quantities A, B, C, if equation (c) be made to assume the form

$$\int p^2 dm = C + (A - C)\cos^2\alpha + (B - C)\cos^2\beta,$$

it is evident that $\int p^2 dm$ is constantly greater than C.

In the particular case in which the three quantities A, B, C are equal, A is also equal to $\int p^2 dm$, whatever be the direction of the axes to which the moment of inertia $\int p^2 dm$ is referred; therefore in this case, the moments of inertia are equal for all axes passing through the origin of the coordinates. This is the case of a homogeneous sphere, or of one composed of concentrical strata, when the origin of the coordinates is placed at its centre; it obtains also in the case of the cube, of the regular octaedron, and of the other regular homogeneous solids, the origin of whose coordinates being always supposed to be at their centre of figure, the three principal moments of inertia cannot differ from each other.

If we have only $A = B$, equation (c) will be reduced to

$$\int p^2 dm = A\sin^2\gamma + C\cos^2\gamma;$$

and as this value of $\int p^2 dm$ is independent of the angles $\alpha$ and $\beta$, the moment of inertia will be the same for all axes drawn through the origin of the coordinates, and which make the one same angle with the axis of $z$. This is the case of a solid of revolution when this right line is its axis of figure.

It appears from No. 374, that the least of all moments of inertia appertaining to the same body, is that which belongs to one of the three principal axes that intersect in its centre of gravity. Thus, for example, the least of all moments of

inertia of a homogeneous ellipsoid, is that which belongs to the greatest of its three rectangular conjugate diameters.

377. In what has been stated in the preceding numbers, it has been taken for granted, that in every body three axes exist, which possess the properties that have been ascribed to principal axes; we now proceed to demonstrate their existence, and to determine their direction for each point of a body of any form whatever; but for this purpose, it is necessary to advert to the general formulæ for the transformation of coordinates, which, moreover, we shall have occasion to make use of in other cases. Let $x$, $y$, $z$, be the three coordinates of any point M, referred to the rectangular axes O$x$, O$y$, O$z$ (fig. 2), and let $x_1$, $y_1$, $z_1$ be the coordinates of the same point with respect to three other rectangular axes O$x_1$, O$y_1$, O$z_1$, having the same origin. From the point M, let the perpendiculars MP and MK be let fall on the axis O$x$ and on the plane of the axes of $x_1$ and $y_1$, and from the point K, a perpendicular KH on the axis O$x_1$, so that

$$\text{OP} = x, \quad \text{OH} = x_1, \quad \text{KH} = y_1, \quad \text{MK} = z_1.$$

The projection on the axis O$x$, of the line, which is made up of MK, KH, HO, will be O$p$; and as the projections of its respective parts OH, KH, MK, will be equal to these lines, multiplied by the cosines of the angles which the axes of $x_1$, $y_1$, $z_1$, make with the axis O$x$, by taking their sum, we shall have

$$x = x_1 \cos x\text{O}x_1 + y_1 \cos x\text{O}y_1 + z_1 \cos x\text{O}z_1.$$

In the figure these three angles are supposed to be acute, and the three coordinates $x_1$, $y_1$, $z_1$, to be positive; in which case their projections fall on the direction itself of O$x$, and their sum makes up the absolute magnitude of O$x$; but it is easy to be assured that this equation will subsist in all cases, regard being had to the signs of the coordinates $x_1$, $y_1$, $z_1$, of the cosines, and of the abscissa $x$. For example, it is evident that, if the abscissa $x_1$ is negative, and the angle $x\text{O}x_1$ acute,

or if this abscissa is positive, and this angle obtuse, the projection of OH will fall on the production of O$x$, and the absolute magnitude of this value ought to be subtracted; and, on the other hand, it is evident that it ought to be added, when this abscissa $x_1$ being negative, the angle $x$O$x_1$ is, at the same time, obtuse; in both cases, this agrees with the sign of the product $x_1 . \cos x\text{O}x_1$.

In like manner it is evident that the projections of this same line made up of MK, KH, HO, on the axes O$y$ and O$z$, or on their productions, are always equal to $y$ and $z$. This being so, if we make

$$\cos x\text{O}x_1 = a,\quad \cos x\text{O}y_1 = b,\quad \cos x\text{O}z_1 = c,$$
$$\cos y\text{O}x_1 = a',\quad \cos y\text{O}y_1 = b',\quad \cos y\text{O}z_1 = c',$$
$$\cos z\text{O}x_1 = a'',\quad \cos z\text{O}y_1 = b'',\quad \cos z\text{O}z_1 = c'',$$

we shall have

$$\left.\begin{aligned} x &= ax_1 + by_1 + cz_1, \\ y &= a'x_1 + b'y_1 + c'z_1, \\ z &= a''x_1 + b''y_1 + c''z_1. \end{aligned}\right\} \quad (1)$$

These nine coefficients, $a$, $b$, $c$, $a'$, &c., are connected together by the six following equations:

$$\left.\begin{aligned} a^2 + a'^2 + a''^2 = 1, \quad ab + a'b' + a''b'' = 0, \\ b^2 + b'^2 + b''^2 = 1, \quad ac + a'c' + a''c'' = 0, \\ c^2 + c'^2 + c''^2 = 1, \quad bc + b'c' + b''c'' = 0. \end{aligned}\right\} \quad (2)$$

The first, for example, results from this, that $a, a', a''$, are the cosines of the angles which the same line O$x_1$ makes with the three rectangular axes O$x$, O$y$, O$z$; and the fourth from this, that the line O$x_1$, and the line O$y_1$, are perpendicular to each other. These six equations may likewise be obtained by substituting the values of $x$, $y$, $z$ in the equation

$$x^2 + y^2 + z^2 = x_1^2 + y_1^2 + z_1^2,$$

each member of which is the square of OM, and which should consequently be identical.

Conversely, we can by means of equations (2), deduce from equations (1),

$$\left.\begin{aligned} x_1 &= ax + a'y + a''z, \\ y_1 &= bx + b'y + b''z, \\ z_1 &= cx + c'y + c''z; \end{aligned}\right\} \quad (3)$$

and equations (2) may be replaced by the following :(i)

$$\left.\begin{aligned} a^2 + b^2 + c^2 = 1, \quad & aa' + bb' + cc' = 0, \\ a'^2 + b'^2 + c'^2 = 1, \quad & aa'' + bb'' + cc'' = 0, \\ a''^2 + b''^2 + c''^2 = 1, \quad & a'a'' + b'b'' + c'c'' = 0. \end{aligned}\right\} \quad (4)$$

A comparison of these formulæ with those of No. 277, shews clearly the analogy which subsists between the projections of right lines and those of plane surfaces, from which results the identity of the composition of forces represented by portions of lines, with the composition of moments represented by plane areas.

378. In the transformation of coordinates, six of the nine coefficients, $a, b, c$, &c., should be therefore considered as functions of the three others, which can be determined either by equations (2) or (4); but it will be more convenient to express these nine coefficients by means of three new quantities, by formulæ which will satisfy equations (2) or (4).

For this purpose, let the line NON′ (fig. 3) be the intersection of the plane of the axes of $x_1$ and $y_1$, with the plane of the axes of $x$ and $y$, and let

$$\text{NO}x = \psi, \quad \text{NO}x_1 = \phi, \quad \text{ZO}z_1 = \theta;$$

these three angles $\psi$, $\phi$, $\theta$, will determine, *without any ambiguity*, the position of the axes of $x_1, y_1, z_1$, with respect to those of $x, y, z$, provided that the direction in which these angles are measured is previously agreed on. For greater convenience, we shall suppose that the plane of the axes of $x$ and $y$ is horizontal, and that, consequently, the axis of $z$ is vertical, and that the positive ordinates relating to this last axis, are estimated in the *direction* of gravity.

The angle $\theta$ will extend from zero to 180; and according as it is acute or obtuse, the axis $oz_1$ will be situated below or above the plane of the axes of $x$ and $y$; in the case of $\theta = 0$, $oz_1$ will coincide with $oz$, and when $\theta = 180°$, $oz_1$ will coincide with the production of $oz$.

In the motion of a solid body about the point o, the axes $ox_1$, $oy_1$, $oz_1$, are supposed to be *fixed* in *its interior* and *moveable* with it, and the axes $ox$, $oy$, $oz$ are supposed to be *fixed* in *space*, and *consequently immoveable*. The angles $\psi$ and $\phi$ may then be either positive or negative, and may comprise one or more circumferences; but, at any instant whatever, we shall always have

$$\psi = 2n\pi + u, \quad \phi = 2i\pi + v,$$

in which $n$ and $i$ denote any whole numbers whatever, either positive, negative, or cipher, $u$ and $v$ being positive variables less than $2\pi$. Now, let the angle $u$ be measured, reckoning from the line $ox$, in the direction indicated by the sagitta $s$; so that, for example, the line ON will coincide with $ox$ when $u = 0$, with the production of $oy$ when $u = 90°$, with that of $ox$, when $u = 180°$, and with $oy$ when $u = 270°$. Let the angle $v$ be measured, reckoning from ON, *above* the plane of the axes of $x$ and $y$, so that the axis $ox_1$ may be above this plane, when $v$ is less than 180°, and below it when $v$ is greater than 180°. In the case of $v = 0$, the axis $ox_1$ will coincide with the line ON, and when $v = 180°$, with ON′ the production of ON. In all cases, the angle $\theta$, whether acute or obtuse, will be equal to the solid angle contained by the planes of the axes of $x$ and $y$ and of the axes of $x_1$ and $y_1$, and therefore it is equal to the angle whose edge is ON, and whose faces are the angles NO$x$ and NO$x_1$, reduced to their parts $u$ and $v$. In the figure the three angles $u$, $v$, $\theta$ are supposed to be acute.

This being premised, when the angle $\psi$ is given, let its part $u$ be laid off on the horizontal plane, reckoning from the axis $ox$, in the direction of the sagitta $s$; this will determine

the position of the line ON. The angle $\phi$ being also given, let its part $v$ be first laid off on the horizontal plane, reckoning from the line ON in the direction of the sagitta $s'$, that is to say, in an opposite direction from that of $s$; then, the plane of the angle $\phi$ should be made to revolve about ON, in such a manner that the part of $\phi$ adjacent to ON should be raised above the horizontal plane. When this plane shall have described the given angle $\theta$, the other side of the angle $\phi$ will be the true position of the axis $ox_1$, which will be above or below the horizontal plane, according as $v \angle 180°$, or $v > 180°$. If the angle $v$ be increased, in its plane, by 90°, we shall have the position of the axis $oy_1$, and if a perpendicular be erected to this plane, it will determine the axis $oz_1$, the direction of which will be below or above the horizontal plane, according as the angle $\theta$ is acute or obtuse.

The three axes $ox_1$, $oy_1$, $oz_1$, being thus completely determined with respect to the axes $ox$, $oy$, $oz$, by means of the angles $\psi$, $\phi$, $\theta$, it is requisite that the nine cosines $a$, $b$, &c., should be functions of these three angles; and, in fact, if the directions, which have been just explained, be assigned to these angles, there results

$$\left.\begin{aligned}
a &= \cos\theta \sin\psi \sin\phi + \cos\psi \cos\phi, \\
b &= \cos\theta \sin\psi \cos\phi - \cos\psi \sin\phi, \\
c &= \sin\theta \sin\psi, \\
a' &= \cos\theta \cos\psi \sin\phi - \sin\psi \cos\phi, \\
b' &= \cos\theta \cos\psi \cos\phi + \sin\psi \sin\phi, \\
c' &= \sin\theta \cos\psi, \\
a'' &= -\sin\theta \sin\phi, \\
b'' &= -\sin\theta \cos\phi, \\
c'' &= \cos\theta.
\end{aligned}\right\} \qquad (5)$$

It is easy to verify these values of $a$, $b$, &c., by shewing that they render equations (3) and (4) identical, and that no

relation between the angles $\psi$, $\phi$, $\theta$, can be deduced from them.

379. Although these formulæ (5) are generally known, it still may not be useless to point out how they may be obtained. From a known property of spherical triangles, it is evident that if $\alpha, \beta, \gamma$, be its three sides, and A the spherical angle opposite to the side $\alpha$, then

$$\cos\alpha = \cos A \sin\beta \sin\gamma + \cos\beta \cos\gamma.$$

Now, if a sphere be conceived to be described from the point O as centre, and with any radius whatever, there will be traced on its surface a triangle formed by the three arcs which measure the angles N$Ox$, N$Ox_1$, $xOx_1$, in which the angle opposite to the last side will be equal to $\theta$; therefore, as N$Ox = \psi$ and N$Ox_1 = \phi$, we shall have

$$\cos xOx_1 = a = \cos\theta \sin\psi \sin\theta + \cos\psi \cos\phi.$$

As this equation obtains for any values whatever of $\phi$, and $\psi$, it must obtain when $\phi$ becomes $\phi + 90$; then, the axis $Oy_1$ will take the place of $Ox_1$, the angle $xOx_1$ will become $xOy_1$, and we shall have

$$\cos xOy_1 = b = \cos\theta \sin\psi \cos\phi - \cos\psi \sin\phi.$$

In like manner, by substituting $\psi + 90$ in place of $\psi$, in the preceding equation, the axis $Ox$ will be changed into the axis $Oy$, and the angle $xOx_1$, will become $yOx_1$, so that we shall have

$$\cos yOx_1 = a' = \cos\theta \cos\psi \sin\phi - \sin\psi \cos\phi.$$

If we substitute at once, in the preceding equation, $\psi + 90^\circ$ and $\phi + 90^\circ$ in place of $\psi$ and $\phi$, the angle $xOx_1$ will be replaced by the angle $yOy_1$, and there will result

$$\cos yOy_1 = b' = \cos\theta \cos\psi \cos\phi + \sin\psi \sin\phi.$$

In like manner, in the case of the spherical triangle, the three sides of which measure the angles N$Ox$, N$Oz_1$, $xOz_1$, the angle opposite to the last side is $90^\circ - \theta$; more-

over, $\text{NO}x = \psi$ and $\text{NO}z_1 = 90°$. Therefore, by substituting $90° - \theta$ for A, $\psi$ for $\beta$, $90°$ for $\gamma$, and $xoz_1$ for $a$, in the general equation, there results

$$\cos xoz_1 = c = \sin\theta \sin\psi;$$

hence likewise we infer,

$$\cos yoz_1 = c' = \sin\theta\cos\psi,$$

by substituting $\psi + 90°$ for $\psi$; in which case the axis $ox$ is changed into $oy$, and the angle $xoz_1$ into $yoz_1$.

Finally, in the spherical triangle, the sides of which measure the angles $\text{NO}z$, $\text{NO}x_1$, $zox_1$, the angle opposite to this last side is equal $\theta + 90°$, and we have $\text{NO}z = 90°$ and $\text{NO}x_1 = \phi$. If, therefore, we make $\text{A} = 90° + \theta$, $\beta = 90°$, $\gamma = \phi$, $a = zox_1$ in the general equation, we shall have

$$\cos zox_1 = a'' = -\sin\theta\sin\phi.$$

If, in this result, $\phi + 90$ be substituted in place of $\phi$, the axis $ox_1$ will be changed into $oy_1$, and the angle $zox_1$ into $zoy_1$; consequently, we shall also have

$$\cos zoy_1 = b'' = -\sin\theta\cos\phi.$$

With respect to the ninth cosine $c''$, we have evidently

$$c'' = \cos zoz_1 = \cos\theta.$$

(380.) If we now suppose that the axes of the coordinates $x_1, y_1, z_1$, are principal axes, which intersect at the point o; by their definition (No. 368), we shall have

$$\textstyle\int x_1y_1dm = 0, \quad \int x_1z_1dm = 0, \quad \int y_1z_1dm = 0; \qquad \text{(a)}$$

and it will be necessary for us to prove that these three equations furnish always real values for the angles $\psi$, $\phi$, $\theta$.

By making, in order to abridge,

$$\text{X} = x\cos\psi - y\sin\psi,$$
$$\text{X} = x\cos\theta\sin\psi + y\cos\theta\cos\psi - z\sin\theta,$$

and substituting formulæ (5) in equations (3), we shall have $(k)$

$$x_1 = \text{Y} \sin \phi + \text{X} \cos \phi,$$
$$y_1 = \text{Y} \cos \phi - \text{X} \sin \phi,$$
$$z_1 = x \sin \theta \sin \psi + y \sin \theta \cos \psi + z \cos \theta.$$

By means of these values, the first equation (a) assumes the form

(1) $$\sin 2\phi \int (\text{X}^2 - \text{Y}^2)\, dm = 2 \cos 2\phi \int \text{XY}\, dm, \tag{b}$$

and the two last become

$$\cos \phi \int \text{Y} z_1 dm - \sin \phi \int \text{X} z_1 dm = 0,$$
$$\sin \phi \int \text{Y} z_1 dm + \cos \phi \int \text{X} z_1 dm = 0.$$

If these last equations be multiplied by $\cos \phi$ and $\sin \phi$, or by $-\sin \phi$ and $\cos \phi$, and then added, they will be replaced by the following, namely,

$$\int \text{Y} z_1 dm = 0, \quad \int \text{X} z_1 dm = 0,$$

which do not contain the angle $\phi$. By substituting for X, Y, $z_1$, their respective values, and making

$$\int x^2 dm = f, \quad \int y^2 dm = g, \quad \int z^2 dm = h,$$
$$\int yz dm = f', \quad \int zx dm = g', \quad \int xy dm = h';$$

these six integrals being supposed to extend to the entire mass of the body in question, there results(*l*)

$$(f \sin^2 \psi + 2h' \sin \psi \cos \psi + g \cos^2 \psi - h) \sin \theta \cos \theta$$
$$+ (g' \sin \psi + f' \cos \psi)(\cos^2 \theta - \sin^2 \theta) = 0,$$

$$[h' (\cos^2 \psi - \sin^2 \psi) + (f - g) \sin \psi \cos \psi] \sin \theta$$
$$+ (g' \cos \psi - f' \sin \psi) \cos \theta = 0;$$

now, if we make(*m*)

$$\tan \psi = u, \quad \sin \psi = \frac{u}{\sqrt{1+u^2}}, \quad \sin \psi = \frac{1}{\sqrt{1+u^2}},$$

the second of these two equations will give

(2) $$\text{tang.}\, \theta = \frac{(f' u - g' \sqrt{1+u^2})}{h'(1-u^2) + (f-g) u}, \tag{c}$$

and the first will assume the form

$$[(f-h)u^2+2h'u+g-h]\frac{\text{tang}\,\theta}{\sqrt{1+u^2}}$$
$$+g'u+f'=(g'u+f')\,\text{tang}^2\,\theta.$$

By substituting for tang $\theta$ its value, it becomes

$$[(f-h)u^2+2h'u+g-h](f'u-g')$$
$$+[h'(1-u^2)+(f-g)u](g'u+f')=\frac{(g'u+f')(f'u-g')^2(1+u^2)}{h'.(1-u^2)+(f-g).u};$$

its first member is the same thing as

$$[hg'-gg'+f'h'-(hf'-ff'-g'h')u](1+u^2);$$

consequently, we shall have finally,

$$\left.\begin{array}{c}[gg'-hg'-f'h'+(hf'-ff'+g'h')u][h'(1-u^2)+(f-g)u]\\ +(g'u+f')(f'u-g')^2=0.\end{array}\right\}\text{(d)}$$

Thus, equations (a) are replaced by equations (b), (c), (d). Now, as this last is of the third degree, it will have at least one real root; there will be, therefore, one real value of $u$ or tang $\psi$, to which will correspond an angle $\psi$ less than $\frac{1}{2}\pi$, and another equal to the first increased by $\pi$, which will appertain respectively to ON and ON$'$, the two parts of the intersection of the unknown plane of the axes of $x_1$ and $y_1$, with the given plane of the axes of $x$ and $y$. By reason of the radical $\sqrt{1+u^2}$, equation (c) will then give two values of tang $\theta$, that are equal but of opposite signs, which appertain to an acute angle and its supplement, and, consequently, to the axis O$z_1$, and to its production. Finally, from equation (b), there results a real value of tang $2\phi$, to which two values of $\phi$ will correspond, one of which will be less than $\frac{1}{2}\pi$, and the other will be equal to the first increased by $\frac{1}{2}\pi$. The first being taken for the value of the angle NO$x_1$, the second will be that of NO$y_1$; and, in fact, every thing being similar with respect to the axes O$x_1$ and O$y_1$, they ought to be determined by the same equation.

Thus it appears, that the three roots of equation (d) must be real, and that they will represent the tangents of the angles comprised between the axis o$x$ and the three lines, which are the intersections of the planes of the axes of the coordinates $x_1$, $y_1$, $z_1$ with the plane of the coordinates $x$ and $y$; for these three tangents must be furnished by the same equation, since in the calculation there is no difference in the manner in which the three principal axes, the position of which is sought, are expressed.

From the preceding analysis it follows, that for a given point o there always exist three principal rectangular axes, which intersect in this point, and that in general this system of principal axes is unique. In order that there should be several such systems, the degree of equation (d) should be higher than the third, and it should have three times as many real roots as there are systems; however, in certain cases, the equations on which the values of $\psi$, $\phi$, $\theta$ depend, become identical, and then the number of the principal axes is infinite. These particular cases might be determined by a particular discussion of the equations in question, but they can be obtained with greater facility by means of the following considerations.

381. By formulæ (1) and equations (a), which characterize the principal axes o$x_1$, o$y_1$, o$z_1$, we have evidently

$$\int xydm = aa'\int x_1^2dm + bb'\int y_1^2dm + cc'\int z_1^2dm,$$
$$\int zxdm = aa''\int x_1^2dm + bb''\int y_1^2dm + cc''\int z_1^2dm,$$
$$\int yzdm = a'a''\int x_1^2dm + b'b''\int y_1^2dm + c'c''\int z_1^2dm.$$

Now, if the three integrals $\int x_1^2dm$, $\int y_1^2dm$, $\int z_1^2dm$ are equal, these values of $\int xydm$, $\int zxdm$, $\int yzdm$ become zero, in virtue of equations (4); consequently, in this case, the lines o$x$, o$y$, o$z$ constitute a second system of principal axes; and as their direction with respect to the axes o$x_1$, o$y_1$, o$z$ is entirely undetermined, it follows that all systems of rectangular

axes which can be drawn through the point o, are principal axes. In this case the three principal moments of inertia are equal, for, since

$$\int x_1^2 dm = \int y_1^2 dm = \int z_1^2 dm,$$

it follows that

$$\int (x_1^2 + y_1^2)\, dm = \int (x_1^2 + z_1^2)\, dm = \int (y_1^2 + z_1^2)\, dm.$$

If only two of three principal moments of inertia are equal, for example, those which refer to the axes $ox_1$ and $oy_1$, so that

$$\int (y_1^2 + z_1^2)\, dm = \int (x_1^2 + z_1^2)\, dm,$$

there will still exist an infinite number of systems of principal axes, which will have all one axis in common, namely, $oz_1$. In fact, in this case, the two integrals $\int y_1^2 dm$ and $\int x_1^2 dm$ will be equal; and, in virtue of the last equations (4), the values of $\int xydm$, $\int zxdm$, $\int yzdm$ may be made to assume the form ($n$)

$$\begin{aligned} \int xydm &= cc'(\int z_1^2 dm - \int x_1^2 dm), \\ \int zxdm &= cc''(\int z_1^2 dm - \int x_1^2 dm), \\ \int yzdm &= c'c''(\int z_1^2 dm - \int x_1^2 dm). \end{aligned}$$

Now, if the axis $oz$ be supposed to coincide with the principal axis $oz_1$, the angles $xoz_1$ and $yoz_1$ will be right, and we shall have $c = 0$, $c' = 0$, consequently

$$\int xydm = 0, \quad \int zxdm = 0, \quad \int yzdm = 0.$$

Therefore, in this case, every system which consists of the axis $oz_1$ and of two other rectangular axes drawn arbitrarily through the point o, in the plane $y_1ox_1$, will be a system of principal axes.

Finally, when the three principal moments of inertia are unequal, we may be certain that there is only one system of principal axes: for A, being the greatest of these three unequal moments, if there was a second system of principal axes, and if A′ be the greatest of the three moments which refer to it, by theorem of No. 376, we should have, at the same time,

A > A′ and A′ > A, which is impossible; consequently, the supposition of a second system of principal axes is inadmissible.

382. Those points of a body, for which the three principal moments of inertia (when there are such) and, consequently, all moments of inertia, are equal, possess, as will be seen in the sequel, a remarkable property with respect to the rotation of this body; it will therefore be useful to determine them, which can be effected in the following manner.

Let the origin of the coordinates $x$, $y$, $z$ be placed at the centre of gravity of the body, and let these coordinates be referred to the three principal axes which intersect at this point; if A, B, C denote the moments of inertia relative to the axes of $x$, $y$, $z$, we shall have (No. 368)

$$\int x dm = 0, \quad \int y dm = 0, \quad \int z dm = 0,$$

$$\int yz dm = 0, \quad \int zx dm = 0, \quad \int xy dm = 0,$$

$$\int (y^2 + z^2) dm = \text{A}, \quad \int (x^2 + z^2) dm = \text{B}, \quad \int (x^2 + y^2) dm = \text{C};$$

in all these expressions the integrals are supposed to extend to the entire mass of the body.

$\alpha$, $\beta$, $\gamma$ being the unknown coordinates of one of the required points referred to the axes of $x$, $y$, $z$, so that for this particular point $x = \alpha$, $y = \beta$, $z = \gamma$, if the origin of the coordinates be transferred thither, without changing their direction, then the coordinates of $dm$ will become $x - \alpha$, $y - \beta$, $z - \gamma$. But if the moments of inertia relative to all right lines which pass through this new origin be equal, all these lines must be principal axes, for, by the preceding number, one of these conditions is a necessary consequence of the other. Therefore, the axes of the coordinates $x - \alpha$, $y - \beta$, $z - \gamma$, being principal axes, we shall have

$$\int (x-\alpha)(y-\beta) dm = \int xy dm - \alpha \int y dm - \beta \int x dm + \alpha\beta \int dm = 0,$$

$$\int (z-\gamma)(x-\alpha) dm = \int xz dm - \gamma \int x dm - \alpha \int z dm + \gamma\alpha \int dm = 0,$$

$$\int (y-\beta)(z-\gamma) dm = \int yz dm - \beta \int z dm - \gamma \int y dm + \beta\gamma \int dm = 0;$$

which, in virtue of the preceding equations, are reduced to

$$\alpha\beta = 0, \quad \gamma\alpha = 0, \quad \beta\gamma = 0.$$

Now, in order to satisfy these equations, two of the three quantities $\alpha$, $\beta$, $\gamma$, should be cipher. If, therefore, the required point exists, it must be found on one of the axes of the coordinates $x$, $y$, $z$, that is to say, on one of the three principal axes which intersect in the centre of gravity.

If we suppose $\beta = 0$, $\gamma = 0$, which implies that the required point exists on the axis of $x$, at a distance $a$ from this centre, then the moments of inertia relative to this point, will be $\text{A}$ with respect to the axis of $x$, and, in virtue of theorem of No. 374, $\text{B} + \text{M}a^2$ and $\text{C} + \text{M}a^2$, with respect to parallels to the axes of $y$ and $z$, $\text{M}$ denoting, as before, the mass of the body. Consequently, by the condition of the problem, we shall have

$$\text{B} + \text{M}a^2 = \text{C} + \text{M}a^2 = \text{A};$$

but in order that these equations may be possible, we must have $\text{B} = \text{C}$; and when these two quantities are in point of fact equal, we shall have

$$\text{M}a^2 = \text{A} - \text{C};$$

therefore, in order that the quantity $a$ may be real, we must have $\text{A} > \text{C}$; if this be the case, $a$ will have two real values, equal and affected with contrary signs, namely,

$$a = \pm\sqrt{\frac{\text{A} - \text{C}}{\text{M}}}; \qquad \text{(c)}$$

consequently, there are two points which possess the required property, and they will be situated on the axis of $x$, at equal distances on opposite sides from the centre of gravity.

Thus it appears that when $\text{A}$, $\text{B}$, $\text{C}$, the three moments of inertia of a body, relative to the principal axes which intersect in the centre of gravity, are equal, there is no other point of the body for which all the moments of inertia are equal; but if two of these three moments are equal, and if the unequal

moment is the greatest of the three, there exists, on the axis of the greatest moment, two points, for which all moments of inertia are equal, and the position of these points is determined by formula (e).

383. If these results be applied to the homogeneous ellipsoid, it will appear at once, that in the case of an ellipsoid, the principal diameters of which are unequal, there is no point, either within or without this body, with respect to which all the moments of inertia are equal; but if the body be an ellipsoid of revolution generated by the revolution of an ellipse about its lesser axis, there are two points on this axis or on its production, which possess the property in question; for, in this case, two of the three moments relative to the principal diameters will be equal, and as the unequal moment corresponds to the least diameter, it will be the greatest of the three.

If $a$ and $b$ denote the semiaxes of the generating ellipse, and if $a$ be less than $b$, so that $a$ may be the semiaxis of revolution, we shall have ($o$)

$$\text{A} = \tfrac{2}{5}\text{M}b^2, \quad \text{B} = \text{C} = \tfrac{1}{5}\text{M}(a^2 + b^2).$$

If in the formulæ of No. 370 we suppose $b = c$, then we shall have by formula (e)

$$a = \pm \sqrt{\frac{b^2 - a^2}{5}},$$

for the distance of the required points from the centre of the ellipsoid. According as $b^2 > 6a^2$, or $b^2 \angle 6a^2$, these points will be found without the body on the production of the axis of revolution, or within this body on the axis itself; in the case of $b^2 = 6a^2$, these two points will be found on the surface, and they will coincide with the poles of the ellipsoid.

The principal axes of a rectangular parallelopiped, which intersect at its centre of gravity, are evidently parallel to its sides. Therefore, if $a$, $b$, $c$ denote respectively half the lengths

of its three adjacent sides, the moments of inertia relative to these axes may be deduced from those of No. 369, which refer to its sides, by substituting in them $2a$, $2b$, $2c$, in place of $a$, $b$, $c$, and then, by the theorem of No. 374, subtracting from them the products $\text{M}(b^2+c^2)$, $\text{M}(c^2+a^2)$, $\text{M}(a^2+b^2)$. By this means, we shall have

$$\text{A}=\tfrac{1}{3}\text{M}(b^2+c^2),\quad \text{B}=\tfrac{1}{3}\text{M}(c^2+a^2),\quad \text{C}=\tfrac{1}{3}\text{M}(a^2+b^2);$$

M denoting always the mass of the parallelopiped, the volume of which is $8abc$. Therefore, if in order to render the moments of inertia B and C equal, we suppose $b=c$, and, moreover, $a \angle b$, in order that A may be greater than B or C, equation (e) will then give

$$\alpha=\sqrt{\frac{b^2-a^2}{3}};$$

and according as $b>2\alpha$, $b=2\alpha$, or $b \angle 2\alpha$, the two required points will be situated without the body, at its surface, or in its interior($p$).

## CHAPTER III.

### OF THE MOTION OF A SOLID BODY ABOUT A FIXED AXIS.

#### I. *Uniform Motion of Rotation.*

384. When a system of material points, connected together in an invariable manner, turns about a fixed axis, to which they are also invariably attached, they describe circles perpendicular to this axis, the centres of which exist on this line. The arcs described in the same time by two different points, are similar and contain the same number of degrees; their absolute velocities are to each other as their distances from this axis; and, by the *angular velocity* of the system, is meant the absolute velocity of those points, whose distance from the axis is unity. If it be denoted by $\omega$, and the distance of any point whatever from the axis of rotation by $r$, the absolute velocity of this point will be $r\omega$. This quantity $\omega$, that is common to all the points, will vary with the time, when the points of the system are acted on by motive forces, which will produce a variable motion; it will remain constant in the case of uniform motion, produced by percussions simultaneously made on different parts of the system, and which is then abandoned to itself. It is this last case which we propose first to discuss.

385. Let $m, m', m''$, &c., be the masses of the material points which are considered, and $r, r', r''$, &c. their distances from the axis of rotation. Then, if simultaneous percussions be made on all these points, (each of them should be decomposed into two others, one parallel to the axis, the other acting in the direction of a plane perpendicular to this line;) as

the effect of the first will be destroyed by the resistance of the fixed axis, it is not necessary to take it into account. Let $v, v', v''$, &c., be the velocities in the directions of planes perpendicular to the fixed axis, which would be impressed on $m, m', m''$, &c., if these material points were entirely free. If the angular velocity of the system, which results from their connexion, be denoted by $\omega$, these points will be actuated by the velocities $r\omega, r'\omega', r''\omega''$, &c., whose directions are perpendicular to the axis and to the radii $r, r', r''$, &c.; and then, by the principle of No. 353, there will be an equilibrium between the quantities of motion $mv, m'v', m''v''$, &c., estimated in their respective directions, and the quantities of motion $mr\omega, m'r'\omega'$, $m''r''\omega$, &c., taken in a direction opposite to that of the actual motion of the system.

In order to obtain the equation of this equilibrium, let the points $m, m', m''$, &c., and the directions of the velocities which are considered, be projected on a plane perpendicular to the fixed axis. Let O$z$ be this axis (fig. 4), and let the plane of projection pass through the point O; likewise, let P be the projection of $m$ on this plane, PA the projection of the velocity $v$, PN that of the velocity $r\omega$, which, by supposition, is perpendicular to the radius $r$ or OP. Likewise let OF, the perpendicular let fall from O on the line PA, be denoted by $p$; the moments of the forces $mv$ and $mr\omega$, projected in the directions of PA and PN, will be $mvp$ and $mr^2\omega$; and if $p', p''$, &c., denote in the same manner the perpendiculars let fall from O on the projections of the other velocities $v', v''$, &c., (by No. 267) the required equation of equilibrium will be

$$mr^2\omega + m'r'^2\omega + m''r''^2\omega + \&c.$$
$$= mvp + m'v'p' + m''v''p'' + \&c.$$

When some of the simultaneous percussions tend to make the system to turn in one direction, and others in an opposite direction, the moments of each of these must be affected with contrary signs in the second member of this equation. The

system will turn in the direction of the forces, which, independently of the consideration of the sign, will furnish the greatest sum of moments. If the sum, whether positive or negative, of all these moments affected with suitable signs, be denoted by L, we shall obtain, by means of the preceding equation,

$$\omega = \frac{L}{\Sigma mr^2};$$

in which $\Sigma$ indicates a sum that extends to all the points of the system.

If all the velocities $v, v', v''$, &c., are equal, L will be the product of their common value $v$ and of the sum $mp + m'p' + m''p'' +$ &c.; if, besides, all these velocities are parallel to each other, and if through the axis $oz$, a plane is drawn parallel to their common direction, $p, p', p''$, &c., will be the distances of the points $m, m', m''$, &c., from this plane; and, with regard to the signs with which the terms of L are affected, these perpendiculars must be considered to be positive or negative, according as the points $m, m', m''$, &c., are situated on the one or other side of this plane. Hence if the sum of the masses $m, m', m''$, &c., be denoted by M, and if the distance of its centre of gravity from this plane (which may be either positive or negative) be denoted by $q$, we shall have (No. 65)

$$mp + m'p' + m''p'' + \text{\&c.} = Mq;$$

consequently $L = Mvq$, and the value of $\omega$ will become

$$\omega = \frac{Mvq}{\Sigma mr^2}.$$

If the velocity $v$ is only impressed on some of the points of the system, and if no velocity is directly communicated to the other points, we shall have, in the same manner,

$$\omega = \frac{\mu vf}{\Sigma mr^2};$$

$\mu$ being the sum of the masses actuated by the velocity $v$, and $f$ the distance of the centre of gravity of the[illegible]

tem, from the plane drawn through the fixed axis, parallel to the direction of $v$.

386. If the system of points $m$, $m'$, $m''$, &c., be supposed to constitute a solid body, it is then sufficient, in the preceding formulæ, to change the mass $m$ of any point whatever, into the differential element of the mass of the body, (which, as usual, we shall denote by $dm$), and the sum $\Sigma$ into an integral. Hence the quantity $\Sigma mr^2$ will become the integral $\int r^2dm$, extended to the entire mass of the body, and it will express its moment of inertia with respect to the axis of rotation; consequently, the last formula will be changed into the following:

$$\omega = \frac{\mu v f}{\int r^2 dm}. \qquad (1)$$

This formula is applicable to the case of a solid body retained by a fixed axis, and struck by another body, which after the impact attaches itself to the first, so that then the two bodies constitute only one, turning about the fixed axis with an angular velocity $\omega$. The mass of the striking body is $\mu$, its velocity before the impact, which is common to all its points and perpendicular to the direction of the fixed axis, is $v$, and $f$ expresses the distance of its centre of gravity from a plane parallel to the direction of this velocity, and passing through the axis of rotation. The integral $\int r^2dm$ should be extended to the two masses which are united together after the impact. If the striking body does not remain attached to the other after the impact, the determination of the angular velocity of this last is a different problem, the solution of which will be given in a subsequent chapter.

When the body retained by a fixed axis is struck simultaneously by several masses, such as $\mu$, $\mu'$, $\mu''$, &c., actuated by the velocities $v$, $v'$, $v''$, &c., whose directions are perpendicular to that of the axis, and which are united to this body after the impact, the expression for $\omega$, the resulting angular velocity, will be

$$\omega = \frac{\mu v f + \mu' v' f' + \mu'' v'' f'' + \&c.}{\int r^2 dm};$$

in which expression, the integral must be extended to the total mass, and $f, f', f''$, &c., denote the distances of the centres of gravity of $\mu, \mu', \mu''$, &c., from planes drawn through the axis of rotation, parallel to the directions of the velocities $v, v', v''$, &c. If the first term of the numerator of this formula be affected with the sign $+$, the other terms should be affected with the signs $+$ or $-$, according as the corresponding percussions tend to produce a revolution, in the direction of that which corresponds to the first term, or in a contrary direction. When $\omega = 0$, the system will remain at rest, and all the percussions will constitute an equilibrium. If the percussions, instead of being simultaneous, were successive, the value of $\omega$, after all the impacts had ceased, would be still furnished by the preceding formula; for after the first impact the motion is the same at each instant, as if this impact took place then; consequently, we may suppose that it took place at the instant of the second impact, in order to determine the angular velocity after the second percussion; and so on in succession.

387. When the rotatory motion commences about a *fixed* axis, this line experiences percussions which it is important to determine; they are due to the quantities of motion that are lost, at each epoch, by the different points of the body, which quantities of motion, as they constitute an equilibrium by means of the fixed axis, must consequently be reduced to percussions, whose directions either meet this axis, or are parallel to it. The parallel percussions may be immediately determined; they are the components, in this direction, of the quantities of motion which have been impressed on the body; as has been stated in No. 385, we shall not take them into account; and we shall suppose that the rotatory motion has been produced by an impact perpendicular to the direction of the fixed axis, to which formula (1) refers. Let the point P be the centre

of gravity of $\mu$, the line PA the direction of its velocity before the impact, so that OF, the distance of this line from the axis O$z$, may be denoted by $f$. In the plane perpendicular to this fixed axis, and comprising the line PA, let there be drawn through the point O of this axis, two other rectangular axes O$x$ and O$y$. Let $x$, $y$, $z$ be the three rectangular coordinates of $dm$, referred to the axes O$x$, O$y$, O$z$; as $r\omega$, the velocity of this material point, is perpendicular to the radius $r$, and parallel to the plane of the axes of $x$ and $y$, it is easy to perceive that the cosines of the angles which it makes with these three axes are $-\frac{y}{r}, \frac{x}{r}$, and zero, the rotation being supposed to have place in the direction of the sagitta $s$; it may consequently be decomposed into two velocities $-y\omega$ and $x\omega$, parallel to the axes O$x$ and O$y$($a$). Hence the components of the quantities of motion of all the points of the body along these directions will be $-\omega\int ydm$ and $\omega\int xdm$, or what comes to the same thing, $-\omega\mathrm{M}y_1$, and $\omega\mathrm{M}x_1$, in which M denotes, as before, the entire mass after the impact, and $x_1$ and $y_1$ the values of $x$ and $y$, which belong to its centre of gravity. The sums of the moments of all these quantities of motion, with respect to the plane of the axes of $x$ and $y$, will be $-\omega\int yzdm$ and $\omega\int xzdm$; by No. 54, they should be equal to the moments of the total forces $-\omega\int ydm$ and $\omega\int xdm$, with respect to the same plane, that is to say, to $-\omega\mathrm{M}y_1z'$ and $\omega\mathrm{M}x_1z''$, $z'$ and $z''$ denoting the distances of these two forces from this plane; consequently we shall have

$$\mathrm{M}y_1z' = \int yzdm, \quad \mathrm{M}x_1z'' = \int xzdm, \tag{2}$$

by means of which, these two quantities $z'$ and $z''$ (which may be either positive or negative) can be determined.

This being established, the quantities of motion lost by all the points of the mass M, and which constitute an equilibrium by means of the fixed axis, may be replaced by a force $\omega\mathrm{M}y_1$, parallel to the axis of $x$, and situated at the distance $z'$ from the plane of the axes of $x$ and $y$, by a force $-\omega\mathrm{M}x_1$, parallel

to the axis of $y$, and situated at the distance $z''$ from this plane, and by the force $\mu v$, which retains its proper direction, namely, from the point P towards the point A. These three forces can be at least reduced to two, which will meet the fixed axis, and they express the percussions that it experiences, perpendicularly to its length. When these three forces are reducible to one, the axis will experience an unique percussion, and it is sufficient, in order that the axis may resist it, that the point where it will be met by this force, should be supposed fixed.

388. If the line $oz$ be one of the three principal axes of M, which intersect in the point O, we shall have

$$\int xzdm = 0, \quad \int yzdm = 0.$$

Hence in this case the distances $z'$ and $z''$ are cipher, and the forces $\omega My_1$, $-\omega Mx_1$, and also the force $\mu v$, will all exist in the plane of the axes of $x$ and $y$, in consequence of which, the unique percussion to which they will be reduced, will pass through the point O. In order to determine its magnitude and position, if R be this force, $a$ and $b$ the angles that it makes with the axes of $x$ and $y$, $\alpha$ and $\beta$ the angles which the line PA makes with parallels to these axes, drawn through the point P, we shall have

$$\begin{aligned} R\cos a &= \mu v\cos\alpha + \omega My_1, \\ R\cos b &= \mu v\cos\beta - \omega Mx_1; \end{aligned}$$

and as the value of $\omega$ is given by formula (1), and $x_1$ and $y_1$, the coordinates of the centre of gravity of M, are also known, every thing is known in these values of the two components of the force R.

Since this resultant must pass through the point O, the sum of the moments of its three components, with respect to this point, must be equal to cipher; now if $x'$ and $y'$ be the distances of the forces $\omega My_1$ and $-\omega Mx_1$ from the axes $ox$ and $oy$, respectively parallel to these lines, there will result, re-

gard being had to the direction in which these two forces and the percussion $\mu v$ tend to make the body to turn about the point O,

$$\omega \mathrm{M} y_1 y' + \omega \mathrm{M} x_1 x' - \mu v f = 0;$$

$f$ denoting, as before, OF the perpendicular let fall from the point O on the line PA. In fact, it is easy to verify this result; for, considering the moments, with respect to the planes of the axes of $x$ and $z$, and of $y$ and $z$, of all the quantities of motion parallel to these planes, the sums of which are $-\omega \mathrm{M} y_1$ and $\omega \mathrm{M} x_1$, we shall have

$$\mathrm{M} y_1 y' = \int y^2 dm, \quad \mathrm{M} x_1 x' = \int x^2 dm;$$

and these values, combined with those of $\omega$, render the preceding equation identical($b$).

In order that the fixed axis may not experience any percussion, it is easy to perceive that the distances $z'$ and $z''$ should in the first place be cipher, and in virtue of equation (2), this cannot be the case, unless the line O$z$ be one of the principal axes which intersect in the point O, as we have supposed. This condition being satisfied, it is moreover necessary that the force R should be cipher; this implies that

$$\mu v \cos \alpha = -\omega \mathrm{M} y_1, \quad \mu v \cos \beta = \omega \mathrm{M} x_1.$$

From which there results

$$\frac{x_1 \cos \alpha}{y_1 \cos \beta} + 1 = 0;$$

this equation shews that the line PA must be perpendicular to the plane passing through the fixed axis and through the centre of gravity of M. Moreover, if $r_1$ denotes the distance of this point from this axis, the preceding equations will give

$$\mu^2 v^2 = \omega^2 \mathrm{M}^2 (x_1^2 + y_1^2) = \omega^2 \mathrm{M}^2 r_1^2;$$

and by substituting for $\omega$ its value furnished by equation (1), there will result

$$f = \frac{\int r^2 dm}{\mathrm{M} r_1}.$$

Hence, when a solid body retained by a fixed axis, is struck by a second body, the mass of which remains attached to the first, in order that the fixed axis may experience no percussion, it is necessary, first, that the direction of the shock be in the plane of the two lines, which, with the fixed axis, constitute a rectangular system of principal axes of the body made up of the two masses united together; secondly, that its direction be perpendicular to the plane passing through the centre of gravity of this body and the fixed axis; thirdly, that this direction [illegible] should meet this plane in a point of which, $f$, the distance from the axis of rotation, is furnished by the preceding formula. This point is what is termed *the centre of percussion*.

389. While a solid body turns about a fixed axis, the centrifugal forces of its different points produce on this axis pressures which we now proceed to determine; they are the only pressures that have place when the motion is uniform, in which case the points of the body are not solicited by any motive forces.

If the preceding notations be retained, $r\omega^2 dm$ will express the centrifugal force of the element $dm$ which describes a circle, whose radius is $r$, with a velocity $r\omega$; and as this force acts in the direction of the production of $r$, the cosines of the angles which it makes with the axes of $x$, $y$, $z$, will be $\frac{x}{r}$, $\frac{y}{r}$, and zero. If, therefore, it be transferred to the point where its direction meets the axis $oz$, it may be replaced by two forces comprised in the planes $xoz$ and $yoz$, respectively parallel to the axes $ox$ and $oy$, and equal to $x\omega^2 dm$, $y\omega^2 dm$. As the same thing obtains for all the other elements of the body, it follows that the axis $oz$ will be urged along these directions, by forces, whose values are $\omega^2 \int x dm$, $\omega^2 \int y dm$, or, what is the same thing, $\omega^2 M x_1$, $\omega^2 M y_1$, as is evident from the preceding notation. It appears also that $z'$ and $z''$, the distances of these two total pressures, from the plane of the axes of $x$ and $y$, are given by the equations

$$\text{M}x_1z' = \int xzdm, \quad \text{M}y_1z'' = \int yzdm, \tag{3}$$

which are the inverse of equations (2) that are relative to the percussions($c$).

When $z' = z''$, the two pressures $\omega^2\text{M}x_1$ and $\omega^2\text{M}y_1$ will be applied to the same point of the axis o$z$, they can then be reduced to one force perpendicular to this line, which acts in the direction of a plane that comprises the centre of gravity of the body, and its value will be $\omega^2\text{M}r_1$; $r_1$ denoting the distance of this centre from the fixed axis.

This will always be the case when o$z$ is one of the principal axes that intersect in the point o; for then the second members of equations (3) will be cipher, and we shall have $z' = 0$ and $z'' = 0$. The unique pressure, which, during the rotatory motion, the axis experiences, will therefore pass through the point o; hence, in order that the pressure may be destroyed, and that the axis may be immoveable, it will be sufficient if this point is fixed. Therefore for every point o belonging to a solid body, or invariably connected with it, there are always three rectangular axes passing through this point, about which the body can turn, without these axes undergoing any displacement, just as if they were entirely fixed.

Such is the property relative to the uniform motion of rotation, which the lines that have been termed *principal axes* possess. It belongs to them exclusively; for if the body turns about a line o$z$, which is not one of the principal axes relative to the fixed point o, the two pressures $\omega^2\text{M}x_1$ and $\omega^2\text{M}y_1$ will, in general, be irreducible to one; or if they are reducible to one, because $z' = z''$, this unique pressure will pass through a point different from o; consequently, in order that the pressures due to the centrifugal forces may be destroyed, and that the axis of rotation be not displaced during the motion, it is necessary that a second point be supposed fixed, but this implies that the entire axis is fixed. When a body retained by a fixed point o, and not acted on by any motive force, com-

mences to turn about one of these axes which intersect in this point, the motion will continue indefinitely about this line. This will be the case, for example, when the body is put in motion by an impact, the direction of which is in the plane passing through the other two principal axes relative to this point o, for then it appears, by what has been established in the preceding number, that the percussions which the axis experiences, the first instant, will be reduced to one which, passing through this fixed point, will be destroyed by its resistance. If o be one of the particular points, for which all moments of inertia are equal, the axis about which the body will begin to turn, will be necessarily a principal axis; and in *whatever* manner the body is set in motion about this fixed point, the axis of rotation will remain immoveable. Thus, an ellipsoid of revolution, or a parallelopiped retained by one of the points, the position of which has been determined already in No. 383, will always turn about a fixed axis. This will also be the case with respect to a sphere when its centre is fixed, or a cube retained by the point situated at the intersection of its three diagonals; and, (moreover, in these two last cases, the fixed point being the centre of gravity, the axis of rotation will remain immoveable, although the body should be supposed to be acted on by gravity.) In general, the moang forces that act on the body, will, like the centrifugal and z es, produce pressures on the axis of rotation, which will direc to displace it, when they are not reducible to one sole com e, passing through the fixed point.

th 390. If the line $oz$ is one of the three principal axes which intersect at G, the centre of gravity of the body that is considered, it will be also one of the three principal axes of this same body for any point whatever of its direction. In fact, if OG, the distance of this centre from the point O, be denoted by $\gamma$, and if, without changing the preceding direction of the coordinates $x$, $y$, $z$, their origin be transferred to the point G, those of $dm$ any element whatever will become $x$, $y$, $z-\gamma$,

$$\int x(z-\gamma)\,dm = \int xzdm - \gamma\int xdm = 0,$$
$$\int y(z-\gamma)\,dm = \int yzdm - \gamma\int ydm = 0.$$

Moreover, as the point G is on the axis of the ordinate $z$, we have $\int xdm = 0$, and $\int ydm = 0$; consequently, we shall have $\int yzdm = 0$, $\int xzdm = 0$, from which it appears, that O$z$ is one of the three principal axes that intersect at the point O.

The direction of the two other principal axes, in the plane of the axes of $x$ and $y$, can be determined by the transformation of the coordinates. Let $x'$ and $y'$ be those of $dm$ with respect to these two other axes, we must have

$$\int x'y'dm = 0, \quad \int x'zdm = 0, \quad \int y'zdm = 0;$$

and if $\theta$ be the angle contained between the axis of $x'$ and that of $x$, we shall have

$$x' = x\cos\theta - y\sin\theta, \quad y' = x\sin\theta + y\cos\theta;$$

now, if these values be substituted in the preceding equations, the two last must disappear, because $\int xzdm = 0$, and $\int yzdm = 0$; the first becomes

$$(\cos^2\theta - \sin^2\theta)\int xydm + \sin\theta\cos\theta\left(\int x^2dm - \int y^2dm\right) = 0,$$

from which the value of $\theta$ may be obtained. The values of the integrals which this equation contains, may change with the position of the point O; so that the direction of one of the principal axes being supposed to be that of the axis O$z$, the directions of the two others will not, in general, always remain parallel to themselves.

As the four equations

$$\int xdm = 0, \quad \int xzdm = 0, \quad \int ydm = 0, \quad \int yzdm = 0,$$

obtain simultaneously, it follows from the two first, that the parallel pressures existing in the plane of the axes of $x$ and $z$, which arise from the centrifugal forces, may be reduced to two equal and directly opposite forces; and in virtue of the

two last equations, the same thing is true with respect to the components existing in the plane of the axes of $y$ and $z$. Consequently, when the body turns about one of the three principal axes which intersect in its centre of gravity, the centrifugal forces of all its points produce no pressure on the axis of rotation, and if the motion commences about such an axis, it will continue indefinitely, although this axis has no fixed point, provided that the body is not acted upon by any motive force. This result is evident in the case of an ellipsoid, which turns about one of its three axes of figure; for every thing being supposed symmetrical about each of these lines, there is no reason why it should experience a pressure in one direction rather than in the opposite.

## II. *Variable Motion of Rotation.*

391. Let the element of the mass of the body which revolves about the fixed axis o$z$ (fig. 5) be denoted by $dm$, through a point o taken arbitrarily on this line, let two other fixed axes o$x$ and o$y$ be drawn perpendicular to each other and to the axis o$z$; and at the end of $t$ any time whatever, let $x$, $y$, $z$, denote the three coordinates of $dm$ referred to these axes o$x$, o$y$, o$z$. At the same instant, the components of the velocity parallel to these lines will be $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$. Now, whatever be the nature of the forces applied to the element $dm$, they may be decomposed parallel to these same axes; and, at the end of the time $t$, let the components of the given accelerating force which acts on this material point, estimated in the positive directions of $x$, $y$, $z$, be X, Y, Z, respectively. If this point was free, the components of the velocity, namely, $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, would be increased by X$dt$, Y$dt$, Z$dt$, during the instant $dt$ (No. 147); but these functions of the time are really increased by their differentials $d.\frac{dx}{dt}$, $d.\frac{dy}{dt}$, $d.\frac{dz}{dt}$; con-

sequently, the velocities lost during the instant $dt$, in the directions of the coordinates, will be

$$\text{X}dt - d.\frac{dx}{dt}, \quad \text{Y}dt - d.\frac{dy}{dt}, \quad \text{Z}dt - d.\frac{dz}{dt}.$$

Multiplying them by $dm$, and then dividing by $dt$, there will result

$$\left(\text{X} - \frac{d^2x}{dt^2}\right)dm, \ \left(\text{Y} - \frac{d^2y}{dt^2}\right)dm, \ \left(\text{Z} - \frac{d^2z}{dt^2}\right)dm,$$

which will be the values of the components of the force lost by the element $dm$, during this instant $dt$, in consequence of its connexion with the other points of the body and with the axis of rotation. Therefore, in virtue of the principle of No. 350, an equilibrium must obtain about this fixed axis, between similar forces applied to all the points of the body.

In order to obtain the equation of this equilibrium, it will be sufficient to substitute the two first of the three preceding forces in the place of $\text{P}\cos\alpha$ and $\text{P}\cos\beta$, in the first term of equation (5) of No. 266, and then to put equal to cipher, the sum of the values of this quantity for all the points of the body, which sum will be an integral extended to the entire mass. If the differentials be placed in the first member, and the given forces in the second member of this equation, the required equation will be

$$\int\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right)dm = \int\left(x\text{Y} - y\text{X}\right)dm, \qquad (1)$$

which will be that of the motion of rotation about the axis of the ordinate $z$.

392. Let the angular velocity at the end of the time $t$ be denoted by $\omega$, which we shall consider to be positive or negative, according as the motion of rotation is in the direction of the sagitta $s$, or in the opposite direction; likewise, let $r$ be the radius of the circle described by $dm$, or the perpendicular let fall from this point on the axis $\text{O}z$; $r\omega$ will be equal to

the absolute velocity, and the values of its two components in the direction of $x$ and $y$ will be

$$\frac{dx}{dt} = -y\omega, \quad \frac{dy}{dt} = x\omega.$$

In fact, if P be the projection of $dm$ on the plane of the axes of $x$ and $y$, and if the radius OP be drawn, and also QPP′, perpendicular to OP, it will intersect the axes O$x$ and O$y$ in Q and Q′, and we shall have

$$\text{OP} = r, \quad \cos x\,\text{QP} = -\frac{y}{r}, \quad \cos y\,\text{Q}'\text{P}' = \frac{x}{r};$$

now as the direction of the velocity $r\omega$ is parallel to the line QPP′, it must be multiplied by these cosines, in order to obtain the value of its components $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

From the equation $x^2 + y^2 = r^2$, we obtain($d$)

$$xdy - ydx = r^2\omega dt. \qquad (2)$$

If this last be differenced with respect to $t$, there will result, because the radius $r$ is constant,

$$xd^2y - yd^2x = r^2 d\omega dt;$$

and as $d\omega$ is a quantity common to all the points of the body, it must be considered as constant in the integration relative to $dm$, consequently, equation (1) becomes

$$\frac{d\omega}{dt} \int r^2 dm = \int (x\text{Y} - y\text{X})\, dm; \qquad (3)$$

by means of which, the value of $d\omega$, corresponding to a given position of the body, can be determined.

Let the accelerating force which acts on $dm$ be decomposed into two others, one parallel to the axis O$z$, and the other comprised in a plane perpendicular to this line; as the first does not contribute to produce the motion of rotation, it is not necessary to take it into account; and the second, which we shall denote by $\phi$, will be the resultant of the forces X and Y.

If X, Y, $\phi$, be projected on the plane of the axes of $x$ and $y$, and if PC be the direction of $\phi$ thus projected; and $h$ the perpendicular OH, let fall from O on PC or on its production; by considering the moments with respect to the point O, we shall have (No. 46)

$$xY - yX = \pm h\phi,$$

the sign of the second member of this equation will be + or —, according as the force $\phi$ tends to make the body to turn in the direction of the sagitta $s$, or in the opposite direction. If we denote by $\delta$ the angle Q′PC which the line PC makes with PQ′ the perpendicular to the radius OP, and drawn in the direction indicated by the sagitta $s$, this angle will be acute or obtuse, according as the second member of the preceding equation is affected with the upper or lower sign; and as OP $= r$, we shall, therefore, have always

$$\pm h = r \cos \delta;$$

by means of these values, equation (3) will become

$$\frac{d\omega}{dt} \int r^2 dm = \int r\phi \cos \delta \, dm.$$

Now, if the given forces act on the body only during a very short interval of time, and if notwithstanding, they are capable of producing in this short interval, given velocities which are of a finite magnitude, and moreover, if during this same time, neither the directions of these forces, nor the positions of the points of the body are sensibly changed, we shall obtain by integrating the two members of this equation with respect to $t$,

$$\omega \int r^2 dm = \int rv \cos \delta \, dm;$$

$v$ being the integral of $\int \phi dt$ during the continuance of the action of the forces, that is to say, the velocity which the percussion exerted on $dm$ would impress on this material point, if it was entirely free. This equation is that of uniform

motion of rotation, and the formulæ given in No. 386 may be obtained from it, without any difficulty.

In the case of variable motion, equation (3) becomes a differential equation of the second order, on which the position of the body at each instant, and its velocity in a function of the time, depends, as we shall see immediately by an example; but it is expedient to determine previously the pressures which the axis of rotation experiences during the continuance of the motion.

393. We shall only consider the pressures perpendicular to this axis $oz$, which are due to the components, parallel to the axes $ox$ and $oy$, of the forces lost by all the points of the body, the resultants of which must intersect the fixed axis.

Denoting the sums of all these forces by U and V, we shall have, by No. 391,

$$\mathrm{U} = \int\left(\mathrm{X} - \frac{d^2x}{dt^2}\right)dm, \quad \mathrm{V} = \int\left(\mathrm{Y} - \frac{d^2y}{dt^2}\right)dm;$$

and if $u$ and $v$ be the distances of the total forces U and V from the plane of the axes of $x$ and $y$, we shall likewise have (No. 54)

$$\mathrm{U}u = \int\left(\mathrm{X} - \frac{d^2x}{dt^2}\right)zdm, \quad \mathrm{V}v = \int\left(\mathrm{Y} - \frac{d^2y}{dt^2}\right)zdm.$$

From the preceding values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$, we obtain

$$\frac{d^2x}{dt^2} = -y\frac{d\omega}{dt} - \omega\frac{dy}{dt} = -y\frac{d\omega}{dt} - x\omega^2,$$

$$\frac{d^2y}{dt^2} = x\frac{d\omega}{dt} + \omega\frac{dx}{dt} = x\frac{d\omega}{dt} - y\omega^2.$$

If $\frac{d\omega}{dt}$ be eliminated by means of equation (3), and if the values of $x$ and $y$, which refer to the centre of gravity of the body, be denoted by $x_1$ and $y_1$, and its mass by M, so that we may have

$$\int xdm = \mathrm{M}x_1, \quad \int ydm = \mathrm{M}y_1;$$

then by substituting the values of $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ in the preceding formulæ, they will become (e)

$$\text{U} = \text{M}x_1\omega^2 + \int \text{X}dm + \text{M}y_1 \frac{\int(x\text{Y} - y\text{X})dm}{\int r^2dm},$$

$$\text{V} = \text{M}y_1\omega^2 + \int \text{Y}dm - \text{M}x_1 \frac{\int(x\text{Y} - y\text{X})dm}{\int r^2dm},$$

$$\text{U}u = \omega^2\int xzdm + \int z\text{X}dm + \frac{\int yzdm\int(x\text{Y} - y\text{X})dm}{\int r^2dm},$$

$$\text{V}v = \omega^2\int yzdm + \int z\text{Y}dm - \frac{\int xzdm\int(x\text{Y} - y\text{X})dm}{\int r^2dm}.$$

When the angular velocity $\omega$ is known, these equations will determine U and V, the pressures parallel to the axes o$x$ and o$y$, which the axis o$z$ sustains; and also $u$ and $v$ the distances comprised between the point o and the points of application of U and V on this axis. When the forces X and Y are cipher, these results coincide with those of No. 389. When the line o$z$ passes through the centre of gravity of the body, $x_1 = 0$, and $y_1 = 0$, consequently

$$\text{U} = \int \text{X}dm, \quad \text{V} = \int \text{Y}dm;$$

so that the total pressure on the fixed axis is the same as in the state of equilibrium; but, in general, it is differently distributed. If the fixed axis is, besides, one of the principal axes which intersect in the centre of gravity, we have likewise $\int xzdm = 0$ and $\int yzdm = 0$; consequently

$$\text{U}u = \int z\text{X}dm, \quad \text{V}v = \int z\text{Y}dm;$$

and the pressure on the axis is then found to be distributed, in the state of motion, as in the state of equilibrium.

394. Let us now apply equation (3) to the case of a heavy body, revolving about an horizontal axis.

If the force of gravity be denoted by $g$, and if the axis o$y$ be supposed to be vertical, and to be drawn in the direction of this force, we shall have

and, because $\int x dm = \mathrm{M}x_1$, equation (3) will be reduced to

$$\frac{d\omega}{dt}\int r^2 dm = g\mathrm{M}x_1. \tag{4}$$

If $\theta$ denotes the angle which, at the end of the time $t$, the moveable plane that passes through the centre of gravity of the body makes with the fixed plane of the axes of $y$ and $z$, which angle will be considered as positive or negative, according as the moveable plane exists, relatively to the fixed plane, on the side of the axis of the positive $x$s, or on the opposite side; and if $a$ be the constant distance of the centre of gravity from the axis $\mathrm{O}z$, we shall have

$$x_1 = a\sin\theta;$$

and, according to the direction of the velocity $\omega$, whether positive or negative, we shall likewise have

$$\omega = -\frac{d\theta}{dt};$$

which may also be deduced from equation (2), applied to the centre of gravity, that is to say, to the values $a\sin\theta$, $a\cos\theta$, $a$, of $x$, $y$, $r$. Finally, let $\mathrm{M}k^2$ be the moment of inertia of the body with respect to an axis passing through its centre of gravity, and parallel to $\mathrm{O}z$; $k$ will be a line of a given magnitude, and by the theorem of No. 374, we shall have

$$\int r^2 dm = \mathrm{M}(a^2 + k^2)$$

for the value of the moment of inertia with respect to the axis of rotation.

By means of these values of $x_1$, $\omega$, $\int r^2 dm$, equation (4) becomes

$$\frac{d^2\theta}{dt^2} = -\frac{ga\sin\theta}{a^2 + k^2}.$$

Hence, multiplying by $2d\theta$, and integrating, we shall have

$$\frac{d\theta^2}{dt^2} - \frac{2ga\cos\theta}{a^2 + k^2} = c,$$

$c$ being an arbitrary constant. If, at the commencement of the motion,

$$\theta = a, \frac{d\theta}{dt} = -\Omega,$$

the value of this constant will be

$$c = \Omega^2 - \frac{2ga\cos a}{a^2 + k^2};$$

and therefore, at any instant whatever, we shall have

$$\frac{d\theta^2}{dt^2} + \frac{2ga(\cos a - \cos\theta)}{a^2 + k^2} = \Omega^2. \qquad \text{(a)}$$

This equation is that of the motion of a pendulum of any form whatever, turning about an horizontal axis. In its state of equilibrium, the plane passing through its centre of gravity and through the fixed axis is vertical, and we have $a = 0$, $\Omega = 0$, $\theta = 0$. If the body is made to deviate from this position, so that these two planes may comprise a given angle $a$, and if then it is remitted to itself, we shall have $\Omega = 0$; if, on the contrary, the body experiences a percussion at the commencement of its motion, the initial velocity $\Omega$ must be determined by the rules of No. 386, or given in some other manner; and, in all cases, equation (a) will make known the angular velocity of the body at any instant whatever, when the position of its centre of gravity is known. If this equation be resolved with respect to $dt$, and then integrated, the value of $t$ in a function of $\theta$ will be obtained, and conversely, this will determine the variable position of this centre, and, consequently, that of the body at each instant.

395. If this heavy body is reduced to a material point, attached to the axis $oz$ by an inextensible and inflexible thread, destitute of weight, and whose direction is perpendicular to $oz$, we shall have the case of the simple pendulum, as is evident from its definition given in No. 179. If its length be denoted by $l$, we shall have $a = l$; M will be the

mass of the material point; and the moment of inertia $M(a^2+k^2)$ must be reduced to the product of this mass, and of the square of $l$ its distance from the fixed axis. Consequently, we shall have $k=0$, and equation (a) applied to this particular case, will become

$$\frac{d\theta^2}{dt^2}+\frac{2g}{l}\cos(a-\cos\theta)=\Omega^2; \tag{b}$$

and, in fact, it is easy to shew that it agrees with equation (1) of No. 180, relative to the motion of the simple pendulum. It appears from a comparison of equations (a) and (b), that this motion will coincide with that of any pendulum whatever, whenever the coefficients $\frac{2g}{l}$ and $\frac{2ga}{a^2+k^2}$, by which these two equations differ, are equal, that is to say, when

$$l=a+\frac{k^2}{a}. \tag{c}$$

It is, therefore, by this formula that we determine, as has been stated in No. 179, the length of the simple pendulum, whose time of vibration is equal to that of a given pendulum; when the form of this compound pendulum is known, the two quantities $a$ and $k$ can be determined by the known rules, either accurately, or by approximation.

When this pendulum makes very small oscillations, the same will be the case with respect to the corresponding simple pendulum. Therefore, if the duration of an entire oscillation be denoted by $T$, we shall have (No. 182)

$$T=\pi\sqrt{\frac{l}{g}},\quad g=\frac{\pi^2 l}{T^2}.$$

If the number of oscillations which the compound pendulum performs during a considerable time be reckoned, the value of $T$ will be had by dividing the entire time by this number; and by substituting it in this last formula, the measure of the gravity $g$ will be obtained with great accuracy,

when the length of $l$ corresponding to the pendulum that is employed is known, as has been already explained in No. 192. As the lengths of simple pendulums are to each other as the squares of the times in which very small oscillations are performed, if the length of the pendulum which vibrates seconds be denoted by $\lambda$, we shall obtain (the second being taken as the unit of time)

$$\lambda = \frac{l}{\text{T}^2},$$

by means of which equation, the value of $\lambda$ can be determined when those of $l$ and T are known.

396. If in the interior of the compound pendulum, a line be traced below its centre of gravity, in the plane passing through this centre and the axis of rotation, parallel to this axis, and at the distance $l$ from this same axis, the motion of the points of this parallel will neither be accelerated or retarded by their connexion with the other points of the body. Among all the points of this line, that is *properly* termed *the centre of oscillation*, which exists on the same perpendicular to the axis as the centre of gravity.

Let ABD (fig. 6) be the section of the pendulum perpendicular to the axis of rotation, and passing through its centre of gravity, G this centre, and C the point where this section is cut by the axis; let the line CG be produced to O, so that we may have ($g$)

$$\text{CG} = a, \quad \text{GO} = \frac{k^2}{a},$$

and, consequently,

$$\text{CO} = a + \frac{k^2}{a} = l.$$

The point O will be the centre of oscillation; and if, after having caused the given pendulum to oscillate about the axis perpendicular to the section ABD, and passing through the point C, we then reverse it, and cause it to oscillate about the

axis passing through the point O, and perpendicular to this same section, the point C will become the centre of oscillation. This theorem is commonly expressed by stating that the centres of oscillation and suspension are reciprocal, so that when the centre of oscillation is made the centre of suspension, the latter becomes the centre of oscillation.

In fact, in each case, the moment of inertia $Mk^2$ is the same, since it always respects the axis perpendicular to ABD, and passing through the point G; so that the quantity $k$ will not be changed. Moreover, if O′ be the point of the production of OG, which is the centre of oscillation, when O becomes the centre of suspension, and if the distance OO′ be denoted by $l'$, its value can be deduced from formula (c), by substituting OG in place of CG, that is to say, $\frac{k^2}{a}$ in place of $a$; therefore we shall have

$$l' = \frac{k^2}{a} + a = l, \quad OO' = CO;$$

and, consequently, the point O′ will coincide with the point C.

397. The duration of very small oscillations about two axes perpendicular to ABD, and passing through the points C and O, is the same, and equal to $\pi\sqrt{\frac{l}{g}}$; $l$ being always the distance CO. Conversely, if the duration of very small oscillations is the same about two parallel axes, the plane of which contains the centre of gravity G, and which are not equally distant from it, their mutual distance will be equal to $l$, the length of the simple pendulum which also oscillates in the same time.

In fact, let $a$ and $a'$ be the unequal distances of the centre of gravity from these two parallel lines, and, consequently, $a + a'$ their mutual distance; likewise let $Mk^2$ be the moment of inertia with respect to the parallel axis passing through the centre of gravity, since the duration of the oscillations about the two lines is the same, we must have

$$a' + \frac{k^2}{a'} = a + \frac{k^2}{a};$$

hence we obtain

$$a' = a, \text{ or } a' = \frac{k^2}{a};$$

therefore, as by supposition the first value of $a'$ must be rejected, we shall have

$$a + a' = a + \frac{k^2}{a}.$$

Consequently, if $a + a'$, the distance between the two synchronous axes be measured, the length of the simple pendulum which oscillates in the same time as each of them will be obtained.

This method, which has the advantage of not requiring any computation relative to the form of the compound pendulum to be made, has been successfully employed in England, to determine the length of the compound pendulum ($l$).

398. There are an infinite number of different axes about which the duration of small oscillations of the same body are equal.

In the first place, it is evident that the value of $l$ and the duration of the oscillations will be the same for all axes of suspension parallel to each other, and equally distant from the centre of gravity, since, for all these axes, the quantities $k$ and $a$, which occur in equation (c), do not vary.

The direction of these axes, and their distance from the centre of gravity, may also be changed without the value of $l$ undergoing any change; for if the angles, that the parallel to the axis of suspension, drawn through the centre of gravity, makes with the three principal axes, which intersect in this point, be denoted by $\alpha$, $\beta$, $\gamma$, and if A, B, C be the moments of inertia relative to these axes, and if $Mk^2$ be, as before, that which refers to this parallel, by equation (c) of No. 375, we shall have

$$Mk^2 = A\cos^2 a + B\cos^2\beta + C\cos^2\gamma,$$

and, consequently,

$$l = a + \frac{A\cos^2 a + B\cos^2\beta + C\cos^2\gamma}{Ma}.$$

Now, it is evident that we can assign to $a$, $a$, $\beta$, $\gamma$, an infinite number of different values, for which this value of $l$ will remain the same.

If it be proposed to determine when this function $l$ is a *minimum* with respect to the variables $a$, $a$, $\beta$, $\gamma$, it follows from its form, that A being supposed to be the least of the three constant quantities A, B, C, we must have $a = 0$, $\beta = 90°$, $\gamma = 90°$, and, consequently,

$$l = \frac{Ma^2 + A}{Ma};$$

from which, by the common rule there results $a = \sqrt{\frac{A}{M}}$ for the value of $a$ that corresponds to the *minimum*, and the actual value of this *minimum* is $l = 2\sqrt{\frac{A}{M}}$ (*i*).

399. It was demonstrated in No. 190, that the resistance of a medium does not influence the *duration* of small oscillations of a pendulum of a given length; but it is also necessary to prove, that this force does not change the *length* of the *simple* pendulum, whose motion is the same in the air as that of a given pendulum, in this same fluid. Now, in order to determine this motion, to the forces $X dm$ and $Y dm$, which compose the second member of equation (3), and which act on *all* the points of the body, there must be joined the components of the resistance of the air, which only act on the *elements* of its *surface*. Let, therefore, this resistance be supposed to be expressed, generally, by the sum of several powers of the velocity, so that for $v$ any particular velocity whatever, its action on the unit of surface may be denoted by

$$Av^n + A'v^{n'} + A''v^{n''} + \&c.;$$

A, A′, A″, &c., $a$, $a'$, $a''$, &c., being given constants. Let $\rho$ be the distance of M, a point of the surface of the pendulum from the axis of rotation; its velocity at the end of the time $t$, will be $\rho\omega$, $\omega$ denoting as before the angular velocity at this instant. If $\epsilon$ be the angle that its direction makes with the interior part of the normal at this point, then $\rho\omega\cos\epsilon$ will be its component in the direction of this line; and by what has been already stated (No. 365), it is this normal component that should be employed in place of the velocity $v$, in the expression of the resistance at the point M. Therefore, if the differential element of the surface at this same point, be denoted by $d\sigma$, and the resistance exerted on this element by $\text{R}d\sigma$, we shall have

$$\text{R} = \text{A}\rho^a\omega^a\cos^a\epsilon + \text{A}\rho^{a'}\omega^{a'}\cos^{a'}\epsilon + \&\text{c}.$$

If $\mu$ and $\nu$ be the angles which the part of the normal at the point M, that falls within the body, makes with lines drawn through this point, parallel to the axes of $x$ and $y$, the components of the resistance in the direction of these lines will be $\text{R}\cos\mu d\sigma$ and $\text{R}\cos\nu d\sigma$; and if $x'$ and $y'$ be the values of $x$ and $y$ at the point M, then

$$x'\text{R}\cos\nu d\sigma - y'\text{R}\cos\mu d\sigma,$$

will be the part of the second member of equation (3), that is relative to the element $d\sigma$; consequently, if the integral of this quantity for the entire portion of the surface of the body, which experiences the resistance of the medium, be taken, the quantity that should be added to the second member, when this resistance is taken into account, will be obtained. Let, in order to abridge,

$$x'\cos\nu - y'\cos\mu = \zeta,$$

and the expression of this quantity will be

$$\text{A}\omega^a\int\zeta\rho^a\cos^a\epsilon d\sigma + \text{A}'\omega^{a'}\int\zeta\rho^{a'}\cos^{a'}\epsilon d\sigma + \&\text{c}.$$

It is evident that $\zeta$ is the shortest distance between the

axis of suspension and the direction of $\rho\omega$ the velocity of the point M, comprised in a plane perpendicular to this axis; therefore, $\zeta$ does not depend on the time, no more than the angle $\varepsilon$ or the radius $\rho$; consequently, if we make

$$\int \zeta \rho^{\alpha} \cos^{\alpha} \varepsilon d\sigma = \gamma, \quad \int \zeta \rho^{\alpha'} \cos^{\alpha'} \varepsilon d\sigma = \gamma', \ \&c.,$$

these integrals will be constants depending on the form of the body, their values may however be different in two successive oscillations. In order to determine their limits, let there be circumscribed about the body, in its position of equilibrium, a cylinder perpendicular to the vertical plane passing through the fixed axis; the curve of contact of this cylinder with the surface of the body will divide this surface into two parts, one of which will experience the resistance of the air, while the body moves in one direction, and the other, while it moves in the opposite direction; these integrals must, therefore, be extended to one of these two parts for one entire oscillation, and to the other part for the following; and when these two parts are different, the values of $\gamma, \gamma', \gamma''$, &c., will be so also in two consecutive oscillations. When the body performs an entire revolution about the axis these values will not change. This being established, after having added the preceding quantity to the second member of equation (3), or what comes to the same thing, after having subtracted this quantity divided by M $(a^2 + k^2)$, the moment of inertia, from the value of $\frac{d^2\theta}{dt^2}$ given in No. 394, we shall have

$$\frac{d^2\theta}{dt^2} = -\frac{ga \sin\theta}{a^2 + k^2} - \frac{A\gamma}{M(a^2 + k^2)}\omega^{\alpha} - \frac{A'\gamma'}{M(a^2 + k^2)}\omega^{\alpha'} - \&c.,$$

for the equation of the motion of any pendulum whatever in a resisting medium. In like manner, we shall have

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta - B\omega^{\alpha} - B'\omega^{\alpha'} - \&c.,$$

... of the simple pendulum, the

length of which is $l$; B, B′, B″, &c., denoting constant coefficients.

The initial velocity and positions of the two bodies being supposed to be the same, in order that their motions may be so also, it will be sufficient and it is necessary to assume

$$\frac{g}{l} = \frac{ga}{a^2 + k^2}, \quad \text{B} = \frac{\text{A}\gamma}{\text{M}(a^2 + k^2)}, \quad \text{B}' = \frac{\text{A}'\gamma'}{\text{M}(a^2 + k^2)}, \text{ \&c.};$$

by means of which the value of $l$ can be determined, (which will be the same as in formula (c),) and likewise those of B, B′, B″, &c., for the entire duration of each oscillation.

Thus, whatever be the form of a pendulum, and the law of the resistance of the medium in which it moves, it appears, that there is always a simple pendulum, the motion of which is the same as that of the given pendulum; likewise, that the resistance of the medium in which the simple pendulum should move, may be deduced from that of the given medium, and from the form of the compound pendulum; and, finally, that the length of the simple pendulum depends altogether on this *form*, and not at all on the *resistance*.

However, it does not follow that the length of this pendulum, which is isochronous with the given pendulum, is the same in the medium and in a vacuo; the loss of weight that the compound pendulum sustains in the medium, and which is not the same in a state of motion and repose, influences the length of the isochronous simple pendulum that vibrates in a vacuum, as has been already stated (No. 191).

400. In order to determine the motion of the axle in the wheel, and of its two weights, suspended the one to the wheel and the other to the cylinder, the sum of the forces lost by all the points of the machine should be taken as in No. 391, then to this sum there should be added the moments of the forces lost in the same instant by these two weights; and the sum total of all these moments should be put equal to cipher. Now, if a chord be wrapped round the wheel, and attached

at one extremity to a point of its circumference, and if $m$ be the mass of the body which is vertically suspended at the other extremity, likewise, if $m'$ be the mass of the body which is vertically suspended to the extremity of a second cord wrapped round the cylinder and attached to its surface at its other extremity, and if $u$ and $u'$ are distances of the centres of gravity of $m$ and $m'$ from the horizontal plane passing through the axis of the machine, at the end of the time $t$, the forces lost by these masses during the instant $dt$, will be $m\left(g-\frac{d^2u}{dt^2}\right)$, $m'\left(g-\frac{d^2u'}{dt^2}\right)$: and their moments with respect to the axis of the machine will be obtained by multiplying the first by the radius of the wheel, which we shall denote by $c$, and the second by the radius of the cylinder, which we shall denote by $c'$; now, since these forces tend to make the parts of the machine to turn in opposite directions, the moment of one of them must be affected with the sign $+$, and that of the other with the sign $-$. For greater clearness, we shall suppose that the first force tends to make the machine to turn in the direction in which it actually turns, or, in other words, that it is the mass $m$ which descends and the mass $m'$ which ascends; it follows from this that the second member of equation (3), will be increased by $gmc-gm'c'$, and its first member by $m\frac{d^2u}{dt^2}c-m'\frac{d^2u'}{dt^2}c'$; moreover, the integral which contains the second member will be cipher, since it must extend to all the points of the machine, whose centre of gravity exists on the axis of rotation; we shall, therefore, have

$$\frac{d\omega}{dt}\int r^2dm+mc\frac{d^2u}{dt^2}-m'c'\frac{d^2u'}{dt^2}=g\,(mc-m'c');$$

for the equation of the motion of the machine and of the two masses $m$ and $m'$.

During the entire continuance of this motion, $\frac{du}{dt}$ the ve-

locity of $m$ is equal to $c\omega$ the velocity of the point of the wheel where the cord commences to detach itself from its circumference, the radius of which point is horizontal; in the same manner, $\frac{du'}{dt}$ the velocity of $m'$ is equal and contrary to $c'\omega$, the velocity of the point of the surface of the cylinder, whose radius is also horizontal and situated on the other side of the axis, hence we have constantly,

$$\frac{du}{dt} = c\omega, \quad \frac{du'}{dt} = -c'\omega;$$

by means of which, the preceding equation becomes($i$)

$$(\text{M}k^2 + mc^2 + m'c'^2)\frac{d\omega}{dt} = g\,(mc - m'c'),$$

in which equation M denotes the mass of the machine, and $\text{M}k^2$ its moment of inertia with respect to the axis of rotation. If, for greater simplicity, the initial velocities of the machine, and of the masses $m$ and $m'$ be supposed to be cipher, we shall have, at any instant whatever,

$$\omega = \frac{(mc - m'c')\,gt}{\text{M}k^2 + mc^2 + m'c'^2};$$

from which it appears at once, that the motion of the machine is uniformly accelerated.

The tensions of the cords to which the masses $m$ and $m'$ are attached, are measured by the forces lost by these masses; therefore, if they be denoted by T and T$'$, we shall have

$$\text{T} = m\left(g - \frac{d^2u}{dt^2}\right), \quad \text{T}' = m'\left(g - \frac{d^2u'}{dt^2}\right),$$

and, if the weights of these bodies, and of the machine, be denoted by $p, p'$, P, respectively, so that

$$p = mg, \quad p' = m'g, \quad \text{P} = \text{M}g,$$

it follows from the preceding equations, that

$$T = p - \frac{(pc - p'c')pc}{Pk^2 + pc^2 + p'c'^2}, \quad T' = p' + \frac{(pc - p'c')p'c'}{Pk^2 + pc^2 + p'c'^2}.$$

Since by supposition the weight $p$ descends, $pc$ is greater than $p'c'$, consequently, the tension $T$ is less than this weight, which will be its value in the state of equilibrium, and for the same reason, the tension $T'$ is greater than $p'$.

The pressures exerted on the axis of the machine by the centrifugal forces of its different points, are evidently destroyed two by two, in consequence of the symmetrical arrangement of the parts of the machine about this axis. Therefore, the weight of the machine and the tensions $T$ and $T'$ make up the entire pressure, which the axis sustains during the motion, so that if this vertical force be denoted by $\Pi$, we shall have

$$\Pi = P + T + T',$$

and by substituting for $T$ and $T'$ their values, there will result

$$\Pi = P + p + p' - \frac{(pc - p'c')^2}{Pk^2 + pc^2 + p'c'^2};$$

from which it appears, that this pressure is always less than in the state of equilibrium, in which case it is equal to $P + p + p'$.

401. If we make $c = c'$, in these different formulæ, we shall have the case of the machine invented by Atwood, and thus by means of them all the circumstances of the motion of two unequal weights $p$ and $p'$, one of which ascends and the other descends, will be known.

If, for example, $h$ be the height through which the weight $p$ descends in a given time, such as $\theta$, the value of $h$ will be obtained by integrating that of $du$ or of $c\omega dt$, from $t = 0$ to $t = \theta$; which gives

$$h = \frac{\frac{1}{2}(p - p')c^2 g\theta^2}{Pk^2 + (p + p')c^2}.$$

In this expression, the weights P, $p$, $p'$, and also the radiu

of the wheel are given; the quantity $k^2$ can be calculated from knowing the form of the wheel; and the height $h$ can be measured; consequently, if the time $\theta$ is given by observation, this formula will make known the value of $g$. But, however carefully this experiment is made, it can never be susceptible of the same degree of precision as observations on the pendulum, for in this last, the duration of each oscillation is obtained, by dividing the time during which the pendulum oscillates, by the number of oscillations that it makes, and as this number is very great, the error to be apprehended, in the duration of one sole oscillation, must be much less than the inevitable error which is committed in the measurement of the time $\theta$, in the machine of Atwood.

The mass of the thread to which the two weights $p$ and $p'$ are suspended, is not considered in the preceding expression, however it would be easy to take it into account, in the same manner as in the problem of No. 356; but then the law of the motion would be extremely complicated. The resistance which the air opposes to the motions of the two weights $p$ and $p'$ is also neglected; in order to diminish, as much as possible, this effect, and also that we may be enabled to measure the time $\theta$ with greater facility, these motions are rendered very slow, by diminishing the excess of one of these weights above the other.

402. The pendulum has also been employed to determine the velocity of projectiles in gunnery. This machine is called the *pendulum* of *Robins*, from the name of the engineer who first invented it; it consists of a very considerable mass, that vibrates about a fixed horizontal axis firmly fixed. The ball, whose velocity is required to be known, penetrates into this mass without passing through it, and by this means causes the pendulum to vibrate; the magnitude of the arc which a determinate point of the total mass describes is then measured; from which its quantity of motion may be easily obtained,

and, consequently, the velocity of the bullet at the instant it strikes the pendulum.

In fact, let AEBF (fig.7) be a section of the pendulum made by a plane passing through the centre of gravity, and perpendicular to the fixed axis, $G$ its centre of gravity, O the centre of oscillation, (No. 396), C the point where this section cuts the axis, so that, in the state of equilibrium, CGO is a vertical line. Likewise, let B be the point where the production of this line meets the inferior surface of the pendulum, BB′ the arc of the circle that is described by this point B, and of which C is the centre, E the centre of the circular aperture that the ball makes at the surface of the pendulum, or, more generally, the projection of this point on the plane of the section AEBF. Let $\mu$ denote the mass of the ball, $v$ its velocity, at the instant of the impact, $f$ the perpendicular let fall from the point C on the direction of $v$ projected on the plane of AEBF, M the mass of the pendulum and of the ball, $a$ the distance of the centre of gravity of M from the fixed axis, $M(a^2+k^2)$ its moment of inertia with respect to this axis, we shall have (No. 386)

$$\Omega = \frac{\mu v f}{M(a^2+k^2)},$$

for the value of $\Omega$ that should be substituted in equation (a) of No. 394, in which we should likewise make $\alpha = 0$, since the pendulum sets out from its position of equilibrium. It will continue to deviate from this position until the angular velocity is cipher, consequently, if the angle BCB′ be denoted by $\beta$, we shall have, by this equation (a),

$$\frac{\mu^2 v^2 f^2}{M^2(a^2+k^2)} = 2ga(1-\cos\beta);$$

in which $g$ denotes the measure of the force of gravity. If the cord of the arc BB′ be denoted by $b$, and the radius CB by $c$, we shall have ($k$)

$$\cos\beta = 1 - \frac{b^2}{2c^2}.$$

By substituting this value in the preceding equation, there results

$$v^2 = \frac{n^2gab^2}{f^2c^2}(a^2 + k^2);$$

in this equation $n$ denotes the ratio of M to $\mu$, which will be always a very great given number.

All the other quantities contained in this formula will be also known. The distance $c$ can be immediately measured, the cord $b$ is given by means of a riband attached to the point B, and passing through a ring firmly fixed in the ground, the part of this riband that is unrolled, and traverses the ring during the ascent of the pendulum, is evidently equal to $b$. If the direction is horizontal, the quantity $f$ is the distance of the axis of the piece from the axis of rotation; if the direction deviates a little from horizontality, in which case, the line drawn from the point E to the mouth of the cannon is no longer horizontal, it is easy to compute with sufficient accuracy, the quantity which should be added or subtracted from the distance of the two axes, in order to obtain the value of $f$. With respect to the quantities $a$ and $k$, they may be computed from knowing the form of the pendulum and the density of its parts; but their values may be determined likewise by experiment.

If a cord attached to the inferior extremity of the pendulum be made to pass over a fixed bar, situated parallel to the axis of the pendulum, and at the same distance from the horizon, and if a weight M′ attached to the other end of the cord raise the pendulum until its centre of gravity is on the same level as the axis and the bar, then, $a'$ being the given distance of the bar from the axis, we shall have

$$a : a' :: \text{M}' : \text{M},$$

by means of which proportion, the value of $a$ can be accurately determined. If the pendulum be made to perform very small oscillations, and if T denote the duration of one such oscilla-

tion, and $l$ the distance of the centre of oscillation from the axis of suspension, we shall have (No. 395)

$$\mathrm{T} = \pi\sqrt{\frac{l}{g}}, \quad l = \frac{a^2 + k^2}{a},$$

hence we can deduce

$$a^2 + k^2 = \frac{g\mathrm{T}^2 a}{\pi^2};$$

by means of which, when the value of $a$ is known, that of $k$ can be obtained.

By substituting this value of $a^2 + k^2$, a simpler expression will be obtained for $v$, the velocity of the ball, than that already given, namely,

$$v = \frac{ng\mathrm{T}ab}{\pi fc}. \qquad \text{(a)}$$

403. If the mouth of the cannon is not very far from the pendulum, the value of $v$ given by this formula will differ very little from the velocity of *projection* of the ball; and if the coefficient of the resistance of the air was supposed to be known, it would be easy to calculate by means of formula (5) of No. 212, the quantity by which this quantity $v$ should be increased, in order to obtain the velocity of projection. But, the magnitude of this last velocity can be obtained immediately, by attaching the cannon firmly to the pendulum; the quantity of motion impressed on the pendulum, thus constituted, will be then equal to the mass of the ball multiplied by the velocity which it has at the mouth of the cannon, the recoil of which will not, in consequence of the compressibility of the matter, sensibly commence before the projectile has traversed the length of the piece; consequently, the value of $v$, furnished by formula (a), will be that of the velocity of projection without any correction, and without the necessity of knowing the coefficient of the resistance($l$).

By firing the same cannon, loaded in the same manner, at different given distances from the pendulum, so many values

of $v$ will be obtained, the differences between which and that which results when the cannon makes a part of the pendulum, will enable us to verify the law of the resistance of the air, on which formula (5) of No. 212 is founded, and also to determine the coefficient of this resistance.

A great number of experiments were made in England, with Robin's pendulum, in which the two methods pointed out above were employed. One of the most general consequences which has been deduced from them consists in this, that every thing else being the same, the squares of the velocities of projection are very nearly as the weights of the charges, and this ratio is so much the more accurate, according as the length of the charge is less considerable relative to that of the cannon($m$).

## CHAPTER IV.

### OF THE MOTION OF A SOLID BODY ABOUT A FIXED POINT.

### I. *Preliminary Formulæ.*

404. LET us, in the first place, consider by itself, and independently of the forces which produce it, the motion of rotation of a solid body of any figure whatever, about a fixed point which either appertains to this body, or is invariably attached to it.

Let O (fig. 3) be this point; O$x$, O$y$, O$z$, three *fixed* rectangular axes arbitrarily selected; O$x_1$, O$y_1$, O$z_1$ three other rectangular axes, fixed in the body, and *moveable* with it about the point O. In the sequel, we shall suppose that these last lines are the principal axes of the body; but for the present, we shall consider their directions as entirely arbitrary. Likewise, let $x$, $y$, $z$ be the coordinates of M any point whatever of the body referred to the first axes, and $x_1$, $y_1$, $z_1$ its coordinates referred to the axes O$x_1$, O$y_1$, O$z_1$. If the notations of No. 377 be retained, we shall have

$$x = ax_1 + by_1 + cz_1,$$

$$y = a'x_1 + b'y_1 + c'z_1,$$

$$z = a''x_1 + b''y_1 + c''z_1;$$

and the nine coefficients $a$, $b$, &c., will be connected together by equations (2), or by equations (4), of this number.

It is evident that these quantities $a$, $b$, &c., are the same, at each instant, for all the points of the body($a$); but they vary during the motion, so that they must be considered as functions of the time. On the contrary, the coordinates $x_1$, $y_1$, $z_1$

vary from one point to another of the body; but remain constantly the same for the same point, and do not vary with the time. Therefore, if the time be denoted by $t$, we shall obtain, by differentiating with respect to this variable,

$$\frac{dx}{dt} = x_1 \frac{da}{dt} + y_1 \frac{db}{dt} + z_1 \frac{dc}{dt},$$

$$\frac{dy}{dt} = x_1 \frac{da'}{dt} + y_1 \frac{db'}{dt} + z_1 \frac{dc'}{dt},$$

$$\frac{dz}{dt} = x_1 \frac{da''}{dt} + y_1 \frac{db''}{dt} + z_1 \frac{dc''}{dt}.$$

These values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ will express, at any instant whatever, the components of the velocity of the point M resolved parallel to the axes O$x$, O$y$, O$z$. If, therefore, it was required to know those points of the body whose velocity vanishes at this instant, they will be determined by equalling these quantities to cipher, from which there results

$$\left.\begin{aligned} x_1 da + y_1 db + z_1 dc &= 0, \\ x_1 da' + y_1 db' + z_1 dc' &= 0, \\ x_1 da'' + y_1 db'' + z_1 dc'' &= 0. \end{aligned}\right\} \quad (1)$$

Now, by adding these equations together, after having multiplied them by $c$, $c'$, $c''$, respectively, then making, in order to abridge,

$$cdb + c'db' + c''db'' = pdt, \quad cda + c'da' + c''da'' = -qdt,$$

and observing that the equation $c^2 + c'^2 + c''^2 = 1$, gives $cdc + c'dc' + c''dc'' = 0$, there results

$$py_1 - qx_1 = 0.$$

If the same equations (1) be added, after being multiplied by $b$, $b'$, $b''$, we obtain

$$rx_1 - pz_1 = 0_1,$$

by making, in order to abridge,

$$bda + b'da' + b''da'' = rdt,$$

and observing that the equation $b^2 + b'^2 + b''^2 = 1$, gives

$$bdb + b'db' + b''db'' = 0;$$

and that, in consequence of the equation $bc + b'c' + b''c'' = 0$ of No. 377, we have also

$$bdc + b'dc' + b''dc'' = -cdb - c'db' - c''db'' = -pdt.$$

Finally, if equations (1) are multiplied by $a$, $a'$, $a''$, respectively, and then added together, there will result

$$qz_1 - ry_1 = 0;$$

for since $a^2 + a'^2 + a''^2 = 1$, $ada + a'da' + a''da'' = 0$; and, moreover, from equations

$$ba + b'a' + b''a'' = 0, \text{ and } ca + c'a' + c''a'' = 0,$$

of the number cited above, there results

$$adb + a'db' + a''db'' = -bda - b'da' - b''da'' = -rdt,$$
$$adc + a'dc' + a''dc'' = -cda - c'da' - c''da'' = qdt.$$

In this manner, in place of equations (1), we shall have

$$py_1 - qx_1 = 0, \quad rx_1 - pz_1 = 0, \quad qz_1 - ry_1 = 0. \qquad (2)$$

Each of these equations results from the two others; and they appertain to a right line passing through o, the origin of the coordinates. (It follows therefore from this analysis, that all the points of the body, whose velocity is cipher at any instant whatever, exist on a right line, passing through the centre of rotation) This line may be considered as immoveable during an infinitely short space of time, therefore, during this instant, the body turns about this line as about a fixed axis; and the motion of rotation of a solid body about a fixed point, may be represented, as having place at each instant about an axis which remains immoveable during an infinitely short interval of time. In general, the position of this axis changes

from one instant to another, during the motion; and, for this reason, it is termed *the instantaneous axis of rotation.*

405. Let us suppose that the line IOI′ is this axis at the end of the time $t$; equations (2) will be those of its projections on the three planes of the coordinates $x_1$, $y_1$, $z_1$, hence it is easy to infer ($b$)

$$\left.\begin{aligned} \cos \text{IO}x_1 &= \frac{p}{\sqrt{p^2+q^2+r^2}}, \\ \cos \text{IO}y_1 &= \frac{q}{\sqrt{p^2+q^2+r^2}}, \\ \cos \text{IO}z_1 &= \frac{r}{\sqrt{p^2+q^2+r^2}}. \end{aligned}\right\} \quad (3$$

When, therefore, the three quantities $p, q, r$, are known, the position of the instantaneous axis with respect to the *moveable* axes $\text{O}x_1$, $\text{O}y_1$, $\text{O}z_1$, can be assigned, and when the sign of the radical is also given, OI, the part of this line to which these formulæ appertain, will be completely determined; henceforth, we shall always consider this radical to be positive.

Whenever the quantities $p, q, r$, are constant, the axis of rotation will continue fixed in the body, that is to say, it will constantly traverse it in the same points. Now, as the points of the body, whose velocity is cipher at each instant, are always the same, they will remain immoveable during the entire continuance of the motion, consequently, in this case, the axis of rotation will be also a right line fixed in space.

It follows from equations (2) of No. 9, and the notations adopted in No. 377, that

$$\begin{aligned} \cos \text{IO}x &= a \cos \text{IO}x_1 + b \cos \text{IO}y_1 + c \cos \text{IO}z_1, \\ \cos \text{IO}y &= a' \cos \text{IO}x_1 + b' \cos \text{IO}y_1 + c' \cos \text{IO}z_1, \\ \cos \text{IO}z &= a'' \cos \text{IO}x_1 + b'' \cos \text{IO}y_1 + c'' \cos \text{IO}z_1; \end{aligned}$$

therefore, in virtue of equations (3), we shall have

$$\left.\begin{aligned}\cos \text{IO}x &= \frac{ap+bq+cr}{\sqrt{p^2+q^2+r^2}},\\ \cos \text{IO}y &= \frac{a'p+b'q+c'r}{\sqrt{p^2+q^2+r^2}},\\ \cos \text{IO}z &= \frac{a''p+b''q+c''r}{\sqrt{p^2+q^2+r^2}};\end{aligned}\right\} \quad (4)$$

by means of which, the instantaneous axis of rotation can be determined, relatively to the *fixed* axes O$x$, O$y$, O$z$. It appears from these equations, that when $p, q, r$, are constant quantities, the numerators of these formulæ are independent of $t$; which will, in fact, be verified in the sequel($c$).

406. Since at each instant the motion takes place about the line IOI′ as about a fixed axis, it follows, that during an infinitely short time, all the points of the body have the same angular velocity about this axis (No. 384). In order to determine its value, let us consider the point of the axis O$z_1$, whose distance from the point O is equal to unity, we shall have relatively to this point $x_1 = 0$, $y_1 = 0$, $z_1 = 1$, consequently, its absolute velocity will be

$$\sqrt{\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}} = \sqrt{\frac{dc^2+dc'^2+dc''^2}{dt^2}},$$

as is evident from the values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, given above, and because $x_1 = 0$, $y_1 = 0$, $z_1 = 1$; now as the distance of this point from the axis of rotation is

$$\sin \text{IO}z_1 = \sqrt{1-\cos^2 \text{IO}z_1} = \frac{\sqrt{p^2+q^2}}{\sqrt{p^2+q^2+r^2}};$$

if the absolute velocity be divided by this distance, we shall obtain

$$\frac{\sqrt{dc^2+dc'^2+dc''^2}}{\sqrt{(p^2+q^2)}\,dt}\sqrt{p^2+q^2+r^2}$$

for the expression of the angular velocity. But since

$$-pdt = bdc + b'dc' + b''dc'', \quad qdt = adc + a'dc' + a''dc'',$$

we can obtain, in consequence of the equations of No. 377 $(d)$

$$(p^2 + q^2)\,dt^2 = dc^2 + dc'^2 + dc''^2 - (cdc + c'dc' + c''dc'')^2;$$

which expression is reduced to $dc^2 + dc'^2 + dc''^2$, because $cdc + c'dc' + c''dc'' = 0$. Therefore, if the angular velocity at the end of the time $t$ be denoted by $\omega$, we shall have, by considering it as a positive quantity,

$$\omega = \sqrt{p^2 + q^2 + r^2}.$$

Hence it appears, that this velocity will be constant, whenever the position of the axis of rotation is invariable; but the converse of this proposition is not equally true; and it is possible that the instantaneous axis may change its position, without the value of the angular velocity undergoing any change, or, in other words, it is possible that the quantities $p, q, r$, may be variable, at the same time that the value of $\omega$ remains constant.

407. $p, q, r$, are termed the rectangular components of $\omega$ the velocity of rotation about the axes $ox_1$, $oy_1$, $oz_1$; and each of these three quantities is the angular velocity of the body about the corresponding axis.

Now, equations (3) can be replaced by

$$p = \omega \cos \mathrm{I}ox_1, \quad q = \omega \cos \mathrm{I}oy_1, \quad r = \omega \cos \mathrm{I}oz_1;$$

and equations (4) may be written in the following form,

$$\begin{aligned}\omega \cos \mathrm{I}ox &= ap + bq + cr,\\ \omega \cos \mathrm{I}oy &= a'p + b'q + c'r,\\ \omega \cos \mathrm{I}oz &= a''p + b''q + c''r;\end{aligned}$$

hence it appears, that the decomposition of the velocities of rotation are subject to the same laws as that of velocities of translation, the directions of these last being replaced by the directions of the axes of rotation.

As the resultant $\omega$ is a positive quantity, when a determinate part of the line IOI′, such as OI, is taken for the axis to which it is referred, the components $p$, $q$, $r$, whose axes are $\mathrm{o}x_1$, $\mathrm{o}y_1$, $\mathrm{o}z_1$, will be positive or negative, according as these lines make acute or obtuse angles with the axis OI; and, generally, the components of $\omega$ referred to the two parts of the same line, or of which the axes will be the production, the one of the other, should be regarded as equal, but affected with opposite signs.

408. The three quantities $p, q, r$, not only enable us to determine the angular velocity of the body, and the position of its axis of rotation with respect to the moveable axes $\mathrm{o}x_1$, $\mathrm{o}y_1$, $\mathrm{o}z_1$, but we can also express in terms of these quantities, the velocities and accelerating forces of its different points, decomposed in the directions of these three axes; this will enable us to find in the most direct manner, the equations of its motion of rotation, as we shall see very soon.

In fact, the components of the velocity of the point M, being $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, with respect to the fixed axes $\mathrm{o}x$, $\mathrm{o}y$, $\mathrm{o}z$, it follows that the components of the same velocity with respect to the axes $\mathrm{o}x_1$, $\mathrm{o}y_1$, $\mathrm{o}z_1$, will be

$$a\frac{dx}{dt} + a'\frac{dy}{dt} + a''\frac{dz}{dt},$$

$$b\frac{dx}{dt} + b'\frac{dy}{dt} + b''\frac{dz}{dt},$$

$$c\frac{dx}{dt} + c'\frac{dy}{dt} + c''\frac{dz}{dt};$$

as is evident from the notations of No. 377, and because the composition of velocities is subject to the same laws as that of forces. Now, if the values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, of No. 404, be substituted in these expressions, and if the reductions already indicated in this number, be performed, we shall find$(e)$

$$a\frac{dx}{dt}+a'\frac{dy}{dt}+a''\frac{dz}{dt}=qz_1-ry_1,$$

$$b\frac{dx}{dt}+b'\frac{dy}{dt}+b''\frac{dz}{dt}=rx_1-pz_1,$$

$$c\frac{dx}{dt}+c'\frac{dy}{dt}+c''\frac{dz}{dt}=py_1=qz_1;$$

consequently, the three quantities $qz_1-ry_1$, $rx_1-pz_1$, $py_1-qx_1$, which are cipher for all points of the body that are situated on the instantaneous axis of rotation, express for any other point M, the components of its velocity, parallel to the lines O$x_1$, O$y_1$, O$z_1$.

From these last equations we deduce, in consequence of those of No. 377 ($f$),

$$\frac{dx}{dt}=a\,(qz_1-ry_1)+b\,(rx_1-pz_1)+c\,(py_1-qx_1),$$

$$\frac{dy}{dt}=a'(qz_1-ry_1)+b'(rx_1-pz_1)+c'(py_1-qx_1),$$

$$\frac{dz}{dt}=a''(qz_1-ry_1)+b''(rx_1-pz_1)+c''(py_1-qx_1);$$

and by differentiating with respect to $t$, there results

$$\frac{d^2x}{dt^2}=a\,(z_1dq-y_1dr)+b\,(x_1dr-z_1dp)+c\,(y_1dp-x_1dq)$$
$$+(qz_1-ry_1)\,da+(rx_1-pz_1)\,db+(py_1-qx_1)\,dc,$$

$$\frac{d^2y}{dt^2}=a'(z_1dq-y_1dr)+b'(x_1dr-z_1dp)+c'(y_1dp-x_1dq)$$
$$+(qz_1-ry_1)\,da'+(rx_1-pz_1)\,db'+(py_1-qx_1)\,dc',$$

$$\frac{d^2z}{dt^2}=a''(z_1dq-y_1dr)+b''(x_1dr-z_1dp)+c''(y_1dp-x_1dq)$$
$$+(qz_1-ry_1)\,da''+(rx_1-pz_1)\,db''+(py_1-qx_1)\,dc''.$$

The quantities $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$, are the components of the accelerating force of the point M, resolved parallel to the fixed

axes o$x$, o$y$, o$z$; therefore, if $p_1$, $q_1$, $r_1$, be the components of the same force parallel to the axes o$x_1$, o$y_1$, o$z_1$, we shall have

$$p_1 = a\frac{d^2x}{dt^2} + a'\frac{d^2y}{dt^2} + a''\frac{d^2z}{dt^2},$$

$$q_1 = b\frac{d^2x}{dt^2} + b'\frac{d^2y}{dt^2} + b''\frac{d^2z}{dt^2},$$

$$r_1 = c\frac{d^2x}{dt^2} + c'\frac{d^2y}{dt^2} + c''\frac{dz}{dt^2}.$$

Now, if the preceding values of $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$, be substituted in these values of $p_1, q_1, r_1$, and if reductions similar to those of No. 404 be made, we shall find($g$)

$$p_1dt = z_1dq - y_1dr + (py_1 - qx_1)qdt + (pz_1 - rx_1)rdt,$$
$$q_1dt = x_1dr - z_1dp + (qz_1 - ry_1)rdt + (qx_1 - py_1)pdt,$$
$$r_1dt = y_1dp - x_1dq + (rx_1 - pz_1)pdt + (ry_1 - qz_1)qdt;$$

and dividing by $dt$, the values of $p_1$, $q_1$, $r_1$, expressed by means of the variables $p, q, r$, and of their differentials, will be obtained.

409. With respect to the quantities of motion, with which all the points of the body are actuated at any instant whatever, their moment relative to each of the three axes o$x_1$, o$y_1$, o$z_1$, may, according to the definition of No. 273, also be expressed by means of the quantities $p$, $q$, $r$.

In order to shew this, let $dm$ be the differential element of the mass of the body at the point M, whose coordinates are $x_1, y_1, z_1$; the components of its quantity of motion parallel to the axes o$x_1$, o$y_1$, o$z_1$, will be the products of the velocities $qz_1 - ry_1$, $rx_1 - pz_1$, $py_1 - qx_1$, multiplied by $dm$; therefore, if the moments with respect to the axes o$z_1$, o$y_1$, o$x_1$, of the quantities of motion of all the points of the body, be denoted by L, M, N, we shall have, by what has been established in No. 274,

$$\text{L} = \int [(rx_1 - pz_1)x_1 - (qz_1 - ry_1)y_1]\, dm,$$
$$\text{M} = \int [(qz_1 - ry_1)z_1 - (py_1 - qx_1)x_1]\, dm,$$
$$\text{N} = \int [(py_1 - qx_1)y_1 - (rx_1 - pz_1)z_1]\, dm;$$

in which expressions, the integrals are supposed to extend to the entire mass of the body. These values will be very much simplified, if $\text{O}x_1$, $\text{O}y_1$, $\text{O}z_1$, be the three principal axes of the body which intersect in the point O; for then the three integrals $\int x_1y_1dm$, $\int z_1x_1dm$, $\int y_1z_1dm$, will be cipher; and if the three moments of inertia with respect to these principal axes be denoted by A, B, C, so that

$$\int (y_1^2 + z_1^2)\, dm = \text{A},$$
$$\int (z_1^2 + x_1^2)\, dm = \text{B},$$
$$\int (x_1^2 + y_1^2)\, dm = \text{C},$$

we shall have simply

$$\text{L} = \text{C}r, \quad \text{M} = \text{B}q, \quad \text{N} = \text{A}p;$$

therefore, the quantities $p$, $q$, $r$, will have constantly the same signs as L, M, N; consequently, their signs will depend on the direction in which the body turns about each of the three principal axes; for example, according as the body turns parallel to the plane of $x_1\text{O}y_1$, from $\text{O}x$ towards $\text{O}y$, or in the opposite direction, the moment L (No. 274), and, consequently, the velocity $r$, will be positive or negative quantities, and, conversely, the sign of $r$ will make known at each instant, the direction of rotation about $\text{O}z_1$.

It appears from the theorems of No. 281, that if the principal moment of the quantities of motion which have been considered be denoted by G, we shall have

$$\text{G} = \sqrt{\text{A}^2p^2 + \text{B}^2q^2 + \text{C}^2r^2};$$

(this radical being assumed to be a positive quantity); if the right line $\text{O}m$ (fig. 8) be the axis of this motion, its direction with respect to the *moveable* axes $\text{O}x_1$, $\text{O}y_1$, $\text{O}z_1$, will be determined by the formulæ

$$\cos mox_1 = \frac{\text{A}p}{\text{G}},\ \cos moy_1 = \frac{\text{B}q}{\text{G}},\ \cos moz_1 = \frac{\text{C}r}{\text{G}}; \qquad (5)$$

and its direction with respect to the *fixed* axes o$x$, o$y$, o$z$, will be determined by means of the following equations:

$$\left.\begin{aligned} \text{G}\cos mox &= \text{A}pa + \text{B}qb + \text{C}rc, \\ \text{G}\cos moy &= \text{A}pa' + \text{B}qb' + \text{C}rc', \\ \text{G}\cos moz &= \text{A}pa'' + \text{B}qb'' + \text{C}rc''; \end{aligned}\right\} \qquad (6)$$

it is evident that the second members of these equations are the moments of the quantities of motion of the body with respect to the fixed axes o$x$, o$y$, o$z$.

410. The position of this body at each instant, with respect to the fixed axes, depends on the three angles $\psi$, $\theta$, $\phi$ of No. 378; for by means of these angles, the three sections of the body which have been taken for the moveable planes of the coordinates $x_1$, $y_1$, $z_1$, are determined in position with respect to these fixed planes; and it is even sufficient to know the position of two sections, which are not parallel, of a solid body, in order that the positions of all the points of this body may be entirely known. Moreover, when the angles $\psi$, $\theta$, $\phi$ are known, the coefficients $a$, $b$, &c., will be known also, and consequently, $x$, $y$, $z$, the coordinates of any point whatever of the body, will be completely determined. The problem of the motion of rotation about a fixed point will, therefore, eventually resolve itself into the determination of the values of $\psi$, $\theta$, $\phi$ in functions of the time.

Now, when the values of $p$, $q$, $r$ are known, those of these three angles depend on three equations of the first order, which will be obtained by substituting the values of $a$, $b$, &c. (No. 378), and those of their differentials, in functions of $\psi$, $\theta$, $\phi$, in the values of $pdt$, $qdt$, $rdt$, namely,

$$pdt = -bdc - b'dc' - b''dc'',$$

$$qdt = adc + a'dc' + a''dc'',$$

$$rdt = bda + b'da' + b''da''.$$

As the values of $c, c', c''$ do not contain the angle $\phi$, it follows that those of $pdt$ and $qdt$ will not contain its differential; and as the values of $b, b', b''$ may be deduced from those of $a, a', a''$, by increasing $\phi$ by a right angle, the value of $-pdt$ may likewise be deduced from that of $qdt$. The coefficient of $d\phi$ will be equal to unity in the value of $rdt$; for by the formulæ of No. 378, we have($h$)

$$\frac{da}{d\phi} = b, \quad \frac{da'}{d\phi} = b', \quad \frac{da''}{d\phi} = b'';$$

from which there results

$$b\frac{da}{d\phi} + b'\frac{da'}{d\phi} + b''\frac{da''}{d\phi} = b^2 + b'^2 + b''^2 = 1,$$

for the value of this coefficient. After all reductions, we obtain, by substituting the values of $a, b$, &c., in those of $pdt, qdt, rdt$ ($i$),

$$\left.\begin{aligned} pdt &= \sin\phi\sin\theta d\psi - \cos\phi d\theta, \\ qdt &= \cos\phi\sin\theta d\psi + \sin\phi d\theta, \\ rdt &= d\phi - \cos\theta d\psi. \end{aligned}\right\} \quad (7)$$

It is remarkable, that the angle $\psi$ does not occur in these formulæ; and, in fact, as the angle $\psi$, or NO$x$, is reckoned from the axis O$x$, which is entirely arbitrary, the values of $p, q, r$ should not undergo any change, when this angle is increased or diminished by a constant quantity.

Since $r$ is the angular velocity of the body about the axis O$z_1$, it follows that $rdt$ must be the angle described in the plane of the axes of $x_1$, $y_1$, during the instant $dt$, by each of the axes O$x_1$ and O$y_1$; this angle would be $d\phi$, if the line ON, from which the angle $\phi$ is reckoned in this same plane, was immoveable; but in the instant $dt$, the angle NO$x$ is increased by $d\psi$, the projection of which on the plane of the axes of $x_1$ and $y_1$ is $\cos\theta d\psi$, and it is easy to perceive, that, according as the angle $\theta$ is acute or obtuse, the differential $d\phi$ should be

increased or diminished by this projection, in order to have the displacement of $ox_1$ or $oy_1$, with respect to a fixed line, in the plane of these axes. Consequently, we shall have in all cases $rdt = d\phi - \cos\theta d\psi$, as has been obtained above.

411. There exists between the cosines $a$, $b$, &c., and the quantities $p$, $q$, $r$, relations which will be useful on several occasions to know; they are expressed by the following differential equations:

$$\left.\begin{aligned} dc &= (aq-bp)dt, & dc' &= (a'q-b'p)dt, & dc'' &= (a''q-b''p)dt, \\ db &= (cp-ar)dt, & db' &= (c'p-a'r)dt, & db'' &= (c''p-a''r)dt, \\ da &= (br-cq)dt, & da' &= (b'r-c'q)dt, & da'' &= (b''r-c''q)dt. \end{aligned}\right\} \quad (8)$$

If after having multiplied the equations

$$adc + a'dc' + a''dc'' = qdt,$$
$$bdc + b'dc' + b''dc'' = -pdt,$$
$$cdc + c'dc' + c''dc'' = 0,$$

either by $a, b, c$, or by $a', b', c'$, or by $a'', b'', c''$, respectively, they be added together, there will result, by taking into account what has been established in No. 377($k$), the three first of equations (8). The three next will be obtained, by operating in a similar manner on equations

$$cdb + c'db' + c''db'' = pdt,$$
$$adb + a'db' + a''db'' = -rdt,$$
$$bdb + b'db' + b''db'' = 0\ ;$$

and by operating in the same manner on equations

$$bda + b'da' + b''da'' = rdt,$$
$$cda + c'da' + c''da'' = -qdt,$$
$$ada + a'da' + a''da'' = 0,$$

we shall obtain the three last of equations (8).

The following equations also obtain, namely,

$$\begin{cases} pda + qdb + rdc = 0, \\ pda' + qdb' + rdc' = 0, \\ pda'' + qdb'' + rdc'' = 0; \end{cases}$$

these can be immediately deduced from equations (8), and they enable us to verify the invariability of the numerators of formulæ (4), when $p$, $q$, $r$ are constant quantities ($l$).

## II. *Equations of the Motion of Rotation about a fixed Point.*

412. The preceding preliminary formulæ being established, let us now suppose that any given motive forces act on all the points of the moveable, and taking these forces into account, let us investigate the differential equations of its motion about the fixed point O.

Let $X_1 dm$, $Y_1 dm$, $Z_1 dm$, be the three components parallel to the principal axes $Ox_1$, $Oy$, $Oz_1$ of the motive force of the element $dm$, at the end of $t$ any time whatever. If this material point was free, these forces would impress on it in the instant $dt$, in their respective directions, the velocities $X_1 dt$, $Y_1 dt$, $Z_1 dt$. The increments of velocity which it actually receives in these directions, are the quantities $p_1 dt$, $q_1 dt$, $r_1 dt$, of No. 408; consequently, the components of the force lost by the element $dm$, during the instant $dt$, are

$$(X_1 - p_1)dm, \quad (Y_1 - q_1)dm, \quad (Z_1 - r_1)dm.$$

The body will therefore be in equilibrio (No. 350), on the supposition that all its elements are solicited by similar forces. Now, the number of equations of equilibrium of a solid body, about a *fixed* point, is three (No. 266), which, relatively to these forces, will be

$$\int [(Y_1 - q_1)x_1 - (X_1 - p_1)y_1]\, dm = 0,$$
$$\int [(X_1 - p_1)z_1 - (Z_1 - r_1)x_1]\, dm = 0,$$
$$\int [(Z_1 - r_1)y_1 - (Y_1 - q_1)z_1]\, dm = 0;$$

in which the integrals are supposed to extend to the entire body.

The consideration of principal axes simplifies the terms which result from the substitution of the values of $p_1$, $q_1$, $r_1$, under the signs $\int$. For, as in this case, the integrals $\int x_1y_1dm$, $\int x_1z_1dm$, $\int y_1z_1dm$ are cipher, if the three principal moments of inertia be denoted by A, B, C, they represent the same integrals as in No. 409, and we have, consequently,

$$\int(x_1^2 - y_1^2)dm = \text{B} - \text{A},$$
$$\int(z_1^2 - x_1^2)dm = \text{A} - \text{C},$$
$$\int(y_1^2 - z_1^2)dm = \text{C} - \text{B};$$

so that the three preceding equations will become $(m)$

$$\left.\begin{array}{l} \text{C}dr + (\text{B}-\text{A})pqdt = \text{R}dt, \\ \text{B}dq + (\text{A}-\text{C})rpdt = \text{Q}dt, \\ \text{A}dp + (\text{C}-\text{B})qrdt = \text{P}dt, \end{array}\right\} \quad \text{(a)}$$

in which, in order to abridge, we make

$$\int(x_1\text{Y}_1 - y_1\text{X}_1)dm = \text{R},$$
$$\int(z_1\text{X}_1 - x_1\text{Z}_1)dm = \text{Q},$$
$$\int(y_1\text{Z}_1 - z_1\text{Y}_1)dm = \text{P}.$$

413. As $\text{X}_1$, $\text{Y}_1$, $\text{Z}_1$, are the components of the *given* forces acting in the direction of the moveable axes $\text{O}x_1$, $\text{O}y_1$, $\text{O}z_1$, their values will depend on the direction of these lines in space, or on the three angles $\psi$, $\theta$, $\phi$; hence the quantities P, Q, R will be functions of $\psi$, $\theta$, $\phi$, which will be given in each particular case; consequently, the problem of the motion of rotation of a solid body about a fixed point, leads to six differential equations of the first order, between the six unknown quantities $p$, $q$, $r$, $\psi$, $\theta$, $\phi$, and the variable $t$, namely, the three equations (a), joined to the three equations (7) of No. 410. If, in the first equations, namely (a), the three unknown quantities $p$, $q$, $r$ be eliminated by means of equa-

tions (7), three differential equations of the second order relative to $\psi$, $\theta$, $\phi$, will be obtained, which are the unknown quantities that are actually required to be determined; but it is more convenient in practice to retain the six equations of the first order.

The only case which we propose to consider will be that in which gravity is the sole force that acts on the points of the body. If in this case, the axis $oz$ be assumed to be vertical, and drawn in the direction of this constant force, which, as before, we shall denote by $g$; its three components in the direction of the axes $ox_1$, $oy_1$, $oz_1$, will be

$$X_1 = ga'', \quad Y_1 = gb'', \quad Z_1 = gc'',$$

because that by No. 377,

$$a'' = \cos zox_1, \quad b'' = \cos zoy_1, \quad c'' = \cos zoz_1;$$

and if the mass of the body be denoted by M, and the three constant coordinates of its centre of gravity, with respect to these moveable axes, by $\alpha$, $\beta$, $\gamma$, so that we may have

$$\int x_1 dm = M\alpha, \quad \int y_1 dm = M\beta, \quad \int z_1 dm = M\gamma;$$

there will result($n$)

$$\begin{aligned} R &= (\alpha b'' - \beta a'')M\gamma, \\ Q &= (\gamma a'' - \alpha c'')M\gamma, \\ P &= (\beta c'' - \gamma b'')M\gamma. \end{aligned}$$

Equations (a) will therefore become

$$\left.\begin{aligned} C dr + (B-A)pq dt &= (\alpha b'' - \beta a'')Mg dt, \\ B dq + (A-C)rp dt &= (\gamma a'' - \alpha c'')Mg dt, \\ A dp + (C-B)qr dt &= (\beta c'' - \gamma b'')Mg dt, \end{aligned}\right\} \quad (b)$$

to which must be joined equations (7), and the following (No. 378)

$$a'' = -\sin\theta\sin\phi, \quad b'' = -\sin\theta\cos\phi, \quad c'' = \cos\theta. \quad (c)$$

414. Equations (b) can be easily integrated, when their

second members are cipher; this is the case, when the force of gravity is not taken into account, or, which comes to the same thing, when the fixed point o is the centre of gravity of the body, in which case $\alpha = 0$, $\beta = 0$, $\gamma = 0$.

Equations (b) are then reduced to

$$\left.\begin{array}{l} \text{C}dr + (\text{B}-\text{A})pqdt = 0, \\ \text{B}dq + (\text{A}-\text{C})rpdt = 0, \\ \text{A}dp + (\text{C}-\text{B})qrdt = 0. \end{array}\right\} \quad (d)$$

Now, if these be multiplied by $r$, $q$, $p$ respectively, and then added together, there results

$$\text{C}rdr + \text{B}qdq + \text{A}pdp = 0,$$

and, by integrating, we obtain

$$\text{C}r^2 + \text{B}q^2 + \text{A}p^2 = h, \quad (e)$$

$h$ being an arbitrary constant. If after having multiplied the same equations by $\text{C}r$, $\text{B}q$, $\text{A}p$ respectively, they be added together, there results

$$\text{C}^2rdr + \text{B}^2qdq + \text{A}^2pdp = 0;$$

from which we obtain, by integrating,

$$\text{C}^2r^2 + \text{B}^2q^2 + \text{A}^2p^2 = k^2; \quad (f)$$

$k^2$ being a second arbitrary constant, which must be positive, as well as the preceding.

From equations (e) and (f) we can deduce

$$p^2 = \frac{k^2 - \text{B}h + (\text{B}-\text{C})\text{C}r^2}{(\text{A}-\text{B})\text{A}}, \quad q^2 = \frac{k^2 - \text{A}h + (\text{A}-\text{C})\text{C}r^2}{(\text{B}-\text{A})\text{B}}.$$

By substituting these values of $p$ and $q$ in the first equation (d), there results, by resolving it with respect to $dt$,

$$dt = \frac{\pm\sqrt{\text{AB}}\,\text{C}dr}{[k^2 - \text{B}h + (\text{B}-\text{C})\text{C}r^2]^{\frac{1}{2}} \times [\text{A}h - k^2 + (\text{C}-\text{A})\text{C}r^2]^{\frac{1}{2}}}. \quad (g)$$

In this expression we shall consider the denominator as always

positive, consequently, the numerator must be affected with the sign $+$ or $-$, according as the differential $dr$ is positive or negative, in order that as the time always goes on increasing, its differential may be always positive.

By integrating this formula (g), the value of $t$ will be obtained in a function of $r$, and, conversely, the value of $r$ in a function of $t$, therefore, the values of the three quantities $p, q, r$, may be assumed to be known in functions of this variable, or at least they will depend on only one integral, which may be always reduced to a case of elliptic functions.

When two of the three moments of inertia A, B, C, are equal, or when the constant $k^2$ is equal to one of the three quantities A$h$, B$h$, C$h$, this integal may be obtained in a finite form, without the aid of these functions($o$).

415. If the form of equations (d) be attentively considered, other equations, immediately integrable, may be deduced from them, by the aid of formulæ (8) of No. 411.

In fact, if equations (d) be multiplied by $c, b, a$, respectively, and then added together, there results

$$\begin{gathered}[cdr + (aq - bp)\,rdt]\,\mathrm{C} + [bdq + (cp - ar)\,qdt]\,\mathrm{B} \\ + [adp + (br - cq)\,pdt]\,\mathrm{A} = 0,\end{gathered}$$

or, in consequence of the three first formulæ (8)($p$),

$$\mathrm{C}d.cr + \mathrm{B}d.bq + \mathrm{A}d.ap = 0.$$

We shall find in like manner,

$$\begin{aligned}\mathrm{C}d.c'r + \mathrm{B}d.b'q + \mathrm{A}d.a'p = 0,\\ \mathrm{C}d.c''r + \mathrm{B}d.b''q + \mathrm{A}d.a''p = 0.\end{aligned}$$

Therefore, by integrating, we shall obtain

$$\left.\begin{aligned}\mathrm{C}rc + \mathrm{B}qb + \mathrm{A}pa = l,\\ \mathrm{C}rc' + \mathrm{B}qb' + \mathrm{A}pa' = l',\\ \mathrm{C}rc'' + \mathrm{B}qb'' + \mathrm{A}pa'' = l'',\end{aligned}\right\}\quad (h)$$

$l, l', l''$, being three arbitrary constants.

These three integrals are not independent of each other, for if their squares be added together, we shall obtain, in consequence of the equations of No. 377(q),

$$C^2r^2 + B^2q^2 + A^2p^2 = l^2 + l'^2 + l''^2;$$

from a comparison of this result with equation (f), it follows that there exists between the constants $k, l, l', l''$, the relation

$$l^2 + l'^2 + l''^2 = k^2.$$

If in these equations (h), there be substituted in place of $a, b$, &c., their values in functions of $\psi, \theta, \phi$, (No. 378), there will be obtained three equations between the six variables $\psi, \theta, \phi, p, q, r$, and the arbitrary constants $l, l', l''$, which must be the integrals of equations (7) of (No. 410); and this is, in fact, what may be easily verified. As these three integrals are only equivalent to two equations really distinct, it follows that there must be a third integral of equations (7); but previously to investigating it, it is necessary to examine what equations (h) signify.

416. Agreeably to what has been observed in No. 409, they indicate that the quantities of motion of all the points of the body, with respect to the fixed axes $ox, oy, oz$, are constant and equal to $l, l', l''$, during the continuance of the motion. If they be compared to formulæ (6) of this number, and if it be observed that in virtue of equation (f), the principal moment G is equal to the constant $k$ regarded as positive, we shall have

$$\cos mox = \frac{l}{k}, \quad \cos moy = \frac{l'}{k}, \quad \cos moz = \frac{l''}{k},$$

by means of which, the direction of $om$ the axis of this moment, which will remain immoveable, and also the plane perpendicular to this line, can be determined. The position of the axis $om$ with respect to the moveable axes $ox_1, oy_1, oz_1$, changes every instant, but it can be found at each instant, by means of formulæ (5) of 409, in which the quantities $p, q, r$, may be supposed to be known. Therefore, we can assign at

any instant whatever, the point where this line meets the surface of the body, and the line in which the moveable plane perpendicular to this axis intersects the surface.

Hence, when a solid body turns about a fixed point, in virtue of one or more *primitive* impulsions, if no motive force acts on its several points, there exists a plane passing through the fixed point, which remains invariable during the motion, and its position can be determined at each instant, with respect to the moveable planes of the principal axes of the body.

We will have occasion, in the sequel, to generalize this theorem; at present, we shall employ it in determining the third integral of equations (7).

417. As the axis o$m$ is immoveable, it may be taken for the fixed axis o$z$, the direction of which is arbitrary; we shall then have

$$\cos mox_1 = \cos zox_1 = a'',$$

$$\cos moy_1 = \cos zoy_1 = b'',$$

$$\cos moz_1 = \cos zoz_1 = c''.$$

Because $\mathrm{G} = k$, there will result in consequence of the formulæ of 409,

$$a'' = \frac{\mathrm{A}p}{k}, \quad b'' = \frac{\mathrm{B}q}{k}, \quad c'' = \frac{\mathrm{C}r}{k};$$

hence, equations (c) will become

$$\sin\theta\sin\phi = -\frac{\mathrm{A}p}{k}, \quad \sin\theta\cos\phi = -\frac{\mathrm{B}q}{k}, \quad \cos\theta = \frac{\mathrm{C}r}{k}; \quad \text{(i)}$$

by virtue of equation (f), they will agree together, and will enable us to determine the angles $\phi$ and $\theta$, in functions of the time, by means of the values of $p, q, r$.

Now, if between the two first equations (7) of (No. 410), $d\theta$ be eliminated, we shall obtain

$$\sin^2\theta d\psi = \sin\theta\sin\phi p dt + \sin\theta\cos\phi q dt,$$

hence there results, in virtue of the preceding equations($r$)

$$d\psi = -\frac{Ap^2 + Bq^2}{k^2 - C^2r^2}kdt;$$

therefore, in consequence of equation (e), we shall have,

$$d\psi = -\frac{h - Cr^2}{k^2 - C^2r^2}kdt; \qquad \text{(k)}$$

and by substituting formula (g) for $dt$, there will result a value of $d\psi$, the integration of which is also reducible to elliptic functions, and which can be obtained in a finite form in the same cases as the integral of $dt$. In this manner, therefore, the value of the third angle $\psi$ will be known in a function of $r$, and, consequently, in a function of $t$. As the quantities $h - Cr^2$, and $k^2 - C^2r^2$, are positive, in virtue of equations (e) and (f), and as $k$ is also a positive quantity, it follows that the angular velocity $\frac{d\psi}{dt}$ will be always negative and that the motion of the line ON will always take place in the same direction. Because the angle $\psi$ is measured in the direction indicated by the sagitta $s$ (No. 378), this motion will be performed in the contrary direction, that is to say, from the axis $ox$ towards the axis $oy$; hence then it appears, that its constant direction depends on that of the axis $oy$, which we shall determine immediately.

418. The values of the six variables $p$, $q$, $r$, $\psi$, $\theta$, $\phi$, resulting from our analysis, will be functions of the time, which will contain, besides, four arbitrary constants, namely $k$ and $h$, and the two constants introduced by the integration of formulæ (g) and (k). The complete integrals of equations (7) and (d), on which these values depend, ought to contain six arbitrary constants; but the selection which we have made, of $om$ the axis of the principal moment, for one of the axes of the coordinates $x$, $y$, $z$, has caused two of these constants to disappear; for as $om$ coincides with $oz$, the angles $mox$ and $moy$ are right, and it follows from the formulæ of No. 416, that in this case $l' = 0$ and $l'' = 0$. Therefore, in

order to effect the complete solution of the problem, it is only necessary to determine, by means of the initial data of the motion, the four remaining constants, and the parts of the lines passing through O, to which, during the continuance of the motion, the variable angles refer.

For this purpose, let the moveable whose rotatory motion is considered, be supposed to consist, as in No. 386, of two bodies, one of which is at rest, and retained by the fixed point O, and the other, being supposed to be actuated by a given velocity, impinges on the first, and remains attached to it after the impact. Let $\mu$ be the mass of the striking body, $v$ the velocity common to all its points before the impact, FE (fig. 8) the initial direction of its centre of gravity, HEF a section of the moveable made by the plane passing through the line FE and the point O, and $f$ the length of OL, a perpendicular let fall from this point on this line. The percussion which produces the motion of rotation, acts in the direction of FE, and is equal to $\mu v$. By the principle of No. 353, if the quantities of motion of all the points of the moveable, which have place immediately after the impact, be taken in a direction opposite to that in which the bodies actually move, there should be an equilibrium between these finite quantities of motion, and the force $\mu v$ estimated in its proper direction; now, in order that this equilibrium may obtain, it is necessary (No. 282) that $\mu f v$ the moment of this force, should be equal to the principal moment of these quantities of motion, and that the axes of these two moments should be the mutual production of one another. Since this principal moment, which has been denoted by G, is always equal to $h$ (No. 416), it is immediately evident that the value of this positive quantity $h$ is equal to $\mu v f$.

Moreover, if through the point O, there be drawn the axis of the moment $\mu v f$, perpendicular to EHK the given section of the moveable, this line will be likewise the axis of the principal moment, which has been assumed to be the axis O$z$; the directions O$x_1$, O$y_1$, O$z_1$, of the three principal axes of the move-

able, will be likewise given at the commencement of the motion, hence the angles which these lines make with $oz$ will be known; and, by the preceding number, we shall have

$$p = \frac{k \cos zox_1}{A}, \quad q = \frac{k \cos zoy_1}{B}, \quad r = \frac{k \cos zoz_1}{C}, \qquad (1)$$

for the initial values of $p$, $q$, $r$. By substituting them in equation (e), we shall have the value of the constant $h$. We are at liberty to take for the lines $ox_1$, $oy_1$, $oz_1$, such portions of the principal axes of the moveable that intersect in the point o, as we please; but after having once selected them, and fixed the points of the surface of the moveable where these portions terminate, they should not afterwards be changed during the motion.

The direction of the percussion made on the moveable, estimated along the line $KE$, will determine that of the rotation about each of the axes $ox_1$, $oy_1$, $oz_1$, at the commencement of the motion, and, consequently, the signs of the initial values of $p$, $q$, $r$ (No. 409). We shall therefore likewise know, by means of the preceding equations, whether the angles $zox_1$, $zoy_1$, $zoz_1$ are acute or obtuse; and it will be sufficient to have regard to one of these angles, whether greater or less than 90°, in order to know the part of the perpendicular to the plane of the section HEK, which should be taken for the axis $oz$ or $om$, and which will be, during the continuance of the motion, the axis of the principal moment of the quantities of motion of all the points of the moveable.

$NON'$ the intersection of the plane of the section HEK, and of the plane of the axes $ox_1$ and $oy_1$, will be likewise known at the commencement of the motion. In order to know ON, the part of this line to which the angles $\psi$ and $\phi$ constantly refer, it will therefore be sufficient to ascertain, if at this epoch, $\phi$ or $NOx_1$ is an acute angle, or an acute angle increased by 180°, and as

$$\cos zox_1 = -\sin\theta \sin\phi, \quad \cos zoy_1 = -\sin\theta \cos\phi,$$

it will be sufficient to consider the sign of one of these cosines, or the initial value of one of the quantities $p$ or $q$. The position of the fixed line O$x$ in the plane of the section HEK is entirely arbitrary. For greater simplicity, we shall suppose that it coincides with the initial position of ON. By making $\psi = 0$, in the values of $a'$, $b'$, $c'$ of No. 378, we shall have, at the commencement of the motion,

$$\cos y\text{o}x_1 = \cos\theta \sin\phi, \quad \cos y\text{o}y_1 = \cos\theta \cos\phi, \quad \cos y\text{o}z_1 = \sin\theta.$$

Therefore, if at the commencement of the motion, it be known whether the initial values of the angles $\theta$ and $\phi$, are acute or obtuse, it will be sufficient to consider the sign of $\cos y\text{o}x_1$ or $\cos y\text{o}y_1$, in order to know the part of the perpendicular to O$x$ or ON, which should be taken for the fixed axis O$y$, and, consequently, the direction of the velocity $\frac{d\psi}{dt}$, which has place always from ON towards O$y$, and remains unchanged during the continuance of the motion.

Moreover, every thing else being supposed to remain the same, if the direction of the primitive impact be the sole thing that is changed, the signs of the initial values of $p$, $q$, $r$ will all three be changed; if the primitive angles $\theta$ and $\phi$ were acute previous to this change, they will become $\pi - \theta$ and $\pi + \phi$; and the lines O$z$ and ON will be changed into their productions. By substituting $\pi - \theta$ and $\pi + \phi$ in place of $\theta$ and $\phi$ in the preceding equations, the initial values of the angles $y\text{o}x_1$, $y\text{o}y_1$, $y\text{o}z_1$, will undergo no change. The line O$y$ will therefore remain the same; but as the angular velocity $\frac{d\psi}{dt}$ is always negative, and directed from O$x$ towards O$y$, the direction of this velocity will change with that of the primitive percussion, because O$x$ now coincides with ON$'$.

Finally, the arbitrary constants which should be added to the integrals of formulæ (g) and (k), will be determined by making $t = 0$ and $\psi = 0$, at the commencement of the motion, that is to say, for the given initial value of $r$.

419. We now proceed to take notice of some general properties of the motion which has been determined.

1st. By the formulæ of No. 408, the expression for the square of the velocity of $dm$ the element of the moveable, will be

$$(qz_1 - ry_1)^2 + (rx_1 - pz_1)^2 + (py_1 - qz_1)^2.$$

If this quantity be multiplied by $dm$, the living force of this material point will be obtained (No. 361); and then, by integrating throughout the entire extent of the mass of the body, the sum of the living forces with which it is actuated at the end of the time $t$ will be determined. Now, if the terms multiplied by $\int x_1y_1dm$, $\int z_1x_1dm$, $\int y_1z_1dm$, be suppressed, because the coordinates $x_1$, $y_1$, $z_1$ are referred to principal axes, and if we take into account the values of the moments of inertia A, B, C, we obtain for this sum($s$)

$$\text{A}p^2 + \text{B}q^2 + \text{C}r^2;$$

hence it appears that, in virtue of equation (e), the sum of the living forces of all the points of the moveable, is constant during the continuance of the motion.

2nd. If $\theta$ denotes the angular velocity about the axis of the principal moment, which axis is supposed always to coincide with O$z$, this component of the velocity $\omega$ relative to the instantaneous axis, may be obtained from this last, by multiplying it by the cosine of the angle which the instantaneous axis makes with the axis O$z$; therefore by No. 407, we shall have

$$\theta = a''p + b''q + c''r;$$

and if there be substituted for $a''$, $b''$, $c''$, their values found in No. 417, we shall obtain($t$), by having regard to equation (e),

$$\theta = \frac{h}{k}.$$

Therefore the angular velocity of the moveable, resolved parallel to the plane in which the primitive percussion was

made, is constant, and equal to the sum of the living forces of all the points of the body, divided by the moment of this percussion with respect to the fixed centre.

3rd. If $x'$, $y'$, $z'$, be the coordinates of any point whatever of the instantaneous axis, referred to the axes $\text{o}x_1$, $\text{o}y_1$, $\text{o}z_1$, and $u$ the distance of this point from their origin o, then as $\frac{p}{\omega}, \frac{q}{\omega}, \frac{r}{\omega}$ are the cosines of the angles which these lines make with the instantaneous axis (No. 407), we shall have

$$x' = \frac{pu}{\omega}, \quad y' = \frac{qu}{\omega}, \quad z' = \frac{ru}{\omega};$$

if therefore equations (e) and (f) be multiplied by $\frac{u^2}{\omega^2}$, they will become

$$\text{A}x'^2 + \text{B}y'^2 + \text{C}z'^2 = \frac{hu^2}{\omega^2},$$

$$\text{A}^2x'^2 + \text{B}^2y'^2 + \text{C}^2z'^2 = \frac{k^2u^2}{\omega^2};$$

and by eliminating $\frac{u^2}{\omega^2}$ between these equations, we shall obtain

$$\text{A}(k^2 - \text{A}h)x'^2 + \text{B}(k^2 - \text{B}h)y'^2 + \text{C}(k^2 - \text{C}h)z'^2 = 0;$$

hence it follows that the *instantaneous* axis of rotation exists always on the surface of a cone of the second degree, which can be traced in the interior of the moveable, when the constants $h$ and $k$ are known. This cone is changed into a plane, when the square of $k$ is equal to one of the products $\text{A}h$, $\text{B}h$, $\text{C}h$; it becomes a right cone with a circular base, the axis of which is one of the three principal axes relative to this point, when two of the coefficients of the preceding equation are equal.

4th. $\text{o}m$ or $\text{o}z$, the axis of the principal moment of the quantities of motion, being immoveable, the series of lines along which it traverses the body during the motion, will exist on a cone whose summit is at the point o. Now this cone is of the second degree as well as the preceding. In fact, if

$x''$, $y''$, $z''$, be the three coordinates of any point whatever of the axis $om$ referred to the axes $ox_1$, $oy_1$, $oz'$, and if the distance of this point from the origin o, be denoted by $u_1$, we shall have

$$x'' = a''u_1, \quad y'' = b''u_1, \quad z'' = c''u_1,$$

and, consequently,

$$\text{A}p = \frac{hx''}{u_1}, \quad \text{B}q = \frac{ky''}{u_1}, \quad \text{C}r = \frac{kz''}{u_1}.$$

By substituting these values in equations (e) and (f), there results

$$\frac{k^2z''^2}{\text{C}} + \frac{k^2y''^2}{\text{B}} + \frac{k^2x''^2}{\text{A}} = hu_1^2,$$

$$z''^2 + y''^2 + x''^2 = u_1^2;$$

and by eliminating $u_1^2$, we obtain

$$\frac{(k^2 - \text{A}h)}{\text{A}}x''^2 + \frac{(k^2 - \text{B}h)}{\text{B}}y''^2 + \frac{(k^2 - \text{C}h)}{\text{C}}z''^2 = 0,$$

which is the equation of the surface of the cone in question.

5th. In order that this cone and the preceding be not imaginary, it is necessary that the three quantities $k^2 - \text{A}h$, $k^2 - \text{B}h$, $k^2 - \text{C}h$, should not be affected with the same sign. This being the case, if A be the greatest, and C the least of the three principal moments of inertia, the two quantities $k^2 - \text{A}h$, and $k^2 - \text{C}h$, must be of opposite signs. Then, according as the sign of the third quantity $k^2 - \text{B}h$ is the same as that of $k^2 - \text{A}h$ or $k^2 - \text{C}h$, the sections of these two cones will be ellipses perpendicular to the axis of the greatest or to the axis of the least moment of inertia. Consequently, during the continuance of the motion, the instantaneous axis of rotation will only deviate from one of these two principal axes by finite quantities, and, at the same time, this principal axis will not deviate except by finite quantities, from $om$ the axis which is perpendicular to the plane passing through the direction of the primitive percussion and the point o.

420. When OI, the instantaneous axis of rotation (fig. 3), deviates very little from one of the three principal axes, for example from the axis $oz_1$, during the *entire* continuance of the motion, its position and that of the moveable at any instant whatever, may be determined in a very simple manner, without having recourse to elliptic functions. Indeed, this other solution of the problem, which we now propose to give, is only an approximation, but it may be carried to any degree of accuracy we please; and we advert to it here particularly, as it enables us to complete what has been stated in No. 389, respecting the mechanical properties of principal axes.

We have (No. 406)

$$\sin \text{IO}z_1 = \frac{\sqrt{p^2+q^2}}{\sqrt{p^2+q^2+r^2}};$$

and since by hypothesis, the angle $\text{IO}z_1$ is very small, $p$ and $q$ will be small fractions of $r$; and if their product be neglected, the first of equations (d) becomes reduced to $dr = 0$, and gives, by integrating, $r = n$, $n$ being an arbitrary constant, which expresses the velocity of rotation of the body, or the value of $\sqrt{p^2+q^2+r^2}$, the squares of $p$ and $q$ being also neglected. The two other equations (d) will become

$$\left.\begin{aligned} \text{B}dq + (\text{A}-\text{C})\,npdt = 0, \\ \text{A}dp + (\text{C}-\text{B})\,nqdt = 0. \end{aligned}\right\} \quad (1)$$

In order to integrate them, let us assume

$$p = \beta \sin(n't+\gamma), \quad q = \beta' \cos(n't+\gamma),$$

$\beta, \beta', n', \gamma$, being constant quantities. If these values of $p$ and $q$ be substituted in equations (1), and if the sine or cosine which occurs as a common factor to all their terms, be suppressed, there results

$$\text{B}\beta'n' - (\text{A}-\text{C})\,\beta n = 0, \quad \text{A}\beta n' - (\text{B}-\text{C})\,\beta' n = 0;$$

hence we obtain($u$)

$$n' = n\sqrt{\frac{(\text{A}-\text{C})(\text{B}-\text{C})}{\text{AB}}},$$

$$\beta' = a\sqrt{\text{A}(\text{A}-\text{C})},$$

$$\beta = a\sqrt{\text{B}(\text{B}-\text{C})};$$

$a$ being a constant, which, as well as $\gamma$, remains arbitrary. If, therefore, in order to abridge, we make

$$\sqrt{\frac{(\text{A}-\text{C})(\text{B}-\text{C})}{\text{AB}}} = \delta,$$

there will result

$$\left.\begin{aligned} p &= a\sqrt{\text{B}(\text{B}-\text{C})}\sin(\delta nt+\gamma), \\ q &= a\sqrt{\text{A}(\text{A}-\text{C})}\cos(\delta nt+\gamma); \end{aligned}\right\} \quad (2)$$

these will be the complete integrals of equations (1).

If the instantaneous axis $\text{OI}$ be projected on the plane of the axes of $x_1$ and $y_1$, and if the angle which this projection makes with the axis of $y_1$ be denoted by $\zeta$, we shall have $(v)$

$$\tan\!\text{g}\,\zeta = \frac{q}{p};$$

moreover, the value of $\sin \text{IO}z_1$, when $p^2$ and $q^2$ are neglected, with respect to $r^2$, becomes reduced to

$$\sin \text{IO}z_1 = \frac{1}{n}\sqrt{p^2+q^2}.$$

Consequently, the preceding values of $p$ and $q$ will make known immediately, at each instant, the position of the axis of rotation in the interior of the moveable. The following consequences result from what has been just established.

421. If at the commencement of the motion, this line coincides exactly with the axis $\text{O}z_1$, then we must have $p=0$ and $q=0$, when $t=0$; in order that this may be the case, it is necessary that the constant $a$ should be cipher. We shall then *always* have $p=0$, $q=0$, and the instantaneous axis

OI will coincide, during the continuance of the motion, with the axis $oz_1$, which will remain immoveable (No. 405). When, therefore, a body retained by the fixed point O commences to turn about one of the three principal axes which intersect in this point, it will continue indefinitely to turn about this axis, as if it was entirely fixed; this property has been already established in No. 389($x$).

But, if at the commencement of the motion, the axis OI deviates ever so little from $oz_1$, the initial values of $p$ and $q$, and, consequently, the constant $a$, will be only very small. Now, in order that the values of $p$ and $q$ may always continue very small quantities, the constant $\delta$ must be real, for when it is imaginary, the sines and cosines contained in equations (2) become, by known formulæ, real exponentials, and the values of $p$ and $q$, which result from them, increase indefinitely with the time $t$ ($y$). The reality of $\delta$ requires that the principal moment C, should be the greatest or least of the three moments of inertia A, B, C. Therefore, when the instantaneous axis of rotation is made to deviate, ever so little, from the principal axis, which refers to the mean moment of inertia, this deviation increases with the time, and does not continue within very narrow limits; and, on the contrary, when it is caused to deviate ever so little from the principal axis to which the greatest or least moment of inertia refers, its elongation from this axis is always a very small quantity, so that it makes only very small excursions at each side of it, during the entire continuance of the motion.

There is, therefore, an essential difference between the three principal axes of the moveable which intersect at the fixed point O; if A be the greatest and C the least of the three quantities A, B, C, the motion of rotation will be stable about the axes $ox_1$ and $oz_1$, and only instantaneous about the axis $oy_1$. If, for example, the moveable be a homogeneous ellipsoid retained by its centre of figure, the motion of rotation is stable about the greatest or least of its three principal diameters, and instable about its mean diameter.

422. In the case of instable motion, formulæ (2) will only express the approximate values of $p$ and $q$ during the first instants of the motion, and while they are very small, as is implied in equations (1), from whence they are deduced. In order to obtain the values of $p, q, r$, at any instant whatever, it is then necessary to recur to the rigorous solution of the problem. In the case of stability, the approximate values of $p$ and $q$ furnished by equations (2), will subsist during the entire continuance of the motion; those of the three angles $\psi, \theta, \phi$, may be determined in the following manner.

We shall suppose as in No. 418, that the motion has been produced by the impact of a mass $\mu$, all whose points are actuated by a velocity $v$ parallel to the line FE passing through the centre of gravity of $\mu$, and comprised in the plane of the axes of $x$ and $y$. Equations (i) will constantly obtain, and if the distance of this line from the point O be denoted always by $f$, the quantity $k$ which occurs in them, will be still equal to $\mu v f$ the moment of the initial percussion. In consequence of $r = n$ and of formulæ (2), these equations($i$) will become($z$)

$$\left.\begin{aligned}\sin\theta\sin\phi &= -a\frac{\text{A}\sqrt{\text{B}(\text{B}-\text{C})}}{\mu v f}\sin(\delta n t+\gamma),\\ \sin\theta\cos\phi &= -a\frac{\text{B}\sqrt{\text{A}(\text{A}-\text{C})}}{\mu v f}\cos(\delta n t+\gamma),\\ \cos\theta &= \frac{\text{C}n}{\mu v f}.\end{aligned}\right\}\quad(3)$$

As the angles $\theta$ and $\phi$ are given at the commencement of the motion, the values of the two constants $a$ and $\gamma$ can be obtained by making $t = 0$ in the two first of these equations. Then, if the constant $n$ was determined, these two equations would make known the values of $\phi$ and $\theta$ at any instant whatever. It is necessary that $a$ should be a very small quantity, in order that the values of $p$ and $q$ furnished by equations (2),

may be very small, as has been supposed. This being the case, $\theta$ will be constantly a very small angle, and the principal axis $oz_1$, from which the instantaneous axis deviates very little, will itself deviate very little from the axis $oz$, which is perpendicular to FE the direction of the primitive percussion($a'$).

If the square of $\theta$ be neglected, the third equation (3), is reduced to($b'$)

$$\mu vf = cn;$$

by means of which the constant $n$ will be known, this is very nearly equal to the angular velocity of the moveable about the instantaneous axis.

In like manner, the third equation (7) of No. 410 will be reduced to

$$ndt = d\phi - d\psi;$$

from which we obtain

$$\psi = c + \phi - nt;$$

in which $c$ is an arbitrary constant that can be determined from knowing the initial values of $\phi$ and $\psi$. By means of this last equation, the value of the angle $\psi$ at any instant whatever can be known; and this completes the solution of the problem.

423. When the moveable is a solid of revolution, the axis of whose figure is $oz_1$, $B = A$; and the first equation (d) becomes $dr = 0$; $r$ is therefore equal to an arbitrary constant $n$; and all the formulæ of No. 420, as also equations (3), rigorously obtain.

It is then no longer necessary that the angle $\theta$ should be very small; but, in virtue of the third equation (3), its value is constant during the motion, so that the axis of figure of the moveable describes about $oz$, which is perpendicular to FE the direction of the primitive impact, a right cone with a circular base. If the constant and given value of this angle $\theta$ be denoted by $\varepsilon$, we shall have

$$\mu vf \cos\varepsilon = cn,$$

by means of which the constant $n$ can be determined. Likewise, from the two first equations (3), we can deduce($c'$)

$$\mu^2 v^2 f^2 \sin^2 \varepsilon = a^2 \mathrm{A}^3 (\mathrm{A} - \mathrm{C}),$$

which will enable us to determine the constant $a$; and in virtue of equations (2), $\omega$ the angular velocity about the instantaneous axis will be (No. 406)

$$\omega = \sqrt{n^2 + \frac{\mu^2 v^2 f^2}{\mathrm{A}^2} \cdot \sin^2 \varepsilon};$$

from which it appears, that this velocity will be constant, so that the moveable will revolve uniformly, as well about the instantaneous axis, in virtue of this velocity, as about its axis of figure, in virtue of the velocity $n$.

From the two first equations (3), we can also obtain

$$\operatorname{tang} \phi = \operatorname{tang} (\delta n t + \gamma), \quad \phi = \delta n t + \gamma.$$

The third equation (7) of No. 410 becomes

$$n dt = \delta n dt - \cos \varepsilon d\psi;$$

from which we obtain($d'$)

$$\psi = c - \frac{(1 - \delta)\, n t}{\cos \varepsilon} = c - \frac{\mu v f}{\mathrm{A}} t,$$

in which $c$ denotes an arbitrary constant introduced by the integration; consequently, the angles $\phi$ and $\psi$, the first measured on a plane perpendicular to the axis of figure, and the second measured on a plane passing through the primitive direction of the impact and the point O, vary uniformly.

424. As the stability of the motion about the principal axes of the greatest and least moments of inertia, has been inferred from equations (2), which, strictly speaking, are only approximations, some doubts may exist as to the accuracy of this conclusion; but the stability in question may be rigorously demonstrated by means of (e) and (f), the exact integrals of the equations of the motion.

In fact, if the first multiplied by C be taken from the second, we obtain

$$\text{A}(\text{A}-\text{C})p^2+\text{B}(\text{B}-\text{C})q^2=\text{D}; \tag{4}$$

in which, for conciseness, the constant $k^2 - \text{C}h$ is denoted by D. If, therefore the instantaneous axis deviates very little from the principal axis $oz_1$, at the commencement of the motion, so that the quantities may be very small at this epoch, the constant D will be likewise very small; hence it follows, that when the signs of the two differences A—C, B—C are the same, the values of $p$ and $q$ must remain very small, while the motion continues; for in virtue of equations (4), it is necessary that their squares, multiplied by quantities having the same sign, and then added together, should give a sum which is always a very small quantity. We can also in this case, assign limits to the values of $p$ and $q$, for it is evident that we shall always have

$$p^2 \angle \frac{\text{D}}{\text{A}(\text{A}-\text{C})}, \quad q^2 \angle \frac{\text{D}}{\text{B}(\text{B}-\text{C})}.$$

But if the differences (A—C) and (B—C) have contrary signs, then, though the constant D may still be supposed to be very small, it is easy to conceive that equation (4) may nevertheless be satisfied, without the necessity of supposing that the values of $p$ and $q$ continue always very small; and in fact, it appears from the analysis of No. 420, that then these values cannot be supposed to be very small during the entire continuance of the motion.

Finally, the principal axes relative to the fixed point O are the only ones which can remain the same in the interior of moveable, and continue at rest, when they are not entirely fixed, as has been already observed in No. 389. This may, in point of fact, be deduced from equations (d). For, in order that the position of the instantaneous axis of rotation may remain always the same, it is necessary that the three quantities $p, q, r$ should be constant. Therefore we have $dp=0$, $dq=$

$dr = 0$; by means of which, equations (d) become reduced to

$$(B - A)pq = 0, \quad (A - C)rp = 0, \quad (C - B)qr = 0.$$

If the three moments of inertia A, B, C are unequal, two of the three quantities $p$, $q$, $r$ must be equal to cipher, in order to satisfy these equations; and then the instantaneous axis will coincide with one of the three axes $ox_1$, $oy_1$, $oz_1$. If two of these three moments of inertia are equal, so that we may have $B = A$, for example, the first equation will disappear, and the two others will be satisfied by making $r = 0$. Consequently, the instantaneous axis will then be situated in the plane of the two axes $ox_1$ and $oy_1$; but we know that in such a case, all lines existing in this plane, and passing through the point O, are principal axes; therefore the immoveable axis of rotation will be still a principal axis. Finally, when $A = B = C$, these three equations are identical, and the values of $p$, $q$, $r$ may be arbitrarily selected; but, in this particular case, all lines which pass through the point O are principal axes; in all cases, therefore, the axis of rotation must, if it remains immoveable, be a principal axis.

### III. *Solution of a particular Case of the Motion of Rotation of a heavy Body.*

425. When the fixed point O is not the centre of gravity of the moveable, we have not hitherto been able, when the action of gravity is taken into account, to integrate the system of equations (7) and (b), except in the case in which the moveable is a solid of revolution, on whose axis the point O exists. It is this particular case which we now proceed to consider.

Let us suppose that the principal axis $oz_1$ is the axis of figure, and that, consequently, $B = A$. Likewise, let us suppose that G, the centre of gravity of the moveable (fig. 9), exists on the axis of the positive $z_1$s, in which case (No.

413), $\alpha = 0$, $\beta = 0$, and $\gamma$ is a given positive quantity, that denotes the distance OG. As the axis O$z$ is vertical, and drawn in the direction in which the force of gravity acts, the angle $\theta$ or $zoz_1$, will be acute or obtuse according as the point G exists below or above the horizontal plane drawn through the point O. In these two cases, equations (b) will become

$$\left.\begin{aligned} &\mathrm{C}dr = 0, \\ &\mathrm{A}dq - (\mathrm{C} - \mathrm{A})rpdt = \gamma a''\mathrm{M}gdt, \\ &\mathrm{A}dp + (\mathrm{C} - \mathrm{A})rqdt = -\gamma b''\mathrm{M}gdt, \end{aligned}\right\} \quad (1)$$

in which the quantities $a''$ and $b''$ are, by equations (c), equal to $-\sin\theta\sin\phi$, $-\sin\theta\cos\phi$, respectively. The section of the moveable perpendicular to its axis of figure, and passing through the point O, is termed its *equator*. Let NEN′E′ be this section, and NON′ the line in which it intersects the horizontal plane passing through this fixed point. As all lines passing through this point and comprised in this section, are principal axes, the angle $\phi$ may be referred to any one of them indifferently; and, E being a determined point of the moveable, the angle NOE may be assumed to be equal to $\phi$. The angle $\psi$, the differential of which occurs in the third equation (7), will be the angle NO$x$ reckoned from the fixed line O$x$, drawn arbitrarily in the horizontal plane. Therefore, at any instant whatever, we shall have

$$zoz_1 = \theta, \quad \mathrm{NOE} = \phi, \quad \mathrm{NO}x = \psi;$$

and it has been already sufficiently explained, in No. 378, how the position of the moveable can be determined without any ambiguity, by means of the three angles $\psi$, $\phi$, $\theta$.

426. By the first equation (1), we shall have $r = n$, in which $n$ denotes an arbitrary constant. Hence it appears, that the motion of rotation, parallel to its equator, will be uniform. In order to define the direction of this motion, we shall suppose that the point N is the *ascending node* of the equator; so

that when the point E attains to the point N, its radius EO will ascend above the horizontal plane, in virtue of the angular velocity $n$, which will be then a positive quantity. In fact, by the third equation (7) of No. 410, we shall have

$$d\phi = ndt + \cos\theta d\psi. \qquad (2)$$

When the point E is at N, the angle $\phi$ is either cipher or a multiple of $2\pi$, and, in the following instant, it will ascend or descend, according as $\phi$ increases or diminishes (No. 378); therefore, in order that the point E may ascend, as it is supposed to do, the motion of the body parallel to its equator being solely taken into account, it is necessary that the first term of the value of $d\phi$ should be positive.

This being the case, if its second term is likewise positive, it will increase the value of $d\phi$, which will therefore be greater than if the node N was immoveable; consequently, its motion projected on the equator will be *retrograde*, or in a direction contrary to that of the motion of the body parallel to this plane. The contrary will be the case, and the motion of the node will be *direct*, when the second term of the value of $d\phi$ is negative. In this second case, if the second term surpasses the first, the complete value of $d\phi$ will be negative, and the point E, after having arrived at N, will descend beneath the horizontal plane, instead of ascending above it; but this should not prevent us from considering the point N as always the ascending node, with respect to the motion of the body about its axis of figure.

Hence then it appears, that the direction of the motion of the ascending node N, will depend on the sign of the product of $\cos\theta$ and $d\psi$ at each instant; and this motion will be direct or retrograde, according as $\cos\theta$ and $d\psi$ have contrary or the same signs.

427. If equations (1) be multiplied by $c''$, $b''$, $a''$, respectively, and then added together, there results, as in No. 415,

$$\text{C}d.rc'' + \text{A}d.qb'' + \text{A}d.pa'' = 0;$$

hence, as $r = n$, $c'' = \cos\theta$, we shall obtain by substituting for $a''$ and $b''$ their values, and integrating,

$$\text{c}n\cos\theta - \text{A}(p\sin\theta\sin\phi + q\sin\theta\cos\phi) = l; \qquad (3)$$

$l$ being an arbitrary constant, which expresses, as in the number cited above, the moment of the quantities of motion of all the points of the body with respect to the axis oz. Therefore, in the motion which we are now considering, the moment of these quantities of motion is a constant quantity; however this is only the case with respect to the vertical axis, and not with respect to all axes passing through the point o.

If the two last equations (1) be respectively multiplied by $q$ and $p$, and then added together, we obtain

$$\text{A}(pdp + qdq) = \gamma(p\sin\theta\cos\phi - q\sin\theta\sin\phi)\text{M}gdt.$$

But it appears from the two first equations (7) of No. 410, that

$$p\sin\theta\cos\phi - q\sin\theta\sin\phi = -\sin\theta\frac{d\theta}{dt};$$

consequently we shall have

$$\text{A}(pdp + qdq) = -\text{M}g\gamma\sin\theta\, d\theta;$$

and by integrating and denoting the arbitrary constant, introduced by the integration, by $h$, there will result

$$\text{A}(p^2 + q^2) = 2\text{M}g\gamma\cos\theta + h. \qquad (4)$$

Moreover, by equations (7), cited above, we have ($e'$)

$$p\sin\theta\sin\phi + q\sin\theta\cos\phi = \sin^2\theta\frac{d\psi}{dt},$$

$$p^2 + q^2 = \sin^2\theta\frac{d\psi^2}{dt^2} + \frac{d\theta^2}{dt^2},$$

by means of which, equations (3) and (4) can be changed into the following:

$$\left.\begin{aligned} &\text{c}n\cos\theta - \text{A}\sin^2\theta\frac{d\psi}{dt} = l, \\ &\text{A}\left(\sin^2\theta\frac{d\psi^2}{dt^2} + \frac{d\theta^2}{dt^2}\right) = 2\text{M}g\gamma\cos\theta + h. \end{aligned}\right\} \qquad (5)$$

The values of $dt$, $d\psi$, $d\phi$ will be furnished by equations (2) and (5), each of them will be of the form $F\theta d\theta$; therefore, in order to obtain the values of $t$, $\psi$, $\phi$ in functions of $\theta$, it will be only necessary to integrate these three differential formulæ; and, in all cases, their three integrals are reducible to elliptic functions. But without having recourse to these functions, we can also obtain approximate values of $\psi$, $\phi$, $\theta$ in functions of $t$, in the examples which will be given farther on, after that $n$, $l$, $h$, the three arbitrary constants contained in the three preceding equations, shall have been determined. The three new constants which are introduced in those last integrations, can be determined by the values of $\psi$, $\phi$, $\theta$, when $t = 0$; that of $\theta$ will be given; and the initial values of $\psi$ and $\phi$ may be assumed to be, if we please, $\psi = 0$, $\phi = 0$.

428. Whatever the quantities of motion with which the points of the body are actuated at the commencement of the motion may be, their principal moment relative to the point O, and the direction of its axis will be known, by means of the percussions impressed on the moveable at this instant, with which these unknown quantities of motion, estimated in a direction opposite to that in which they actually move, should constitute an equilibrium (No. 353). If by the rule of No. 281, this principal moment be decomposed into three other moments, whose rectangular axes may be the part $Oz_1$ of the axis of figure, which contains the centre of gravity G, a right line perpendicular to $Oz_1$, and comprised in a vertical plane passing through $Oz$ and $Oz_1$, and a horizontal line perpendicular to this plane; as these three lines are principal axes, the value of the moment with respect to $Oz_1$, will be $Cr$ or $Cn$ (No. 409); it would therefore make known the value of $n$; but we, on the contrary, will suppose that this velocity is given directly, and assume this moment to be equal to $Cn$.

If the moments of the forces, with respect to the second and third axes, be denoted by $\mu$ and $m$ respectively, then the initial value of the principal moment will be $\sqrt{C^2n^2 + \mu^2 + m^2}$;

and because $B = A$ and $r = n$, its square, at any instant whatever, will be $A^2(p^2 + q^2) + C^2n^2$ (No. 409); therefore if the initial value of the angle $\theta$ be denoted by $\alpha$, we shall have, in virtue of equation (4),

$$(2Mg\gamma\cos\alpha + h)A = \mu^2 + m^2,$$

at the commencement of the motion; from which we obtain

$$h = \frac{\mu^2 + m^2}{A} - 2Mg\gamma\cos\alpha.$$

The axis of the moment denoted by $\mu$ will make with $oz$ an angle equal to $\alpha + 90$; and as the axis of the moment $m$ is perpendicular to this vertical, this moment will not affect the value of $l$, the moment relative to $oz$; therefore by the general expression of $E$ of No. 281, we shall have simply $(f')$

$$l = Cn\cos\alpha - \mu\sin\alpha.$$

The angle $\alpha$ which occurs in these values of $h$ and $l$, will be acute or obtuse, according as $G$, the centre of gravity of the moveable, is at the commencement of the motion, below or above the horizontal plane passing through the point $O$.

429. In order to verify these different formulæ, let $M$, the mass of the moveable, be supposed to be condensed into its centre of gravity, by which means it is changed into a simple pendulum the length of which is $\gamma$.

In this case, it will not be necessary to consider the angle $\phi$, and the motion will depend solely on the angles $\psi$ and $\theta$. If at the commencement, the material point $G$ is actuated by a velocity $k'$ perpendicular to $GO$ and directed in the plane $GOz$, and by a velocity $k$ perpendicular to this plane, we shall have

$$\mu = M\gamma k, \quad m = M\gamma k'.$$

We shall likewise have

$$C = 0, \quad A = M\gamma^2;$$

from which there will result

$$h = (k^2 + k'^2 - 2g\gamma\cos a)\text{M}, \quad l = -\text{M}\gamma k\sin a;$$

equations (5) will become

$$\gamma\sin^2\theta\frac{d\psi}{dt} = k\sin a,$$

$$\gamma^2\left(\sin^2\theta\frac{d\psi^2}{dt^2} + \frac{d\theta^2}{dt^2}\right) = k^2 + k'^2 + 2g\gamma(\cos\theta - \cos a);$$

and it is easy to make them to coincide with equations (5) and (6) of No. 205.

The first multiplied by $\frac{1}{2}\gamma dt$, indicates that the area described about the point O during the instant $dt$, by the horizontal projection of GO the radius vector of the moveable, is constant and equal to its initial value $\frac{1}{2}\gamma k\sin a\,(g')$. The first member of the second equation is the square of the velocity of this material point at the end of the time $t$; and as $k^2 + k'^2$ is the square of the velocity at the commencement of the motion, this equation is in fact the formula of No. 159.

430. In the case of a body which is not reduced to a material point, if the moveable is made to deviate from its position of equilibrium, and if after a velocity of rotation is impressed on it about its axis of figure, it is then remitted to itself, the two quantities $\mu$ and $m$ will be cipher, and we shall have

$$l = \text{C}n\cos a, \quad h = -2\text{M}g\gamma\cos a;$$

by substituting these values of $l$ and $h$ in equations (5), they will become

$$\left.\begin{aligned} \sin^2\theta\frac{d\psi}{dt} &= \frac{\text{C}n}{\text{A}}(\cos\theta - \cos a), \\ \sin^2\theta\frac{d\psi^2}{dt^2} + \frac{d\theta^2}{dt^2} &= \frac{2\text{M}g\gamma}{\text{A}}(\cos\theta - \cos a). \end{aligned}\right\} \quad (6)$$

It appears from the second, that the difference $\cos\theta - \cos a$ is always positive, and from the first, that the differential

$d\psi$ is so likewise; consequently (No. 426), the motion of the ascending node N will be direct when $\cos\theta$ is negative, which implies that G the centre of gravity is above the horizontal plane passing through the point O; and this motion will be retrograde, when G is below this plane, in which case $\cos\theta$ is positive.

When $n$ is cipher, the differential $d\psi$, and, consequently, the differential $d\phi$ given by equation (2), will be cipher; therefore, the angles $\psi$ and $\phi$ will be constant and may be assumed to be equal to zero; the motion will be changed into that of the common pendulum about an horizontal axis, relatively to which the moment of inertia is A; and in fact, if $d\psi$ be made equal to cipher in the second equation (6), it is reduced to equation (a) of No. 394, when in this last, the initial velocity is supposed to be cipher.

If $\frac{d\psi}{dt}$ be eliminated between the two equations (6), there results

$$\sin^2\theta\frac{d\theta^2}{dt}=\frac{2g}{\lambda}[\sin^2\theta-2\beta^2(\cos\theta-\cos\alpha)](\cos\theta-\cos\alpha), \quad (7)$$

in which, for conciseness, we make($h'$)

$$\frac{\text{M}\gamma}{\text{A}}=\frac{1}{\lambda}, \quad \frac{\text{C}^2n^2}{\text{A}^2}=\frac{4g\beta^2}{\lambda};$$

in this value of $\sin^2\theta\frac{d\theta^2}{dt^2}$, $\lambda$ is the length of the simple pendulum, which would perform its oscillations in the same time as the moveable, if the velocity $n$ was cipher. At the same time, the first equation (6) will become

$$\sin^2\theta\frac{d\psi}{dt}=2\beta\sqrt{\frac{g}{\lambda}}(\cos\theta-\cos\alpha), \quad (8)$$

in which $\beta$ must be regarded as a given positive quantity.

The approximate values of $\theta$ and $t$ which are deduced from these equations (7) and (8), and that of $\phi$ which results from

equation (2), may be easily expressed in a finite form, in the two cases that have been discussed.

431. If in the first place, $oz_1$ the part of the axis of figure which contains G the centre of gravity, is supposed to deviate by a very small quantity from the vertical $oz$ at the commencement of the motion, so that the angle $a$ may be very small, then the angle $\theta$ will be also very small, for we have always $\cos\theta > \cos a$; hence, if the fourth powers of $a$ and $\theta$ be neglected in the expansions of $\cos a$ and $\cos\theta$, equations (7) and (8) will become($i^7$)

$$\left.\begin{aligned} \theta^2\frac{d\theta^2}{dt^2} &= \frac{g}{\lambda}\left[(1+\beta^2)\,\theta^2 - \beta^2 a^2\right](a^2-\theta^2), \\ \theta^2\frac{d\psi}{dt} &= \beta\sqrt{\frac{g}{\lambda}}\,(a^2-\theta^2). \end{aligned}\right\} \quad (9)$$

It appears from the first of these equations, that $\theta$, which must be always a positive quantity (No. 378), can never be greater than $a$, or less than $\dfrac{\beta a}{\sqrt{1+\beta^2}}$. By resolving it with respect to $dt$, we obtain

$$dt = \frac{\mp\sqrt{\dfrac{\lambda}{g}}\,.\,\theta d\theta}{\sqrt{\left[(1+\beta^2)\,\theta^2-\beta^2 a^2\right](a^2-\theta^2)}},$$

and as the denominator is to be always regarded as a positive quantity, the numerator must be affected with the inferior or superior sign, according as $\theta$ increases or decreases.

Let us assume, in order to facilitate the integration,

$$\theta = a\sin u, \quad d\theta = a\cos u\,du;$$

there results by substituting these values in the preceding equation,

$$\sqrt{\frac{g}{\lambda}}\,dt = \pm\frac{d\,.\cos u}{\sqrt{1-(1+\beta^2)\cos^2 u}}.$$

Therefore, by integrating, we shall have($k'$)

$$t\sqrt{\frac{g}{\lambda}}\sqrt{1+\beta^2} = c \pm \text{arc}\left(\sin = \sqrt{1+\beta^2}\cos u\right);$$

$c$ being an arbitrary constant introduced by the integration. When $t = 0$, we have $\theta = a$ and $\cos u = 0$; the angle which corresponds to the sine cipher, is either cipher or some multiple of $\pi$; therefore, we have $c = i\pi$, in which $i$ denotes an integral number either positive, negative, or cipher; by substituting in the preceding expression for $\cos u$ its value, we shall have

$$t\sqrt{\frac{g}{\lambda}}.\sqrt{1+\beta^2} = i\pi \pm \text{arc}\left(\sin = \sqrt{\frac{1+\beta^2}{a^2}}\sqrt{a^2-\theta^2}\right).$$

As the angle $\theta$ decreases first from $\theta = a$ to $\theta = \dfrac{\beta a}{\sqrt{1+\beta^2}}$, we should take the superior sign in the preceding equation, and make $i = 0$; and as it then increases from this last value to $\theta = a$, we should take the inferior sign, and make $i = 1$; as it again decreases from $\theta = a$ to $\theta = \dfrac{\beta a}{\sqrt{1+\beta^2}}$, the superior sign should be taken and $i$ made $= 2$, and so on. It is in this manner that we should determine the arbitrary constant, which should be added to an arc of the circle which is supposed to be a function of its sine, but it is simpler to pass from the arc to the sine, previously to this determination.

We shall have at any instant whatever, by means of the preceding equation($l'$),

$$\theta^2 = a^2 - \frac{a^2}{1+\beta^2}\sin^2 t\sqrt{\frac{g(1+\beta^2)}{\lambda}}.$$

Denoting by T the time in which the angle $\theta$ passes from its greatest value $a$ to the least value that immediately follows it, or in which it returns from the least to the greatest, we obtain

$$\text{T} = \tfrac{1}{2}\pi\sqrt{\frac{\lambda}{g(1+\beta^2)}}.$$

By means of this value of $\theta^2$, that of $d\psi$, which is given by the second equation (9), will be($m'$)

$$d\psi = \frac{\beta\sqrt{\frac{g}{\lambda}}(1+\beta^2)\,dt}{\beta^2 + \cos^2 t\sqrt{\frac{g(1+\beta^2)}{\lambda}}} - \beta\sqrt{\frac{g}{\lambda}}\,dt.$$

We obtain by integrating and assuming $\psi = 0$ when $t = 0$,

$$\psi = \text{arc}\left[\text{tang} = \frac{\beta \text{ tang } t\sqrt{\frac{g(1+\beta^2)}{\lambda}}}{\sqrt{1+\beta^2}}\right] - \beta t\sqrt{\frac{g}{\lambda}};$$

by means of this formula, the retrograde motion of the ascending node N, on the horizontal plane passing through the point O, can be determined($n'$). If the constant $\beta$ is not cipher, the values of the arc comprised in this formula will be

$$\text{arc}\,(\text{tang} = \infty) = \tfrac{1}{2}\pi, \quad \text{arc}\,(\text{tang} = 0) = \pi,$$
$$\text{arc}\,(\text{tang} = -\infty) = \tfrac{3}{2}\pi, \text{ \&c.},$$

at the end of the first, second, third, &c., intervals of time T; consequently, the arc described by the point N during T, the first interval of time, will be

$$\psi = \tfrac{1}{2}\pi - \frac{\frac{1}{2}\pi\beta}{\sqrt{1+\beta^2}};$$

during the two first intervals T, it will be

$$\psi = \pi - \frac{\pi\beta}{\sqrt{1+\beta^2}};$$

at the end of the three first, we shall have

$$\psi = \tfrac{3}{2}\pi - \frac{\frac{3}{2}\pi\beta}{\sqrt{1+\beta^2}};$$

and so on.

It appears from what has been now established, that the arcs described by the node N during the successive intervals

T, will be all equal to each other, and that their common value will be

$$\frac{\frac{1}{2}\pi}{(\beta + \sqrt{1+\beta^2})\sqrt{1+\beta^2}},$$

which will always be so much smaller, as the constant $\beta$ is a greater number.

With respect to the value of $\phi$, it appears that if the square of $\theta$ be neglected in equation (2), and if $\phi = 0$ when $t = 0$, we shall have

$$\phi = nt + \psi;$$

the value of $\phi$ being thus known, the position of the moveable at any instant whatever, will be completely determined.

432. Whatever be the magnitude of the angle $a$, let us suppose that the angle $\theta$ continues very nearly constant, and, consequently, that it differs very little from $a$ during the motion; then if we make

$$\theta = a - u, \quad d\theta = -du;$$

the angle $u$ must be considered as a very small variable. By neglecting all powers of $u$ higher than the square, we shall have $(o')$

$$\sin^2\theta = \sin^2 a - u \sin 2a + u^2 \cos 2a,$$

$$\cos\theta - \cos a = u \sin a - \tfrac{1}{2}u^2 \cos a;$$

and at this degree of approximation, equation (7) gives

$$\frac{\lambda}{g}\frac{du^2}{dt^2} = 2u \sin a - u^2(\cos a + 4\beta^2);$$

hence we obtain $(p')$

$$\sqrt{\frac{g}{\lambda}}\, dt = \frac{du}{\sqrt{2u \sin a - u^2(\cos a + 4\beta^2)}}.$$

By integrating, and observing that $u = 0$ when $t = 0$, there results $(q')$

$$t\sqrt{\frac{g}{\lambda}(\cos a+4\beta^2)}=\text{arc}\left[\cos=\left(1-\frac{u(\cos a+4\beta^2)}{\sin a}\right)\right],$$

and, consequently,

$$u=\frac{\sin a}{\cos a+4\beta^2}\left[1-\cos t\sqrt{\frac{g}{\lambda}(\cos a+4\beta^2)}\right].$$

In order that the variable $u$ may be always a very small quantity, as has been supposed, it is necessary that $\beta$ should be very great, and this, in general, requires that there be impressed on the moveable a very great rotatory velocity about its axis of figure. We can then substitute $4\beta^2$ for $\cos a+4\beta^2$, and we shall have more simply($r'$),

$$u=\frac{1}{2\beta^2}\sin a\sin^2\beta t\sqrt{\frac{g}{\lambda}}.$$

If in equation (8), $a-u$ be substituted in place $\theta$, we shall obtain, by neglecting the square of $u$, and assuming that the angle $a$ is not cipher, in which case the factor $\sin^2 a$, common to both members, may be suppressed($t'$),

$$\frac{d\psi}{dt}=\frac{1}{\beta}\sqrt{\frac{g}{\lambda}}\sin^2\beta t\sqrt{\frac{g}{\lambda}};$$

hence there results

$$\psi=\frac{1}{2\beta}t\sqrt{\frac{g}{\lambda}}-\frac{1}{4\beta^2}\sin 2\beta t\sqrt{\frac{g}{\lambda}}.$$

We shall have at the same time, in virtue of equation (2), and by assuming that the angles $\phi$ and $\psi$ are cipher at the commencement of the motion,

$$\phi=nt+\psi\cos a;$$

and thus the position of the moveable, at any instant whatever, will be completely determined.

From the equation $\theta=a-u$, and from this value of $\psi$, it may be inferred, 1st, that when the moveable on which a very

considerable rotatory motion has been impressed about its axis of figure, is made to deviate from the vertical direction, the inclination of its equator on the horizontal plane passing through the point o, remains very nearly constant, during the entire continuance of the motion which results; 2ndly, that at the same time the intersection of these two planes acquires a motion very nearly uniform, and very slow relatively to the rotation of the moveable, which, as has been stated in No. 430, is direct or retrograde, according as the centre of gravity of the body is above or below the horizontal plane. Unless in the case when the angle $a$ is cipher, the angle $\psi$ and the motion of the node are independent of its magnitude; $u$ the inequality of the inclination of the equator, and the inequality which has place in the motion of the node, are always less sensible, as the rotation is more rapid, and the quantity $\beta$ more considerable.

Most of the lecture rooms of natural philosophers, are now furnished with the machine invented by Bohnenberger, by means of which the various circumstances of this motion of rotation are accurately represented, in like manner as all the phenomena of the motion of heavy bodies are exhibited by Atwood's machine. The rotatory motion is produced by means of a thread wrapped round the equator of the moveable, and attached to one of its points, which is rapidly unrolled, as when a humming top is made to spin.

It may be remarked that when $a$ is cipher, $\theta$ is so likewise, this renders equation (8) an identical one, and the angle $\psi$ indeterminate; in this case, the angle $\phi - \psi$, that is equal to $nt$, represents the motion of the body about its axis of figure, which continues always vertical.

# CHAPTER V.

## OF THE MOTION OF A SOLID BODY ENTIRELY FREE.

433. In order that the motion of a solid body in space may be more easily understood, philosophers substitute two other motions for it, one of *rotation* about one of the points of the moveable, and the other of *translation*, in which *all* its points participate. This is evidently the same thing, as if, at any instant whatever, the velocity of each point was considered to be the resultant of two other velocities, one of which is equal and parallel to that of the point which has been taken as the centre of the motion of rotation, and of which the other is peculiar to each point of the moveable; by means of these particular velocities, the body turns about the centre, as about a fixed point; and, in virtue of the common velocity, all its points are transferred in space, with a motion in which they all participate, but which does not in any manner affect the motion of rotation.

The motion of translation may be in point of fact one of revolution about another body, which last may be either itself at rest or in motion. (If a given face or section of the moveable remains constantly parallel to itself, the body has no motion of rotation;) if the moveable presents always the same face towards the central body, the rotation is performed in the same time as the revolution about this central body. This second case obtains in the motion of the satellites about their respective primary planets. The moon presents always the same face to the earth, so that the radius vector drawn from the centre of the earth to the centre of the moon, meets the surface of the satellite always in the same point (No. 141)(*a*); hence it

why

follows, that the rotation of the moon on its axis, and its revolution about the earth, are performed in the same time, namely, $27^d,32166$. It is demonstrated in the *Mechanique Celeste*, that this equality between the two motions will always subsist, although the motion of revolution should be accelerated from one century to another (No. 244); so that the motion of rotation must participate in this acceleration, the cause of which has been assigned by Laplace.

If it be merely proposed to decompose the motion of a body into two motions which are simpler and more easily conceived, the centre of the motion of rotation may be assumed to be any point we please; but when our object is to determine each of these two motions in particular, we should assume this point to be the centre of gravity of the moveable, because then, in the *first* instant, these two motions may be determined independently of each other, and in several instances, this will be also the case, during the entire continuance of the motion; the selection of this point, as the centre of rotation, will always render the differential equations of the two motions much simpler, as we now proceed to show.

434. Let the centre of gravity of the body be denoted by G, its mass by M, and any element whatever of M by $dm$. Let $x, y, z$ be the three rectangular coordinates of this material point at the end of the time $t$, which is supposed to be reckoned from the commencement of the motion, and $x_1, y_1, z_1$ those of the point G with respect to the same axes; we shall have

$$Mx_1 = \int x dm, \quad My_1 = \int y dm, \quad Mz_1 = \int z dm,$$

in which the integration is supposed to extend to the entire mass. If these equations be differenced with respect to $t$, the operation can be effected under the signs $\int$. By this means, we shall have

$$M\frac{dx_1}{dt} = \int \frac{dx}{dt} dm, \quad M\frac{dy_1}{dt} = \int \frac{dy}{dt} dm, \quad M\frac{dz_1}{dt} = \int \frac{dz}{dt} dm; \quad (1)$$

and, in order to obtain the components of the initial velocity of the centre of gravity, it will be sufficient to know the values of these last integrals at the commencement of the motion.

For this purpose, let us suppose that at this epoch, $\mu, \mu', \mu''$, &c., the constituent parts of M, are actuated by the velocities $v, v', v''$, &c., and that these velocities are the same for all the points of which each of these parts consists, so that they would acquire the quantities of motion $\mu v$, $\mu' v'$, $\mu'' v''$, &c., if they were free. By the principle of No. 353, there should be an equilibrium between these quantities of motion, estimated in a direction opposite to that in which the motion takes place, and those which all the points of the moveable actually acquire in the first moment, which taken in a direction parallel to the axes of $x$, $y$, $z$ respectively, will be, relatively to $dm$, the initial values of $\frac{dx}{dt}dm$, $\frac{dy}{dt}dm$, $\frac{dz}{dt}dm$. Now, as the directions of the velocities $v$, $v'$, $v''$, &c. are given, we can resolve the quantities of motion which correspond to them, in a direction parallel to these axes. Hence if the sums of these components, taken in the directions of the positive $x$s, $y$s, $z$s, be denoted by P, Q, R, we shall have, in order to the equilibrium in question, as the motion is supposed to be entirely free,

$$\int \frac{dx}{dt}dm = \text{P}, \quad \int \frac{dy}{dt}dm = \text{Q}, \quad \int \frac{dz}{dt}dm = \text{R},$$

for the particular value $t = 0$. Therefore, at the commencement of the motion, equations (1) will become

$$\text{M}\frac{dx_1}{dt} = \text{P}, \quad \text{M}\frac{dy_1}{dt} = \text{Q}, \quad \text{M}\frac{dz_1}{dt} = \text{R}; \qquad (2)$$

from these equations it appears, that the initial velocity of the centre of gravity will be the same, in magnitude and direction, as if M, the entire mass of the moveable, was condensed into it, and all the quantities of motion $\mu v$, $\mu' v'$, $\mu'' v''$, &c., or their components P, Q, R, were applied to it, parallel to their respective directions.

435. We may suppose that $\mu$, $\mu'$, $\mu''$, &c. denote the masses of bodies, which, actuated by the velocities $v$, $v'$, $v''$, &c., impinge simultaneously on another body at rest, and that they then remain attached to it, so as to constitute a total mass M, of which G the centre of gravity has acquired a velocity, the values of whose three components, namely, $\frac{dx_1}{dt}$, $\frac{dy_1}{dt}$, $\frac{dz_1}{dt}$, are furnished by equations (2).

The problem would be different if the impinging bodies do not remain attached to the struck body after the impact. Let a mass M at rest, be struck by another body in motion, which touches M in only one point E of its surface (fig. 10), then if the bodies do not slide on each other, during the continuance of the shock, or at least, if they only do so to a small extent, it is not necessary to take into account the inconsiderable friction to which it can give rise (No. 353); and if, finally, we suppose that EF is the normal to the surface of M at the point E, and comprised in the interior of this body, it will be shown in a following chapter, that the motion of M will be the same, as if $\mu$, a certain part of its mass, whose centre of gravity is situated on EF, should receive in this direction a certain velocity $v$ common to all its points. Hence it follows, that EF is the direction of the impact, and its intensity, that is to say, the quantity of motion $\mu v$ will be determined, in that chapter, when the motion of the impinging body, and the form of the two bodies, whether elastic or non-elastic, are given.

This being agreed on, if V be the velocity with which G the centre of gravity of M is actuated, its direction will be along the line GD, parallel to EF, and its value will be equal to $\mu v$ divided by M, so that we shall have

$$MV = \mu v.$$

Conversely, if V the velocity of the centre of gravity is given by observation, the quantity of motion impressed on the struck body, in the direction of the interior part of the

normal to the surface erected at E, the point where the impact is made, will be obtained, by multiplying V by M; this property belongs exclusively to G the centre of gravity, and in general, will not have place for the velocity with which the point E or any other point of M, situated or not on the direction of the line of impact, is actuated.

436. In order that we may more clearly perceive how the motion of rotation of a body is simplified, when it is referred to its centre of gravity, let us first suppose that it is proposed to determine this motion about C, a determinate point (fig. 11) of this body, which we will then cause to coincide with G, its centre of gravity.

Let CA denote the velocity of the point C, in magnitude and direction, and BD that of B any other point whatever of the moveable. Through the point B, let the line BE be drawn equal and parallel to CA, and let the parallellogram BEDF be completed. We can substitute for the velocity BD its components BE and BF, and if the velocity of all the points of the moveable be decomposed in the same manner, they will all have a common velocity, equal and parallel to CA, and each of them will have, besides, a velocity peculiar to itself. Now, if a velocity equal, parallel, and contrary to CA, be impressed on the point B and all the other points of the body, the point C will be reduced to a state of rest, without any change being produced in the motion of rotation about this point, which arises from the particular velocities of the other points, for example, BF for the point B. Therefore, in order to determine this motion, we may consider the point C as fixed, after having impressed on all the elements of the body, quantities of motion equal to the product of their masses and of the velocity CA, estimated in a direction contrary to CA. But as these forces are parallel and proportional to their respective masses, their resultant will be equal to their sum, and will pass through the centre of gravity, like the resultant of forces which arise from gravity; consequently, if we denote the velocity CA by

U, and the mass of the moveable by M as before, it will be sufficient to add to the given quantities of motion impressed simultaneously on the different parts of M, another quantity of motion, namely, MU acting in the direction of the line GA′, parallel and contrary to CA. The motion of rotation about the point C, can then be determined by the rules of the preceding chapter, as if C was a fixed point. This determination therefore requires that we should know U the velocity of the point C; but if the centre of gravity coincided with this point, it is evident that when the quantity of motion MU, the direction of which passes through the point G, is applied to it, we need not take it into account, for any force whatever which passes through the centre of the motion of rotation, cannot influence in any manner this motion, since it cannot make the body to turn about this point, in one direction, rather than in the contrary. 3

It follows, therefore, that when quantities of motion given in magnitude and direction, are simultaneously impressed on different parts of any solid body, the moveable will commence to turn about the centre of gravity, as about a fixed point, and without our being obliged to add any other quantity of motion to those which are given.

437. By combining this theorem with that which precedes, we can completely determine the *initial* motion of a solid body of any form, whatever be the manner in which it has been produced.

For greater clearness, let us suppose that the moveable, whose mass is M, and centre of gravity G (fig. 10), is struck at the point E of its surface by another body, which after the impact is detached from it. By taking for its motion of translation that of the point G, and having regard to this motion solely, all the points of the moveable will in the first instant describe lines parallel to the normal EF; we can always, as has been just stated, determine their common velocity, which will be the entire velocity of the point G; but, for greater

simplicity, we shall suppose that it is given by observation, and denote it by V. Independently of this motion of translation, the body will turn about the point G as if it was fixed, and a quantity of motion equal to MV was impressed on a part of the mass M, the centre of gravity of which was in the line EF. Consequently, in the first instant, the direction of the instantaneous axis of rotation and the angular motion of the body about this axis, will be determined by equations (1) of No. 418, in which we shall make

$$k = \text{M}\text{V}f;$$

$f$ denoting the perpendicular GL let fall from the point G on the line EF.

For this purpose, let this angular velocity be denoted by $\omega$, and the angles which the instantaneous axis of rotation makes with the three principal axes that intersect at the point G, by $\alpha, \beta, \gamma$; likewise, let HEK be the section of the moveable made by the plane passing through the point G, and the line EF; through G let a perpendicular to this plane be drawn, and let $a, b, c$, be the angles which this line makes with the axes to which the angles $\alpha, \beta, \gamma$, are referred; by formulæ (3) of No. 405, and equations (1), we shall have

$$\text{A}\omega \cos \alpha = k \cos a, \quad \text{B}\omega \cos \beta = k \cos b, \quad \text{C}\omega \cos \gamma = k \cos c;$$

in which A, B, C, denote the three moments of inertia of the moveable, with respect to the same axes. By taking the squares of each of these equations, and then adding them together, we shall obtain, as $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$,

$$\omega^2 = \frac{k^2 \cos^2 a}{\text{A}^2} + \frac{k^2 \cos^2 b}{\text{B}^2} + \frac{k^2 \cos^2 c}{\text{C}^2}.$$

As the values of $a, b, c$, will be given in each particular case, the velocity $\omega$ will be known, and thus, by means of the preceding equations, the angles $\alpha, \beta, \gamma$, can be determined, that is to say, the direction of the instantaneous axis.

If the perpendicular to the section HEK coincides with one of the three principal axes, so that we may have, for example, $a = 90°$, $b = 90°$, $c = 0$, there will result $\alpha = 90°$, $\beta = 90°$, $\gamma = 0$, and the instantaneous axis will coincide with the principal axis; hence it follows, that if a body entirely free, is struck in the plane of two of the three principal axes relative to its centre of gravity, it will commence to turn about the third axis. If we substitute for $k$ its value, that of the initial velocity of rotation will be, in this case($b$),

$$\omega = \frac{\text{MV}f}{\text{C}}.$$

Conversely, it is easy to show by means of the preceding equations, that the moveable cannot commence to turn about the perpendicular to the plane of the section HEK, so that $\alpha$, $\beta$, $\gamma$, may be equal to the angles $a$, $b$, $c$, or to their supplements, unless this perpendicular is one of the principal axes which intersect at the point G ($c'$).

When the moveable is either a homogeneous sphere, or one composed of concentrical strata, the perpendicular EF will pass through the point G, which will be its centre of figure. We shall, therefore, have $f = 0$, $k = 0$, $\omega = 0$; consequently, the moveable will acquire no motion of rotation by the impact. When, as is sometimes the case, a sphere entirely free, is made, by the percussion of another body, to turn on itself, the reason of this is always because the impinging body slides more or less on this sphere, and the motion of rotation is then produced by the friction which takes place during the continuance of the impact.

Whatever may be the form of the struck body, if the striking body remains attached to it, the preceding formulæ will still obtain, by substituting for MV, the quantity of motion which the second had before the shock, and by taking for $f$, the perpendicular let fall from the centre of gravity of the two masses, on the primitive direction of the centre of gravity of the second

body; A, B, C, will then be the principal moments of inertia of the body, composed of the two united masses.

438. We now proceed to discuss the motion of the mass M, at the end of any time $t$, and for greater clearness, we shall consider, successively, its motions of translation and rotation, each of them being referred to its centre of gravity.

1st. At this instant, let X, Y, Z, be the components of the given accelerating force, which acts on any element $dm$, resolved parallel to the axes of $x, y, z$; the forces lost during the instant $dt$, by this material point will be (No. 391), when estimated in directions parallel to these axes,

$$\left(X - \frac{d^2x}{dt^2}\right)dm, \quad \left(Y - \frac{d^2y}{dt^2}\right)dm, \quad \left(Z - \frac{d^2z}{dt^2}\right)dm,$$

As the moveable is entirely free, it is necessary, in order to the equilibrium of the forces lost by all its elements, that the integrals of these quantities extended to the entire mass, should be equal to cipher; consequently, we shall have

$$\int \frac{d^2x}{dt^2}dm = \int Xdm, \quad \int \frac{d^2y}{dt^2}dm = \int Ydm, \quad \int \frac{d^2z}{dt^2}dm = \int Zdm.$$

But by differentiating equations (1) a second time, we obtain

$$M\frac{d^2x_1}{dt^2} = \int \frac{d^2x}{dt^2}dm, \quad M\frac{d^2y_1}{dt^2} = \int \frac{d^2y}{dt^2}dm, \quad M\frac{d^2z_1}{dt^2} = \int \frac{d^2z}{dt^2}dm,$$

hence there will result

$$M\frac{d^2x_1}{dt^2} = \int Xdm, \quad M\frac{d^2y_1}{dt^2} = \int Ydm, \quad M\frac{d^2z_1}{dt^2} = \int Zdm; \qquad (3)$$

from which it appears, that *during the continuance of the motion*, G the centre of gravity of the moveable, moves in the same manner, as if the entire mass M was concentrated in it, and the motive forces which act on all its points, or their components, were applied to it, parallel to their respective directions.

2ndly. If, after the *initial* motion of translation is des-

troyed, as in No. 436, we suppose that at each instant, there is communicated to all its points, infinitely small increments of velocity, equal and directly contrary to that by which the point G is actuated at the same instant, this point will be reduced to a state of rest, during the entire continuance of the motion; and the rotation about this point will not be altered. But, this is evidently the same thing, as if there was applied during the entire continuance of the motion to all the elements of the moveable, accelerating forces equal and contrary to that of the centre of gravity; as the corresponding motive forces are parallel, and proportional to the masses of these material points, their resultant will pass through the centre of gravity G, therefore, they need not be taken into account, in determining the motion of rotation about this centre. Consequently, this motion will be the same at each instant, as if G was a fixed point, and the forces which act, at this instant, on the moveable, were not changed.

These two theorems correspond to those of Nos. 434 and 436, which refer to the *commencement* of the motion; but it does not follow, that during its entire continuance, the motion of translation of the moveable, and its motion of rotation about the centre of gravity, are independent of each other, and can be determined separately, as at this commencement. Equations (3) will be those of the motion of translation, and equations (7) and (a) of Nos. 410 and 412, those of the motion of rotation, the origin of the coordinates in these last, being supposed to be at G, the centre of gravity. Now, when the motive forces applied to the different points of the moveable depend on their absolute positions in space, the coordinates of these points, of which these forces are given functions, will occur at the same time in these two systems of differential equations, which can no longer be integrated separately, and the two motions that depend on these equations, will mutually influence each other. We cannot, in general, integrate these simultaneous differential equations, and determine the two

motions of the moveable, except by approximation, nevertheless, they will be independent of each other in the two particular cases which we now proceed to consider.

439. If the moveable is only subjected to the sole action of gravity, equations (3) will be those of a heavy material point, in a vacuum; whatever be the form of the solid body, and its motion about its centre of gravity, this point will describe in space a parabola, to which the direction of the initial velocity is a tangent; the parameter of this curve depends on the magnitude of this velocity, and its motion on this curve will be the same as that of a detached material point (No. 208). On the other hand, as the weight of the body is a force constantly applied to its centre of gravity, it will not affect the motion of rotation about this point, which is entirely produced by the initial percussions, and is the same, as if the centre of gravity was not displaced.

Let us suppose, for example, that the body is a homogeneous ellipsoid, struck by another body that touches it at the point E of its surface (fig. 10); the line GD, parallel to the normal EF, will be a tangent to the parabola that the point G commences to describe; and this curve can be easily constructed when the initial velocity of the point G, which we shall denote by V, is given. Moreover, let HEK, the section formed by the plane passing through the point G and the line EF, be supposed to comprise two of the axes of figure of the ellipsoid; then, if $2a$ and $2b$ denote these two axes, C the moment of inertia with respect to the third axis, and M the mass of the body, we shall have (No. 370)

$$C = \tfrac{1}{5}M(a^2 + b^2).$$

Now, the moveable must turn about the point G, as if it was destitute of gravity, and had no motion whatever; but in this case, the axis perpendicular to the section HEK should remain altogether immoveable (Nos. 389 and 437); and its angular velocity of rotation should be furnished by the formula relative

to the initial motion of a solid body about a fixed axis. Therefore, if we denote it by $\omega$ as before, we shall have, (GL the perpendicular let fall from the point G on EF being denoted by $f$, and the quantity of motion impressed on the moveable being equivalent to MV,) by formula (1) of No. 386,

$$\omega = \frac{\mathrm{MV}f}{\mathrm{C}},$$

or, by substituting for C its value,

$$\omega = \frac{5\mathrm{V}f}{a^2 + b^2}.$$

Thus the two velocities $\omega$ and V are connected together, since each of them results from the same percussion.

Hence it appears, that the points of the moveable will describe parabolas parallel to the trajectory described by its centre of figure; and, at the same time, the body will turn uniformly about the axis perpendicular to the section HEK, which remains constantly parallel to itself, while it is transferred in space.

440. If the moveable is a homogeneous sphere, or one composed of concentrical strata, all whose points are attracted, or repelled, in the inverse ratio of the square of the distances, by points of other bodies which are either at rest, or in motion, the resultant of all these forces will be the same, as if the entire mass of the moveable was condensed in its centre of gravity, for each of them will be equal and contrary to the reaction of the sphere on the centre from which it emanates. Consequently, the centre of gravity will move as a detached point, subjected to given attractions or repulsions; and the motion of rotation of the moveable will be independent of these forces, and the same as if the centre of gravity remained at rest, so that in this case, the two motions of rotation and translation are independent of each other.

Therefore, if we do not take into account the circumstance

of the earth not being perfectly spherical, it will turn constantly and uniformly about one of its diameters, which will be always the same, and will remain constantly parallel to itself; at the same time, the elliptic motion of its centre of gravity about the sun, though deranged by the action of the other planets, will be rigorously independent of the motion of rotation.

441. This however is no longer the case, when the compression of the terrestrial spheroid is taken into account. For, in the first place, if the axis of rotation does not, at the commencement of the motion, coincide with the axis of figure, the instantaneous axis of rotation will oscillate about this line (No. 421), and will meet the earth successively in different points of its surface. Therefore the poles and equator will be displaced on the surface of the globe, so that the geographical latitudes of places on the earth will be changed. The amplitude of these oscillations will be arbitrary, but their duration will depend on the differences between the moments of inertia of the earth; and, from what we know of these differences($d$), this duration will be a little less than a year. Now, in this interval of time, the most precise observations do not indicate any variation in the *zenith* distance of a determinate place on the earth from the point where the production of the axis of rotation meets the concave surface. It follows therefore, that if the oscillations in question were formerly of a sensible magnitude, they have now become altogether insensible; so that at present the only forces which can cause the direction of the axis of rotation of the earth to vary, are the *permanent* forces which arise from the attractions of the sun, moon, and planets on the terrestrial spheroid.

Now as the strata of the earth, though not spherical, differ very little from this form, the part of these forces, the direction of which does not pass constantly through the centre of gravity of the spheroid, is very small, with respect *to the attractions* on the entire spheroid. It is this part which pro-

duces the perturbations of the motion of rotation, namely, *the precession of the equinoxes, and the nutation of the axis of the earth.*

In virtue of the precession, the annual retrogradation of the equinoxes on the ecliptic, the position of which was fixed at 1800, is about 50″,36482; the annual retrogradation on the plane of the orbit of the earth, which is itself in motion in consequence of the action of the other planets, that is to say, on the true ecliptic, is a little less, and equal to 50″,23427, as has been already stated in a former chapter (No. 219).

The nutation is an oscillation of the axis of the earth, in consequence of which it alternately approaches to and recedes from the perpendicular to the plane of the ecliptic; it arises from the attraction of the moon, and its period is the same as that of the motion of the nodes of the lunar orbit, or about eighteen years; its amplitude amounts to 9″,40 (No. 223), on the supposition that the mass of the moon is equal to a seventy-fifth part of that of the earth.

The actions of the sun and moon on the terrestrial spheroid produce only a very slow variation (which will not be sensible except after a long series of years), in the inclination of the equator to the ecliptic; the annual diminution of the obliquity of the ecliptic, which at the commencement of the present century amounted to 0″,45714, arises from the actions of the planets, which produce a change in the plane of the orbit of the earth (No. 244).

It appears from a careful discussion of the question, that the same forces which produce the variations adverted to above, in the absolute direction of the axis of rotation of the earth, or referred to fixed lines, are altogether incapable of displacing this axis in the interior of the spheroid, or of producing any variation in its velocity of rotation. Therefore the earth turns constantly about the same diameter, which is its axis of figure; and its motion is uniform about this moveable line, the direction of which, in space, is continually

changing. Hence the sidereal day is constant; and consequently, the mean day also (No. 111), or, at least, it is only subject to a variation altogether insensible; and either of these periods may be assumed as the unit of time.

As the principal results of this important theory can only be briefly indicated here, the reader, for their fuller exposition, is referred to a memoir of the author *on the Motion of the Earth about its Centre of Gravity*, inserted in the seventh volume of the *Memoirs of the Academy of Sciences*.

442. The invariability of the day is confirmed by the most ancient observations, from which it appears that its duration has not altered the one-hundredth part of a second, for example, for the last 2500 years, as we now proceed to show.

If the duration of the day was variable, the longitudes and latitudes of the sun, the moon, and the other celestial bodies, computed on the supposition that it was constant, would not agree with the observed longitudes and latitudes; the motion of the moon about the earth would, in consequence of its rapidity, be the fittest to throw light on this point; and, if the variation of the day was progressive, the differences between the results of computation and observation would be so much the greater, as the epochs at which the observations were made, were more remote from the present day.

This being agreed on, let $l$ and $l'$ be the true longitudes of the sun and moon at a determined epoch, then if an eclipse of the sun or moon is recorded to have occurred at this epoch, their difference $l - l'$ must differ from a multiple of $180°$, by a quantity less than the semisum of the diameters of the sun and moon; if, therefore, abstracting from the multiple of $180°$, which this difference may contain, we denote it by $\delta$, it is evident that it cannot exceed the mean value of this semisum, i. e. half a degree, and that, in general, it must be much less than this limit. Now, in the *Connaissance des Tems* for 1800, the values of $\delta$ in the case of twenty-seven eclipses observed by the Chaldeans, Greeks, and Arabians, have

been computed, and the results which are in some cases plus, and in other minus, are invariably very small. The greatest, which amounts to $-27', 41''$, is in the case of an eclipse observed 382 years before the Christian era; for the most ancient eclipse observed by the Chaldeans, 720 years before our era, the value of $\delta$ is only $2''$. This comparison evinces at the same time the accuracy of our present lunar tables, and also the necessity of the secular inequalities introduced by Laplace. It likewise proves, that the duration of the day which was supposed to be constant in the computation of the longitudes of the sun and moon, is not, in fact, subject to any *progressive* variation. But in order to remove every doubt on this important point, we will compute the value of $\delta$ corresponding to the most ancient observed eclipse, which would result from such a variation, if it really existed.

443. Let the interval of time which at present constitutes the mean day be taken as the unit of time, and let us suppose that, from a very remote epoch, this duration has diminished from one day to the following, by a constant quantity $a$. Let $n$ be the mean motion of the moon, that is to say, the number of degrees which it describes in each unit of time, abstracting from the inequalities of its true motion; the arcs described in the day when $n$ is determined and the preceding ones will be $n$, $n(1+a)$, $n(1+2a)$, $n(1+3a)$, &c.; and the arc described in a great number of days such as $t$ will be $nt+\frac{1}{2}ant(t-1)$, or $nt+\frac{1}{2}ant^2$ very nearly $(d)$. The term $nt$ is already comprised in the value of $l$, which is computed from the tables, on the supposition that the day is constant; therefore, in consequence of the variation of the day, the true longitude of the moon will be increased by $\frac{a}{2}nt^2$, at an epoch distant from us by a number of days represented by $t$. That of the sun at the same epoch would be increased by $\frac{1}{2}an't^2$, $n'$ denoting the mean diurnal motion of the sun; therefore we shall have at this epoch, if the variation of the day be solely considered,

$$\tfrac{1}{2}a(n-n')t^2 = \delta; \tag{1}$$

and, with respect to the eclipse observed 720 years before the Christian era, this will be the *entire* value of the difference of the longitudes of the sun and moon, since this difference, computed on the supposition that the day is constant, is only $2''$, or very nearly cipher.

If $i$ be the number of centuries contained in $t$ days, we shall have

$$t = (36525)\,i.$$

Likewise let $\beta = (36525)\,a$, $m = (36525)n$, $m' = (36525)n'$; then equation (1) will be changed into (e) $\frac{1}{2}\beta(m-m')\,i^2 = \delta$, in which $\beta$ will now be the secular diminution of the day, and $m$ and $m'$ will represent the secular motions of the sun and moon. From their values determined by modern observations, it appears that, neglecting fractions of degrees,

$$m - m' = 445268°.$$

Now, if we suppose that the day has diminished a ten millioneth part since the most ancient eclipse recorded by the Chaldeans, we shall have $\beta i = 0{,}0000001$, $i = 25{,}32$, from which there would result $\delta = 34'$, a value of the difference of longitudes that would render the observed eclipse impossible (*f*).

Therefore the duration of the day cannot have diminished by this fraction, which is a little less than the hundredth part of a second, in an interval of time which is greater than twenty-five centuries. If there were any *periodic* variations in the duration of the day, there would result from them illusions in the measure of time, which would produce apparent inequalities in the motions of the stars. It would be easy to distinguish these inequalities, since they all follow the same law, for the sun, the moon, and the planets, and their magnitudes would, in the case of each of these bodies, be proportional to the rapidity of its motion. But astronomers have not recog-

nized any inequality of this kind in the motions of the heavenly bodies. Thus then it appears that observation agrees with theory in proving that the duration of the mean day is not subject to any variation, either periodic or progressive, the magnitude of which is sensible.

444. Let us now return to the particular object of this chapter.

When a solid body moves in the air or any other fluid, the resistances exerted on all the points of its surface must be transferred, parallel to themselves, together with the weight of the body, to its centre of gravity; and the motion of this centre is that of a heavy material point, in a resisting medium; the mass of this point being that of the body, and its motive force arising from the resistance, which is a force whose components will depend on the form of the moveable, on the velocities of the different elements of its surface, and on the condensations or dilatations of the fluid in contact with these elements. At the same time the moveable will turn about its centre of gravity, as if the velocity of this point was cipher, and the velocities of the points of its surface were, nevertheless, those which really have place at each instant. It follows from this, that the resistance of the medium will affect, at once, both the motion of translation and the motion of rotation of the moveable, and that in the case of a body of any form whatever, these two motions will mutually depend on each other, and cannot be determined separately.

If the moveable is a homogeneous sphere, or one composed of concentrical strata, on which no velocity of rotation is impressed at the commencement, then no motion of this kind will arise during the continuance of this motion, which will be merely one of translation, in which all the points of the moveable will be actuated at each instant with equal and parallel velocities. In fact, if there be drawn through the centre of the sphere, a tangent to the curve which it describes, it is evident that every thing will be similar about this line, with

respect both to the velocities of the points of its surface, and also to the condensations or dilatations of the surrounding fluid. Consequently, the resultant of the resistances exercised on all the points of the surface, will constantly pass through the centre of figure, which is also the centre of gravity, so that it cannot produce any motion of rotation. The condensations or dilatations of the fluid in contact with the elements of the surface, will depend on the velocity common to all the points of the moveable, and will, moreover, be different for the different sections perpendicular to the tangent which we have supposed to be drawn through the centre. Consequently, the resultant of the exterior resistances which will coincide with this tangent, can only depend on this velocity, on the extent of the surface, and on the natural elastic force of the fluid; and the motion of the centre of gravity will be that of a detached material point, whose mass is that of the body, and to which is applied the resultant in question, in a direction contrary to that of its velocity, and also the weight of the body. This has been assumed in No. 210, with respect to bodies projected in artillery practice, which are always supposed to be homogeneous, and perfectly spherical.

In this case, the centre of the bullet cannot deviate from the vertical plane passing through the direction of its initial velocity, and relatively to which every thing is alike on all sides. But if the projectile deviates a little from the spherical form, or if it has not the same density throughout its entire extent, the resultant of the exterior resistances which are normal to its surface, will not pass *constantly* through its centre of gravity; it will consequently produce a motion of rotation; and if it is not always comprised in the vertical plane, in which the centre of gravity commences to move, it will cause it to deviate from this plane; so that the trajectory of the projectile, referred to its centre of gravity, will no longer be a *plane curve.* It is easy to conceive all this when the projectile is neither perfectly spherical or homogeneous; but it may be

remarked that, even though the defect of perfect sphericity or of non-homogeneity be not taken into account, still if the bullet is actuated by any velocity of rotation at the mouth of the cannon, the friction of its surface against the air during the motion, may still cause its centre of gravity and of figure to deviate from the vertical plane.

445. In order to show this, let G (fig. 12) be the centre of gravity and figure of a spherical homogeneous body, and AGB the diameter which is a tangent to the trajectory described by G. Let us suppose, for greater simplicity, that the axis of rotation which passes through the point G, is perpendicular to this diameter. Let ABDE be the section of the moveable, perpendicular to this axis, and DGE a diameter of this section, at right angles to AGB. Likewise let the motion of the point G be from A to B, in the direction indicated by the sagitta *s*, and the motion of rotation in the direction indicated by the sagittæ placed at A, D, B, E. The friction of each element of the surface against the air, which arises from the motion of rotation, will be a force acting along the tangent to the surface in a direction contrary to this motion. On each section parallel to ADBE, it will vary from one point to another, in consequence of the difference of density of the fluid. The fluid will be condensed *before* the projectile, and it will be dilated *behind* it; consequently, the friction will be greatest on the side of the point B, and least on that of the point A. Hence it follows, that if all the forces arising from the friction exerted against the surface, be transferred parallel to their directions, to the point G, there will result a force acting along the part GD of the diameter DGE. Let this force be denoted by F, the weight of the body by P, the resistance of the medium transferred to the point G, and acting in the direction of GA, a part of the diameter AGB, by R. The motive force of the point G will be the resultant of the three forces F, P, R, and its accelerating force will be equal to this resultant, divided by the mass of the projectile.

This being established, if the axis of rotation be vertical, and consequently, comprised in the plane of the two forces P and R, GD the direction of the force F will be perpendicular to this plane; it will therefore cause the moveable to deviate from this vertical plane, by urging it on the side of the point D; and, in this case, the trajectory of the point G, will be a *curve of double curvature.* If, on the contrary, the axis of rotation is horizontal, GD the direction of the force F, will be comprised in the vertical plane of the forces P and R, which will be that of the section ADBE, consequently, the point G will not deviate from this plane, and its trajectory will be a plane curve.

446. It appears that in this last case, the vertical component of the force F will increase or diminish the weight P, according as the direction of the sagitta A is upwards or downwards. The force F will be vertical, and this diminution or increase of weight will be a *maximum* when the direction AB is horizontal. Thus, in the horizontal level, if the bullet be supposed to turn about a horizontal axis, perpendicular to the direction of the level, the friction of the projectile against the air will increase its weight and diminish the range, when the anterior part of this body turns upwards from the horizon, and when this part turns towards the horizon, the friction will diminish the weight and increase the range. It may even happen, in this second case, that the trajectory becomes convex to the horizon, for in order that this should take place, it would be only necessary for the rotation to be sufficiently rapid to render the force F greater than the weight P; but then as the friction would diminish the velocity of rotation, the force F would decrease, and eventually become less than P, when the trajectory would again become concave towards the ground as usual.

These considerations, combined with those of the preceding number, show that, independently of the defect of sphericity and homogeneity of the bullet, the friction of its surface

against the air, arising from the motion of rotation, which it may acquire in moving in the interior of the cannon, may affect the accuracy($g$) of the aim, because this friction may cause the centre of the bullet to deviate from the vertical plane of projection, and also produce an inequality in the ranges, as this force may increase or diminish the weight of the moveable. However, none of these effects takes place, if the bullet turns about the diameter AB, in the direction in which it moves; for then the friction is equal and contrary for two opposite elements of each section of the surface perpendicular to the axis of rotation; hence it follows, that the forces arising from the friction exercised on all the elements, destroy one another two by two, when transferred to its centre of gravity; so that the motion of this point will not be deranged by the total friction arising from the rotation.

# CHAPTER VI.

## OF THE MOTION OF A HEAVY SOLID BODY ON A GIVEN PLANE.

### I. *Case in which the Friction is not taken into Account.*

447. For greater simplicity, we shall suppose that the moveable touches the given plane only in one point, which we shall denote by K, and which, in general, may vary both on its surface and on the plane. As the forces lost in each infinitely small instant constitute an equilibrium, they must be reducible to a single force, passing through the point K, normal to the given plane, and so directed, that it may always tend to press the moveable against this plane. This resultant will be the pressure that the plane sustains, and which will be destroyed by its resistance denoted by R. If to the weight of the body, this force R of an unknown magnitude be joined, we need not consider the given plane at all, but regard the moveable as entirely free.

Hence it follows, that the motion of its centre of gravity G will be the same as that of an isolated material point, whose mass is that of the moveable, and to which there is applied, the weight of this body and the force R, parallel to their respective directions. At the end of the time $t$, let $x_1$, $y_1$, $z_1$, be the coordinates of G referred to fixed rectangular axes, and $\lambda$, $\mu$, $\nu$, the angles which the direction of the force R makes with lines drawn parallel to these axes, through the point K. If the axis of the positive coordinates of $z_1$, be supposed to be vertical, and directed upwards from the horizon, we shall have for the three differential equations of the motion of the point G,

$$\left.\begin{aligned} M\frac{d^2x_1}{dt^2} &= R\cos\lambda, \\ M\frac{d^2y_1}{dt^2} &= R\cos\mu, \\ M\frac{d^2z_1}{dt^2} &= R\cos\nu - Mg, \end{aligned}\right\} \quad (1)$$

$g$ denoting as usual the gravity, and M the mass of the body. If the given plane is fixed, the angles $\lambda$, $\mu$, $\nu$, will be constant and given; if it is in motion, we shall suppose this motion to be known, and that it cannot be modified by that of the body; $\lambda$, $\mu$, $\nu$, will then be given functions of $t$. As the moveable is a heavy body, in order that it may never be detached from this plane, whether fixed or in motion, it should be always situated above it, consequently, $R\cos\nu$ the vertical component of the resistance of the plane will be always positive, and $\nu$ will be always an acute angle; the other two given angles $\lambda$ and $\mu$ may be either acute or obtuse. At the same time, the body in virtue of the forces R and M$g$, applied to the points K and G, will turn about G as about a fixed point (No. 438); but as the weight M$g$ will not influence this motion of rotation, the differential equations of this motion will depend solely on the force R. In order to obtain them, we shall take for the second members of equations (a) of No. 412, the moments of the force R with respect to the three principal axes of the moveable, which intersect at the point G, multiplied respectively by $dt$. Denoting the coordinates of the point K referred to these axes, by $a$, $\beta$, $\gamma$, and the angles which the direction of the force R makes with lines drawn parallel to these same axes through K, by $\lambda'$, $\mu'$, $\nu'$, these moments will be

$$\begin{gathered} aR\cos\mu' - \beta R\cos\lambda', \\ \gamma R\cos\lambda' - aR\cos\nu', \\ \beta R\cos\nu' - \gamma R\cos\mu', \end{gathered}$$

and equations (a) will become

$$\left.\begin{aligned} \mathrm{C}dr + (\mathrm{B}-\mathrm{A})\,pqdt &= \mathrm{R}\,(a\cos\mu' - \beta\cos\lambda')\,dt, \\ \mathrm{B}dq + (\mathrm{A}-\mathrm{C})\,rpdt &= \mathrm{R}\,(\gamma\cos\lambda' - a\cos\nu')\,dt, \\ \mathrm{A}dp + (\mathrm{C}-\mathrm{B})\,qrdt &= \mathrm{R}\,(\beta\cos\nu' - \gamma\cos\mu')\,dt\,; \end{aligned}\right\} \quad (2)$$

in which A, B, C, denote the moments of inertia that refer to the same axes respectively, as the angles $\lambda'$, $\mu'$, $\nu'$, and $p$, $q$, $r$, the components of the angular velocity of rotation (No. 407).

To these we must join equations (7) of No. 410, namely,

$$\left.\begin{aligned} pdt &= \sin\theta\sin\phi d\psi - \cos\phi d\theta, \\ qdt &= \sin\theta\cos\phi d\psi + \sin\phi d\theta, \\ rdt &= d\phi - \cos\theta d\psi. \end{aligned}\right\} \quad (3)$$

We shall suppose that the nine cosines $a$, $b$, &c., whose values in functions of $\phi$, $\psi$, $\theta$, have been given in No. 378, are those of the angles which the fixed axes of $x_1$, $y_1$, $z_1$, the coordinates of the point G, make with lines parallel to the principal axes relative to this point, drawn through the origin of these coordinates; then it is evident from equation (2) of No. 9, that

$$\left.\begin{aligned} \cos\lambda' &= a\cos\lambda + a'\cos\mu + a''\cos\nu, \\ \cos\mu' &= b\cos\lambda + b'\cos\mu + b''\cos\nu, \\ \cos\nu' &= c\cos\lambda + c'\cos\mu + c''\cos\nu, \end{aligned}\right\} \quad (4)$$

will be the values of $\cos\lambda'$, $\cos\mu'$, $\cos\nu'$, which should be substituted in equations (2).

The position of the moveable at any instant whatever, with respect to the fixed planes of the axes of $x_1$, $y_1$, $z_1$, will be completely determined by means of these coordinates and of the three angles $\phi$, $\psi$, $\theta$, already defined (No. 378); the position of the instantaneous axis of rotation in the interior of the moveable, and its velocity about this axis will depend, moreover, on the three quantities $p$, $q$, $r$: the solution of the problem will, therefore, consist in our deducing from the nine equations (1), (2), (3), the values of these nine unknown quantities in functions of $t$; but as these equations contain

orce R, and the three coordinates $\alpha, \beta, \gamma$, which are also own, four more equations are required, which may be ned in the following manner.

48. Let L be a given function of $\alpha, \beta, \gamma$, and let us re-ent the equation of the surface of the moveable, referred s principal axes which pass through the point G, by $L = 0$. e make

$$V = \left[\left(\frac{dL}{d\alpha}\right)^2 + \left(\frac{dL}{d\beta}\right)^2 + \left(\frac{dL}{d\gamma}\right)^2\right]^{-\frac{1}{2}};$$

hall have (No. 21),

$$\cos\lambda' = V\frac{dL}{d\alpha}, \quad \cos\mu' = V\frac{dL}{d\beta}, \quad \cos\nu' = V\frac{dL}{d\gamma}, \qquad (5)$$

hich the sign of the quantity V must be such, that the les $\lambda', \mu', \nu'$, may respect the interior part of the normal to surface of the moveable, which will be the superior part he normal to the given plane. As one of these equations be deduced from the two remaining, by substituting for-ae (4), in place of $\cos\lambda'$, $\cos\mu'$, $\cos\nu'$, the resulting equa-s, together with the equation $L = 0$, will furnish three he four required equations.

If $x, y, z$, be the coordinates of any point of the given plane, rred to the same fixed axes as $x_1, y_1, z_1$, and $\lambda, \mu, \nu$, the les which the normal to this plane makes with these axes, shall obtain, for its equation,

$$x\cos\lambda + y\cos\mu + z\cos\nu = \zeta;$$

eing a given quantity, which will be constant when the en plane is fixed, and, generally, a given function of $t$. oreover, if $x, y, z$, are the coordinates of the point K, which sts in this plane, we shall have, by the formula of No. 377,

$$\left.\begin{aligned} x &= x_1 + a\alpha + b\beta + c\gamma, \\ y &= y_1 + a'\alpha + b'\beta + c'\gamma, \\ z &= z_1 + a''\alpha + b''\beta + c''\gamma. \end{aligned}\right\} \qquad (6)$$

The values of $x$, $y$, $z$, should therefore satisfy the preceding equation; by substituting them in this equation, there will result($a$) in consequence of formulæ (4),

$$x_1 \cos\lambda + y_1 \cos\mu + z_1 \cos\nu + a\cos\lambda' + \beta\cos\mu' + \gamma\cos\nu' = \zeta, \quad (7)$$

which will be the fourth equation that is requisite to determine the unknown quantities of No. 447.

If the moveable be terminated by a point, and always touches the given plane with its extremity, $a$, $\beta$, $\gamma$, the coordinates of the point K, will be constant, and they can be determined by means of the position of this point on the surface, and its distances from the planes of the principal axes of the moveable, which intersect at the point G. But equations (5) will not obtain in a point of this nature; however, equation (7), which expresses that this point appertains to the given plane, will always subsist; and it is only necessary to join it to the equations of the preceding number, in order to determine the nine unknown quantities of the problem, and the magnitude of the force R which these equations contain($b$).

449. When the given plane is fixed and horizontal, by assuming it for that of the coordinates $x$ and $y$, there will result

$$\cos\lambda = 0, \quad \cos\mu = 0, \quad \cos\nu = 1, \quad \zeta = 0;$$

which will reduce formulæ (4) to

$$\cos\lambda' = a'', \quad \cos\mu' = b'', \quad \cos\nu' = c''.$$

In virtue of the two first equations (1), the horizontal motion of the point G will be uniform and rectilineal; its velocity parallel to the given plane, will depend($c$) on the horizontal percussion which the moveable experiences at the commencement of the motion.

The third equation (1) will give

$$R = M\left(\frac{d^2 z_1}{dt^2} + g\right);$$

by means of which the value of R will be known, when that of $z_1$ shall have been determined. At the same time, equations (2) will become

$$\left.\begin{aligned} \text{C}\,dr + (\text{B} - \text{A})\,pqdt &= \text{M}\left(\frac{d^2z_1}{dt^2} + g\right)(\alpha b'' - \beta a'')\,dt, \\ \text{B}dq + (\text{A} - \text{C})\,rpdt &= \text{M}\left(\frac{d^2z_1}{dt^2} + g\right)(\gamma a'' - \alpha c'')\,dt, \\ \text{A}dp + (\text{C} - \text{B})\,qrdt &= \text{M}\left(\frac{d^2z_1}{dt^2} + g\right)(\beta c'' - \gamma b'')\,dt; \end{aligned}\right\} \quad (8)$$

and equation (7) will be changed into the following, viz.:

$$z_1 + a''\alpha + b''\beta + c''\gamma = 0, \quad (9)$$

from which the value of $z_1$ may be deduced, in order to substitute it in the preceding equations.

Thus, in this case, the problem will depend on equations (3) and (8), by means of which, $p, q, r, \phi, \psi, \theta$, can be determined in functions of $t$, as in the motion of solid body about a fixed point. If the position of the point K changes on the surface of the moveable, the quantities $\alpha, \beta, \gamma$, should be eliminated from these equations by means of $\text{L} = 0$, and formulæ (5), which will in this case be

$$a'' = \text{V}\frac{d\text{L}}{d\alpha}, \quad b'' = \text{V}\frac{d\text{L}}{d\beta}, \quad c'' = \text{V}\frac{d\text{L}}{d\gamma}. \quad (10)$$

If, on the contrary, the position of the point K on the surface of the moveable is always the same, the constant and given coordinates of this point should be substituted in equations (8), in place of $\alpha, \beta, \gamma$. This second case, is that of the motion of a top on a horizontal plane, in which the friction of the point K against this plane is not taken into account.

In the state of equilibrium of a heavy body on a fixed horizontal plane, the line GK will be vertical; if this state is stable, when the moveable is caused to deviate from it by a small quantity, and is then remitted to itself, it will make very small

oscillations, which we may determine to any degree of approximation we please, by means of the preceding equations. We shall restrict ourselves here to point out this example, as an exercise of the calculus. We may suppose, for greater clearness, and in order to simplify the question, that the moveable is a homogeneous ellipsoid, or a sphere whose centre of gravity does not coincide with the centre of figure.

450. It is easy to obtain the two first integrals of equations (8); for by multiplying them by $c''$, $b''$, $a''$, respectively, and then adding them together, their second members disappear, and, if we then integrate, we obtain, as in No. 415,

$$\text{A}a''p + \text{B}b''q + \text{C}c''r = l;$$

$l$ being an arbitrary constant, which expresses the sum of the moments of the quantities of motion of all the points of the body, with respect to a vertical axis passing through the point G.

In order to obtain a second integral of these same equations (8), let them be multiplied by $r, q, p$, respectively, and then added together, this gives

$$\text{C}rdr + \text{B}qdq + \text{A}pdp = \text{M}\left(\frac{d^2z_1}{dt^2} + g\right)\left[\alpha\,(b''r - c''q)\right.$$
$$\left. + \beta\,(c''p - a''r) + \gamma\,(a''q - b''p)\right]dt,$$

which, in consequence of the three last equations (8) of No. 411, is equal to

$$\text{C}rdr + \text{B}qdq + \text{A}pdp = \text{M}\left(\frac{d^2z_1}{dt^2} + g\right)(\alpha da'' + \beta db'' + \gamma dc'').$$

By differentiating equation (9) with respect to $t$, we obtain

$$dz_1 + \alpha da'' + \beta db'' + \gamma dc'' = -\,(a''d\alpha + b''d\beta + c''d\gamma).$$

Now, when K is always the same point of the surface of the moveable, the second member of this equation is cipher, because its coordinates $\alpha, \beta, \gamma$, are in this case constant. It

is also cipher, when K is displaced on *this surface*, for, in virtue of equations (10), we have

$$a''da + b''d\beta + c''d\gamma = \mathrm{V}\left(\frac{d\mathrm{L}}{da}da + \frac{d\mathrm{L}}{d\beta}d\beta + \frac{d\mathrm{L}}{d\gamma}d\gamma\right) = \mathrm{V}d\mathrm{L};$$

which is cipher, because $\mathrm{L} = 0$, during the continuance of the motion, and, therefore, $d\mathrm{L} = 0$. Consequently, in these two cases, we shall have

$$ada'' + \beta db'' + \gamma dc'' = -dz_1;$$

hence there results,

$$\mathrm{C}rdr + \mathrm{B}qdq + \mathrm{A}pdp + \mathrm{M}\left(\frac{d^2z_1}{dt^2} + g\right)dz_1 = 0,$$

and, by integrating,

$$\mathrm{C}r^2 + \mathrm{B}q^2 + \mathrm{A}p^2 + \mathrm{M}\left(\frac{dz_1^2}{dt_1^2} + 2gz_1\right) = h;$$

$h$ being an arbitrary constant.

These two integrals will be sufficient to enable us to resolve the problem, when the question is respecting a homogeneous solid, terminated by a surface of revolution; this is what has place, for example, in the case of a top. The axis of figure is the right line GK; and if C be the moment of inertia with respect to this axis, we shall have

$$\mathrm{B} = \mathrm{A}, \quad a = 0, \quad \beta = 0;$$

$\gamma$ will be the length of GK; and, from equation (9) and $c'' = \cos\theta$, we shall have

$$z_1 = -\gamma\cos\theta$$

for the value of the vertical ordinate of the point G. The first equation (8) will give $r = n$, $n$ being an arbitrary constant, denoting the velocity of rotation of the moveable about its axis of figure. But, by the values of $a''$ and $b''$ (No. 378), and the two first equations (3), we have

$$a''p + b''q = -\sin^2\theta\frac{d\psi}{dt}, \quad p^2 + q^2 = \sin^2\theta\frac{d\psi^2}{dt^2} + \frac{d\theta^2}{dt^2};$$

consequently, the two integrals which we have obtained will become

$$\text{C}n\cos\theta - \text{A}\sin^2\theta\frac{d\psi}{dt} = l,$$

$$\text{A}\left(\sin^2\theta\frac{d\psi^2}{dt^2}+\frac{d\theta^2}{dt^2}\right)+\text{M}\left(\gamma^2\sin^2\theta\frac{d\theta^2}{dt^2}-2\gamma g\cos\theta\right)=h,$$

in which $-\text{C}n^2$ is comprised in the constant quantity $h$ (*d*).

These two last equations will make known, by means of elliptic functions, the values of $\psi$ and $t$ in functions of $\theta$; the third equation (3) will then give the value of $\phi$; and the problem will be resolved in the same manner as that of No. 425, the solution of which has been already given in detail.

451. The moveable being always a homogeneous solid of revolution terminated by a point, and K, the extremity of this point, being constantly supposed to touch the given plane, let now this plane be in motion, so that the angles $\lambda$, $\mu$, $\nu$, and the quantity $\zeta$ may be given functions of $t$. The moment of inertia belonging to the axis of figure being C, the two other moments A and B will be equal, the coordinates $\alpha$ and $\beta$ will be cipher, and $\gamma$ will express the length of GK. The first equation will in this case give $r=n$, $n$ denoting an arbitrary constant; so that the angular velocity of the moveable about the axis of figure will be constant, as in the case when the plane is fixed. And, in consequence of formulæ (4), the two other equations (2) will become

$$\left.\begin{aligned}\text{A}dq+(\text{A}-\text{C})npdt &= \text{R}\gamma(a\cos\lambda+a'\cos\mu+a''\cos\nu)\,dt,\\ \text{A}dp-(\text{A}-\text{C})nqdt &= -\text{R}\gamma(b\cos\lambda+b'\cos\mu+b''\cos\nu)\,dt.\end{aligned}\right\}\quad(11)$$

In like manner equation (7) will become

$$x_1\cos\lambda+y_1\cos\mu+z_1\cos\nu+\gamma(c\cos\lambda+c'\cos\mu+c''\cos\nu)=\zeta;\quad(12)$$

and equations (1) and (3) will not undergo any change. The system of equations (1), (3), (11), (12) should therefore enable us to determine the nine coefficients $p$, $q$, $\phi$, $\psi$, $\theta$,

$x$, $y$, $z$, R; but the rigorous integration of these equations is impossible in the present state of the science; and, in order to obtain approximate values of the unknown, which are not very complicated, we are obliged to restrict the generality of the question by different hypotheses which we will state according as they will be necessary.

452. We shall suppose, in the first place, that the rotation of the moveable about its axis of figure is very rapid, and that the different motions of the given plane are very slow relatively to this rotation; so that if, for example, the perpendicular to the given plane erected at the point K, oscillates on each side of the vertical which passes through this point, or turns about this line, the duration of each oscillation or each revolution may be very great, relatively to a revolution of the moveable about the axis KG; and that the same will be the case, if the given plane performs oscillations parallel to itself.

We shall suppose, secondly, that the angles $\theta$ and $\psi$ vary very slowly with respect to the angle $\phi$; a supposition which should be confirmed *à posteriori*, by the values which will be obtained for these three angles.

If in a first approximation, the differential $d\psi$ which occurs in the third equation (3) be neglected, and if we suppose that $\phi = 0$ when $t = 0$, we shall have $\phi = nt$, at any instant whatever.

By means of the formulæ of No. 378, we shall then have

$$a\cos\lambda + a'\cos\mu + a''\cos\nu = \text{P}\sin nt + \text{Q}\cos nt,$$
$$b\cos\lambda + b'\cos\mu + b''\cos\nu = \text{P}\cos nt - \text{Q}\sin nt;$$

in which, for the sake of abridging, we make($c$)

$$\text{P} = \cos\lambda\cos\theta\sin\psi + \cos\mu\cos\theta\cos\psi - \cos\nu\sin\theta,$$
$$\text{Q} = \cos\lambda\cos\psi - \cos\mu\sin\psi;$$

and equations (11) will become

$$\text{A}dq + (\text{A}-\text{C})npdt = \text{R}\gamma(\text{P}\sin nt + \text{Q}\cos nt)dt,$$
$$\text{A}dp - (\text{A}-\text{C})nqdt = \text{R}\gamma(\text{Q}\sin nt - \text{P}\cos nt)dt.$$

Now, in the second hypothesis, the quantities P and Q will vary very slowly, this will be also the case with respect to R, as we shall see immediately; therefore in the integration of these equations, we may consider, in a first approximation, that P, Q, R are constant quantities, and, consequently, only take into account the variation of $\sin nt$ and $\cos nt$ in their second members. If $m$ be a very small fraction of $n$, and if the coefficient of $\sin nt$ contains, for example, a term which has $\cos mt$ for a factor, $\sin nt$ should be replaced by $\frac{1}{2}\sin(n+m)t + \frac{1}{2}\sin(n-m)t$; hence as the coefficient of $\sin nt$ is considered as constant, $mt$ must be neglected with respect to $nt$, at least in the first approximation(*f*).

In this manner, the complete integrals of the preceding equations will be(*g*)

$$p = \text{D}\sin\left(\frac{\text{A}-\text{C}}{\text{A}}\right)nt + \text{E}\cos\left(\frac{\text{A}-\text{C}}{\text{A}}\right)nt - \frac{\text{R}\gamma}{\text{C}n}(\text{Q}\cos nt + \text{P}\sin nt),$$

$$q = \text{D}\cos\left(\frac{\text{A}-\text{C}}{\text{A}}\right)nt - \text{E}\sin\left(\frac{\text{A}-\text{C}}{\text{A}}\right)nt + \frac{\text{R}\gamma}{\text{C}n}(\text{Q}\sin nt - \text{P}\cos nt),$$

D and E being two arbitrary constants. In order to determine them, let the instantaneous axis of rotation coincide with the axis of figure at the commencement of the motion; we shall then have $p = 0$ and $q = 0$, when $t = 0$; and if P′, Q′, R′, be the values of P, Q, R, at this same epoch, there will result

$$\text{E} = \frac{\gamma \text{R}'\text{Q}'}{\text{C}n}, \quad \text{D} = \frac{\gamma \text{R}'\text{P}'}{\text{C}n}.$$

Therefore, at any instant whatever, we shall have

$$p = \frac{\gamma}{\text{C}n}\left[\text{R}'\left(\text{P}'\sin\left(\frac{\text{A}-\text{C}}{\text{A}}\right)nt + \text{Q}'\cos\left(\frac{\text{A}-\text{C}}{\text{A}}\right)nt\right.\right.$$
$$\left.\left. - \text{R}\left(\text{P}\sin nt + \text{Q}\cos nt\right)\right)\right],$$

$$q = \frac{\gamma}{\text{C}n}\left[\text{R}'\left(\text{P}'\cos\left(\frac{\text{A}-\text{C}}{\text{A}}\right)nt - \text{Q}'\sin\left(\frac{\text{A}-\text{C}}{\text{A}}\right)nt\right.\right.$$
$$\left.\left. - \text{R}\left(\text{P}\cos nt - \text{Q}\sin nt\right)\right)\right].$$

If we suppose, that in the two first equations (3), $\phi = nt$, we obtain from them($h$)

$$\sin\theta d\psi = (p \sin nt + q \cos nt)\, dt,$$
$$d\theta = (q \sin nt - p \cos nt)\, dt;$$

and by substituting for $p$ and $q$ their values given above, there results

$$\sin\theta d\psi = \frac{\gamma dt}{\mathrm{C}n}\left[\mathrm{R}'\left(\mathrm{P}' \cos\frac{\mathrm{C}nt}{\mathrm{A}} + \mathrm{Q}' \sin\frac{\mathrm{C}nt}{\mathrm{A}}\right) - \mathrm{RP}\right],$$
$$d\theta = \frac{\gamma dt}{\mathrm{C}n}\left[\mathrm{R}'\left(\mathrm{P}' \sin\frac{\mathrm{C}nt}{\mathrm{A}} - \mathrm{Q}' \cos\frac{\mathrm{C}nt}{\mathrm{A}}\right) + \mathrm{RQ}\right].$$

Now, if C is not a very small fraction of A, it is necessary, in order that $\theta$ and $\psi$ may vary very slowly, as has been supposed, that the terms depending on $\sin\frac{\mathrm{C}nt}{\mathrm{A}}$ and $\cos\frac{\mathrm{C}nt}{\mathrm{A}}$ should disappear from these formulæ; which condition will be always fulfilled, by supposing that at the commencement of the motion, RG the axis of figure coincides with the perpendicular to the given plane.

In fact, if at the commencement of the motion, $\varepsilon$ is the angle that the perpendicular to the given plane makes with the vertical, and $\varepsilon'$ the angle that its horizontal projection makes with the line from which the angle $\psi$ is measured; at this epoch, we shall have (No. 8),

$$\cos\nu = \cos\varepsilon, \quad \cos\mu = \sin\varepsilon \cos\varepsilon', \quad \cos\lambda = \sin\varepsilon \sin\varepsilon',$$

and, if $\psi'$ and $\theta'$ be the initial values of $\psi$ and $\theta$, there will result($i$)

$$\mathrm{P}' = \cos\theta' \sin\varepsilon \cos(\varepsilon' - \psi') - \sin\theta' \cos\varepsilon,$$
$$\mathrm{Q}' = \sin\varepsilon \sin(\varepsilon' - \psi').$$

Now, if the axis of figure was perpendicular to the given plane, when the motion commenced, we shall have $\psi' = \varepsilon'$, and $\theta' = \varepsilon$, hence there will result $\mathrm{P}' = 0$, $\mathrm{Q}' = 0$; in consequence of which, the preceding formulæ are reduced to

$$\sin\theta d\psi = -\frac{\text{R}\text{P}\gamma dt}{cn}, \quad \theta = \frac{\text{R}\text{Q}\gamma dt}{cn}. \tag{13}$$

453. Now, it is necessary still to suppose that the perpendicular to the given plane and the axis of figure of the moveable, deviate very little from the vertical direction during the continuance of the motion, the supplement of the angle $\theta$ will, consequently, be constantly very small; for $\theta$ is the obtuse angle comprised between the vertical drawn through the point G upwards, and the line drawn from G towards the point K, which is below G. We shall, therefore, neglect the square of $\sin\theta$, and assume $\cos\theta = -1(k)$. The quantities $\cos\lambda$, $\cos\mu$, will be very small, and if their squares be neglected we shall have $\cos\nu = 1(l)$. Moreover, we have (No. 378)

$$c = \sin\theta\sin\psi, \quad c' = \sin\theta\cos\psi, \quad c'' = \cos\theta = -1.$$

Hence, if the products $\sin\theta\cos\lambda$, $\sin\theta\cos\mu$ be neglected, equation (12) will give

$$z_1 = \gamma + \zeta - x_1\cos\lambda - y_1\cos\mu.$$

If, therefore, independently of the smallness of $\cos\lambda$ and $\cos\mu$, the vertical oscillations of the given plane are likewise supposed to be very small, the variations of $z_1$ will be so likewise; the value of R furnished by the third equation (1), namely

$$\text{R} = \text{M}g + \text{M}\frac{d^2z_1}{dt^2},$$

will differ very little from M$g$; and if the products of $\frac{d^2z_1}{dt^2}$, and of each of the quantities $\cos\lambda$, $\cos\mu$, $\sin\theta$, be neglected, it will be sufficient to substitute M$g$ in place of R, in equations (13). Also, by substituting the values of P and Q, these equations will become$(m)$

$$\sin\theta d\psi = \frac{\text{M}\gamma g}{cn}(\sin\theta - \cos\lambda\sin\psi - \cos\mu\cos\psi)\,dt,$$

$$d\theta = \frac{\text{M}\gamma g}{cn}(\cos\lambda\cos\psi - \cos\mu\sin\psi)\,dt.$$

The two first equations (1) will become, at the same time,

$$\frac{d^2x_1}{dt^2} = g\cos\lambda, \quad \frac{d^2y_1}{dt^2} = g\cos\mu. \tag{14}$$

By differentiating the values of $c$ and $c'$ with respect to $t$, and substituting $d\theta$ in place of $d.\sin\theta$, we obtain

$$dc = \sin\psi d\theta + \cos\psi\sin\theta d\psi,$$
$$dc' = \cos\psi d\theta - \sin\psi\sin\theta d\psi;$$

and if, in these formulæ, there be substituted in place of $\sin\theta d\psi$ and $d\theta$, their values given above, there results

$$\left.\begin{aligned} dc - c'mdt &= -m\cos\mu dt, \\ dc' + cmdt &= m\cos\lambda dt, \end{aligned}\right\} \tag{15}$$

in which, for the sake of abridging, we make $\frac{\mathrm{M}\gamma g}{\mathrm{C}n} = m$. Thus the question is finally reduced to the integration of these linear equations which are of the first order with respect to the unknown quantities $c$ and $c'$, and have constant coefficients. It may be observed, that it was by employing these same unknown quantities, namely, $\sin\theta\sin\psi$ and $\sin\theta\cos\psi$, in the problem of the motion of the moon about its centre of gravity, that Lagrange succeeded in reducing the differential equations of this problem to the linear form; and this enabled him completely to explain the phenomenon of *libration*, which he was not able to accomplish in his first investigations on this subject.

454. By integrating equations (15) in the usual manner, we obtain, by substituting for $c$ and $c'$ what these letters denote($n$),

$$\left.\begin{aligned} \sin\theta\sin\psi = {} & h\sin mt + h'\cos mt \\ & - m\sin mt\int(\cos\mu\sin mt - \cos\lambda\cos mt)\,dt \\ & - m.\cos mt\int(\cos\mu\cos mt + \cos\lambda\sin mt)\,dt, \\ \sin\theta\cos\psi = {} & h\cos mt - h'\sin mt \\ & - m\cos mt\int(\cos\mu\sin mt - \cos\lambda\cos mt)\,dt \\ & + m.\sin mt\int(\cos\mu\cos mt + \cos\lambda\sin mt)\,dt, \end{aligned}\right\} \tag{16}$$

in which $h$, $h'$, denote two arbitrary constants.

If the integrals indicated in these formulæ be supposed to commence with $t$, and if $\theta$ and $\psi'$ denote, as before, the initial values of $\theta$ and $\psi$, we shall obtain, by making $t = 0$,

$$k = \sin\theta' \cos\psi', \quad k' = \sin\theta' \sin\psi',$$

for the values of $k$ and $k'$, which will be cipher, when the axis of the figure of the moveable is vertical at the commencement of the motion, in which case we shall have $\theta' = 0$.

When the actual values of $\cos\lambda$ and $\cos\mu$ are given in functions of $t$, by performing the requisite integrations, equations (16) will make known the values of $\theta$ and $\psi$, and consequently, the position of the axis of figure of the moveable at any instant whatever. At the same time, we shall have, by the third equation (3),

$$\phi = nt - \psi + \psi',$$

by making $\cos\theta = -1$, and assuming $\phi = 0$, when $t = 0$. The two first equations (3), in which $\phi$ is made simply equal to $nt$, will give the values of $p$ and $q$, which, with the equation $r = n$, will enable us to determine the axis of rotation and the angular velocity of the moveable about this axis, at any instant whatever. Finally, the values of $x_1$ and $y_1$ in functions of $t$ can be obtained by two successive integrations of equation (14); from which those of $z_1$ and n may be immediately deduced.

Hence then it appears, that approximate values of all the unknown quantities of the problem can be determined, by means of the given values of $\cos\lambda$ and $\cos\mu$. By means of these approximate values, those of the quantities neglected in this first approximation may be computed; then if in a second approximation, the quantities thus computed be taken into account, we shall arrive at other values of the unknown quantities more accurate than the first, and, by proceeding in this manner, we shall obtain, by the general method of successive approximations, expressions of the unknown quantities, to any degree of accuracy we please. In the preceding articles, the

values of these quantities were not determined beyond a first approximation, there would be no other difficulty attending a second or third approximation, but what arises from their length.

455. $n$ the velocity of rotation, impressed on the moveable about its axis of figure, being supposed to be very great, the quantity $m$ will be generally very small. It follows therefore from formulæ (16), that the variations of $\cos\lambda$ and $\cos\mu$, arising from the small oscillations of the normal to the given plane on which the moveable presses, will be very inconsiderable in the value of $\theta$. Consequently, if the moveable is terminated at its superior part by a plane surface, perpendicular to its axis of figure, and if at the commencement of the motion, this surface was horizontal, and the axis vertical, which would cause the terms multiplied by $h$ and $h'$ to disappear($o$), this surface would sensibly preserve its horizontality during the continuance of the motion, notwithstanding the small oscillations of the given plane, and this so much the more accurately, as the velocity of rotation impressed on the moveable is more considerable. This is the principle of the method which has been proposed for obtaining at sea, independently of the dipping and rolling of the vessel, an artificial horizon which may be used in making astronomical observations.

In consequence of the smallness of $m$, $\sin mt$ and $\cos mt$ may be regarded as constant quantities in the integrals which formulæ (16) contain. Thus, for example, by integrating by parts, we shall obtain ($p$)

$$\int\cos\lambda\cos mtdt = \cos mt\int\cos\lambda dt + m\sin mt\iint\cos\lambda dt^2 + \&c.;$$

and if the variations of $\cos\lambda$ are very rapid relatively to those of $\sin mt$ and $\cos mt$, although very slow with respect to the rotation of the moveable, this series will be very convergent, and may be reduced to its first term; this comes to considering $\cos mt$ as constant in the integral $\int\cos\lambda\cos mtdt$.

In this way, and by supposing $h, h'$ respectively equal to cipher, formulæ (16) will be reduced to

$$\sin\theta \sin\psi = -m\int \cos\mu dt, \quad \sin\theta \cos\psi = m\int \cos\lambda dt.$$

Likewise, if the centre of gravity of the moveable has not received any horizontal velocity at the commencement of the motion, so that $\frac{dx_1}{dt} = 0$, $\frac{dy_1}{dt} = 0$, when $t = 0$, equations (14) will give

$$\frac{dx_1}{dt} = g\int \cos\lambda dt, \quad \frac{dy_1}{dt} = g\int \cos\mu dt;$$

in which the integrals commence with $t$, as in the preceding equations. Therefore by eliminating them, and substituting for $m$ its value, we shall have

$$\sin\theta \sin\psi = -\frac{\text{M}\gamma}{\text{C}n}\,\frac{dy_1}{dt}, \quad \sin\theta \cos\psi = \frac{\text{M}\gamma}{\text{C}n}\,\frac{dx_1}{dt}.$$

The difference of sign in these two formulæ arises from this, that as the angle $\psi$ is increased by 90°, the axis of the positive abscissæ falls on the axis of the negative ordinates, and this last, on the axis of the positive abscissæ (No. 378); so that by putting $\psi + 90°$ for $\psi$ in the first formula, we must, at the same time, change $y_1$ into $-x_1$; which gives the second formula.

By dividing these equations the one by the other, there results

$$\text{tang.}\,\psi = -\frac{dy_1}{dx_1}.$$

Now, if through the point G a plane be drawn perpendicular to GK the axis of figure, $\psi$ is the angle, which the intersection of this equator of the moveable, and of the horizontal plane of the axes of $x_1$ and $y_1$, makes with the axis of $x_1$; moreover, the variables $x_1$ and $y_1$ are the coordinates of the projection of the point G on this horizontal plane; it follows therefore that this intersection is constantly parallel to the tangent to the curve described by the horizontal projection of the centre of gravity of the moveable. If the horizontal velocity of this point be denoted by $u$, so that we may have

$$u^2 = \frac{dx_1^2 + dy_1^2}{dt^2},$$

we shall also have (q)

$$\sin\theta = \frac{\text{M}\gamma u}{\text{C}n},$$

for the sine of the inclination of the moveable on the horizontal plane, which inclination is the supplement of the angle $\theta$.

These different formulæ, and the consequences which may be deduced from them, will subsist as long as $n$ the velocity of rotation of the moveable about its axis of figure is very great; but the resistance of the air and the friction of the point K against the plane on which the moveable presses, will continually diminish this velocity, and when it ceases to be very great, the axis GK will deviate more and more from the vertical direction, and the moveable will at length fall on this plane, as in the case of a common spinning top.

It should likewise be observed, that the vertical oscillations of the given plane, from which the reciprocal variations of the quantity $\zeta$ arise, do not affect the variations of the angles $\psi$ and $\theta$. But if the given plane rises or falls vertically with a motion uniformly accelerated, the quantity $\zeta$ which is comprised in its equation (No. 448), will contain a term $\pm \frac{1}{2} g't^2$, in which $g'$ denotes a positive constant quantity. The value of R made use of in No. 453, instead of being M$g$, will be then M$(g \pm g')$, therefore we should substitute $(g \pm g')$ for $g$ in the value of $m$; so that a motion of this kind will influence the variations of $\theta$ and $\psi$. If the given plane descends, in which case $g'$ should be affected with a negative sign, it is necessary that $g'$ should be less than $g$, otherwise the value of R would become negative, which implies that the moveable ceased to press on the given plane, which would fall quicker than this heavy body, and must consequently be detached from it.

## II. *Case in which the Friction is taken into Account.*

456. In the present state of the science, the laws of the friction of bodies in motion, can only be determined by experiment; therefore, previously to our taking into account this force, which is of a peculiar kind, in the equations of the motion of a body which presses on a given plane, it is necessary to state the general results, which have been furnished by observation on this subject.

1st. The friction of one solid body on another is independent of the velocity of the moveable.

2ndly. It is also independent of the extent of surface of the rubbing body.

3rdly. It is proportional to the total pressure exerted on this surface.

These two last laws also obtain in a state of rest (No. 269), at the instant that the equilibrium gives way, and when the contact of the bodies has continued sufficiently long to enable the friction to attain its *maximum*.

When a fluid flows over a solid body, the laws of its friction are different. For in this case it appears from experiment, that it is proportional to the velocity of the fluid and to the extent of surface, and that it does not depend on the pressure. In the case of an aeriform fluid, there is reason to suppose that the friction increases or diminishes with the density, as has been assumed in No. 444; so that at equal temperatures, it is found to depend indirectly on the magnitude of the pressure.

457. Let us now suppose that a solid body, whose base is a plane surface of any extent whatever, is laid on a fixed horizontal plane, and that the vertical drawn through its centre of gravity $a$ (fig. 13) meets the fixed plane within the area of this base, which we know is the condition that is necessary and sufficient to insure the equilibrium. Let its mass be de-

noted by M, and its weight by P. To any point A of its surface which is situated in the horizontal plane drawn through the point G, let a cord be attached which may pass over the fixed pulley B, in such a manner that BA may be the production of GA. At C the other extremity of this cord, let another body whose mass is denoted by M′, and weight by P′, be suspended, in such a manner that G′ its centre of gravity may exist on the vertical drawn through the point C.

The equilibrium will subsist as long as the weight P′, increased by the weight of the vertical part of the cord, is less than the friction of M on the fixed plane, and of the cord on the pulley B; and if P′ be gradually increased, the equilibrium will give way, at the instant that P′ surpasses the sum of these frictions, diminished by the weight of the vertical cord. If this last quantity be neglected relatively to P′, and if the frictions of M against the fixed plane, and of the cord against the fixed pulley, which obtain immediately before the equilibrium gives way, be denoted by F and F′ respectively, we shall have

$$P' = F + F'.$$

As the pressure exerted on the base of M is P the weight of this body, the friction F is proportional to P. Also, by what has been observed in No. 302, the friction F′ is proportional to the force F; consequently, we have

$$F = fP, \quad F' = f'F,$$

$f$ and $f'$ denoting two fractions independent of the magnitudes of P and F. By means of these values, the preceding equation becomes

$$P' = fP + f'fP;$$

from which we obtain

$$f = \frac{1}{1+f'} \cdot \frac{P'}{P}.$$

As the value of the weight P′, at the instant that the equilibrium gives way, can be known, this equation will determine

the value of $f$ with respect to the body M and to the horizontal plane on which it is laid, when the value of $f'$ with respect to the cord and the throat of the pulley is known. The means pointed out in No. 269 will make known this value of $f$, independently of any other quantity of the same nature, when the angle at which the moveable *commences* to slide down a plane, that is gradually inclined, is known.

458. At the instant that the weight P′, increased by the weight of the vertical cord, exceeds ever so little the quantity F + F′, the equilibrium will give way, and *à fortiori* when the weight P′ is still greater. The body M will slide on the horizontal plane, and M′ will descend vertically. If the friction of M against this plane, during the motion, be denoted by H, it will be a given fraction of the pressure P. With respect to the pulley B, we may either suppose that it is entirely fixed, and remains immoveable, or, that it turns about an horizontal axis, perpendicular to the plane of the cord ABC. In the first case, there will be, during the motion, a certain friction against the pulley, which will be different from F′, and which should be added to H; in the second case, if the cord does not slide on the pulley, there will be no friction of the one against the other; but the motion communicated to the pulley, by the intervention of the cord, should be taken into account, as if it was attached to it. We shall suppose that the second of these two cases is the one that has place.

In order to obtain the equation of motion, let the horizontal and vertical parts of the cord ABC at the end of any time $t$, be denoted by $z$ and $z'$ respectively, and their masses by $\mu$ and $\mu'$. The velocities of M and M′ at this instant, will be $-\frac{dz}{dt}$ and $\frac{dz'}{dt}$; and the resistances of the air exerted against their surfaces, may be denoted by $a\frac{dz^2}{dt^2}$ and $a'\frac{dz'^2}{dt'^2}$, $a$ and $a'$ being two constant quantities, depending on their form and extent. Therefore, if the gravity be denoted by $g$, the motive forces applied to the system will be the weight $(M' + \mu')g$,

diminished by the resistance $a' \frac{dz'^2}{dt^2}$, and the horizontal force H, increased by the resistance $a \frac{dz^2}{dt^2}$. Their moments with respect to the axis of the pulley B, should be subtracted the one from the other; and if the radius of this circular pulley be denoted by $c$, their difference will be equal to

$$c\left[(\text{M}' + \mu')g - a'\frac{dz'^2}{dt^2} - \text{H} - a\frac{dz^2}{dt^2}\right]. \qquad \text{(a)}$$

The motive forces which actually have place, are $(\text{M}' + \mu')\frac{d^2z'}{dt^2}$ in the vertical direction, $-(\text{M} + \mu)\frac{d^2z}{dt^2}$ in the horizontal direction, and those of all the points of the pulley. As the moments of these last forces with respect to the axis of the pulley, must be added to the vertical motive forces$(r)$, and as the angular velocity of the pulley is $\frac{1}{c}\frac{dz'}{dt}$, if we denote its mass by $m$, and its moment of inertia with respect to this axis by $mk^2$, it follows, from what has been established in No. 392, that the moment of these last forces with respect to this axis, is $\frac{mk^2}{c}\frac{d^2z'}{dt^2}$; consequently, the sum of the moments of the effective forces, with respect to this same axis, will be

$$c\left[\left(\text{M}' + \mu' + \frac{mk^2}{c^2}\right)\frac{d^2z'}{dt^2} - (\text{M} + \mu)\frac{d^2z}{dt^2}\right]. \qquad \text{(b)}$$

Now, in order that the motive forces applied to the system may be in equilibrio, by means of the axis of the pulley, with the effective forces taken in a direction opposite to that in which they act, it is necessary that the two expressions (a) and (b) should be equal to each other; hence there results,

$$\left(\text{M}' + \mu' + \frac{mk^2}{c^2}\right)\frac{d^2z'}{dt^2} - (\text{M} + \mu)\frac{d^2z}{dt^2}$$

$$= (\text{M}' + \mu')g - \text{H} - a'\frac{d^2z'}{dt^2} - a\frac{d^2z}{dt^2}.$$

If the cord ABC be supposed to be inextensible, the sum of its parts will be constant, therefore, if the entire length be denoted by $l$, we shall have

$$z+z'=l,\quad \frac{dz'}{dt}=-\frac{dz}{dt},\quad \frac{d^2z'}{dt^2}=-\frac{d^2z}{dt^2}.$$

Let the weight of the entire cord be denoted by $w$, and the weight of $m$ the mass of the pulley by $p$, we shall likewise have

$$g\mu=\frac{wz}{l},\quad g\mu'=w-\frac{wz}{l},\quad gm=p;$$

we have also

$$g\text{M}=\text{P},\quad g\text{M}'=\text{P}';$$

and since the friction H is proportional to the weight P, we have, besides

$$\text{H}=h\text{P},$$

in which $h$ denotes a constant coefficient independent of P.

By means of these different values, the equation of motion will become($s$)

$$\left(\text{P}+\text{P}'+w+\frac{pk^2}{c^2}\right)\frac{d^2z}{dt^2}=g\left[h\text{P}-\text{P}'-w+\frac{wz}{l}+(a+a')\frac{dz^2}{dt^2}\right].$$

459. It cannot be integrated in a finite form, unless the terms which arise from the resistance of the air be neglected, in which case it is reduced to($t$)

$$\frac{d^2z}{dt^2}-g\frac{\beta}{l}z+ga=0,$$

by making, in order to abridge,

$$\frac{\text{P}'-h\text{P}+w}{\text{P}+\text{P}'+w+\frac{pk^2}{c^2}}=a,\quad \frac{w}{\text{P}+\text{P}'+w+\frac{pk^2}{c^2}}=\beta.$$

Its complete integral will be then

$$z=\text{C}e^{t\sqrt{\frac{\beta g}{l}}}+\text{C}'e^{-t\sqrt{\frac{\beta g}{l}}}+\frac{al}{\beta};$$

C and C′ being two arbitrary constants introduced by the integration, and $e$ the base of the Naperian system of logarithms.

If this value of $z$ be substituted in the terms of the equation of motion, which arise from the resistance of the air, and if this equation be integrated again, a more exact value of $z$ will be obtained; however, as the preceding is sufficiently accurate for our purpose, it is not necessary to continue the approximation farther; and, in order to determine the constants C and C′, if $\gamma$ be the initial value of $z$, we shall have $z = \gamma$ and $\frac{dz}{dt} = 0$ when $t = 0$; hence there results($u$)

$$\mathrm{C} + \mathrm{C}' + \frac{al}{\beta} = \gamma, \quad \mathrm{C} - \mathrm{C}' = 0;$$

from which we obtain

$$\mathrm{C} = \mathrm{C}' = \tfrac{1}{2}\gamma - \frac{al}{2\beta};$$

and, consequently,

$$z = \tfrac{1}{2}\left(\gamma - \frac{al}{\beta}\right)\left(e^{t\sqrt{\frac{\beta g}{l}}} + e^{-t\sqrt{\frac{\beta g}{l}}}\right) + \frac{al}{\beta},$$

at any instant whatever.

If $\theta$ be the time that the weight P takes to reach the pulley B, that is to say, to traverse the distance $\gamma$, we shall have at the same time, $t = \theta$ and $z = 0$; hence we infer

$$(al - \beta\gamma)\left(e^{\theta\sqrt{\frac{\beta g}{l}}} + e^{-\theta\sqrt{\frac{\beta g}{l}}}\right) = 2al.$$

When $\theta$ is given by observation, this equation will enable us to determine the value of $a$, and, consequently, that of $h$ the coefficient relative to the friction of the weight P, when it moves on the horizontal plane. In this experiment, the value of P′ may be assumed to be what we please, provided it exceeds the friction which has place in the state of equilibrium. If the weight $w$ is very small relatively to the weights P and P′, $\beta$ will be a very small fraction, and the exponentials may be de-

veloped into very convergent series, according to the powers of $\beta$. In this manner, we shall have(*v*)

$$\gamma = (al - \beta\gamma)\left(\frac{g\theta^2}{2l} + \frac{\beta g^2\theta^4}{24l^2} + \&c.\right);$$

and if the quantity $\beta$ be entirely neglected, we shall have simply

$$\gamma = \tfrac{1}{2}ga\theta^2;$$

which, in fact, we know ought to be the value of $\gamma$, for the motion of the weight P is then uniformly accelerated.

Whatever be the motion of the system under consideration, the tension of the horizontal part of the cord ABC is constantly equal to the least of the two forces which act at its extremities (No. 352), that is to say, to the friction H increased by the resistance that the air exerts against the surface of M. Therefore, if the resistance of the air is not taken into account, it is constant, and equal to H or *h*P, during the entire continuance of the motion; and its value, measured by the extension of a spring placed in the direction of this cord, will enable us to determine the coefficient *h*.

460. The values of this coefficient may be determined, as has been just observed, either by observing $\theta$, the time that the weight P takes to reach the pulley, or by measuring the tension of the cord. They depend in general on the degree of smoothness of the surfaces which rub against one another, and on the material of which they are composed; they do not vary like those of the coefficient *f* (No. 269), with the time during which the bodies have been in contact, before they slide the one on the other. When they attain to their *maximum*, they always surpass the corresponding values of *h*; so that, in the state of motion, the friction H is always less than the friction denoted by F, that has place at the instant in which the equilibrium gives way; and the tension of the cord, when in motion, is also less than that which obtains at the last instant of the equilibrium.

The weight P being supposed to be at rest, the equilibrium subsists as long as the weight P′ is less than F; but it has been remarked that if P′, although less than F, considerably exceeds H, it is merely sufficient, by means of slight percussions, to agitate the horizontal plane a little, in order that the weight P may commence to move.

When the coefficient $h$ is known, it is easy to determine the motion of the weight P on an inclined plane. Let $i$ denote the inclination of this plane to the horizon; it must always be greater than the angle, at which the equilibrium begins to give way, or, in other words, it must be such that $\tan i > f$ (No. 269). $P\sin i$ and $P\cos i$ will be the components of P, respectively parallel and perpendicular to this plane. The first diminished by the friction H, will be the motive force of the moveable whose mass is M; therefore if $z$ be the space traversed at the end of the time $t$, we shall have

$$M\frac{d^2z}{dt^2} = P\sin i - H.$$

Moreover, as the pressure on this plane is the other component $P\cos i$, we shall also have

$$H = hP\cos i;$$

and as $P = Mg$, the preceding equation will consequently become

$$\frac{d^2z}{dt^2} = (1 - h\cot i)g\sin i;$$

from which it appears that the motion will be uniformly accelerated, and the same as if, there being no friction, the sine of inclination was diminished in the ratio of $1-h\cot i$ to unity. This quantity $1-h\cot i$ is positive, since by hypothesis, we have $h \angle f$ and $f\cot i \angle 1$.

461. As the friction H is proportional to the pressure, and independent of the extent of the rubbing surface, it follows that, the weights P and P′ of No. 457 remaining the same, the

motion of P on the horizontal plane will not be changed, whatever be the extent of its base, provided that it always has the same degree of smoothness. Thus, if this body is a rectangular parallelopiped, consisting of homogeneous matter, all whose faces have the same degree of smoothness, its horizontal motion will be always the same, on whichever of its faces it is placed, and this will be also the case when the body descends down an inclined plane, in which case the weight P′ is not required to set it in motion.

Finally, this proposition, that the friction is independent of the extent and contour of the base of P, and simply proportional to the weight P, implies, that at each point of this base, the friction is proportional to the pressure relative to this point. In fact, if we denote this base by $b$, one of its differential elements by $d\sigma$, and the vertical pressure which this element sustains by $pd\sigma$, $p$ being the pressure referred to the unit of surface, the resultant of the pressures exerted on all the elements of $b$, should reproduce the weight P, applied to the centre of gravity G; we should therefore have

$$\int pd\sigma = \text{P}; \tag{a}$$

and if through the projection of the point G on the fixed plane, two rectangular horizontal axes be drawn, of which one may be, for example, the projection of the line GB, and if $x$ be the distance of $d\sigma$ from this projection, and $y$ its distance from the other axis, we should likewise have

$$\int xpd\sigma = 0, \quad \int ypd\sigma = 0; \tag{b}$$

in both these equations and also in equation (a), the integration must extend to the entire base $b$. This being agreed on, if we suppose that the friction of the element $d\sigma$ on the fixed plane, is proportional to $pd\sigma$ the pressure which it experiences, and equal to $hpd\sigma$, $h$ denoting a coefficient independent of $p$, and relative to the nature of the surface $d\sigma$, we shall have

$$\text{H} = \int hpd\sigma,$$

for the value of the entire friction; and as the frictions of all the elements are parallel to GB the direction of the motion, if $x_1$ be the distance of H their resultant, from the vertical plane passing through the line GB, we shall also have

$$\mathrm{H}x_1 = \int hxpd\sigma.$$

Now, if the base of the moveable has the same degree of smoothness throughout its entire extent, the coefficient $h$ will be constant, and there will result

$$\mathrm{H} = h\int pd\sigma = h\mathrm{P}, \quad \mathrm{H}x_1 = h\int xpd\sigma = 0;$$

consequently, the entire friction will depend solely on the weight P, whatever may be the extent and contour of its base; and since $x_1 = 0(x)$, the direction of this force must exist in the vertical plane passing through the line GB, so that it cannot impress any motion of rotation on the moveable, and its motion must consequently be parallel to this plane, as we have supposed.

When the centres of gravity of the weight P and of the base $b$ are situated on the same perpendicular to this base, as in the case of a prism or vertical cylinder, equations (a) and (b) may be satisfied by supposing that the pressure $p$ is constant and equal to $\frac{\mathrm{P}}{b}$. Conversely, when the value of $p$ is constant, equations (b) give($y$)

$$\int xd\sigma = 0, \quad \int yd\sigma = 0;$$

it is evident from these equations that the centre of gravity of $b$ coincides with the horizontal projection of the point G; consequently, when this condition is not satisfied, the pressure $p$ necessarily varies from one point to another of the base of the moveable. The determination of its value at any point whatever, is then an extremely difficult problem, which cannot be resolved without taking into account the flexibility of the material of which the moveable consists, and also that of the horizontal plane on which it rests, for without this reference,

there would be an apparent indetermination in the value of $p$, as in the case of the problem of No. 270.

If the two centres of gravity be supposed to exist on the same vertical, we shall have at once, whatever the coefficient $k$ may be($z$),

$$p = \frac{P}{b}, \quad H = \frac{P}{b}\int k d\sigma, \quad x_1 \int k d\sigma = \int k x d\sigma.$$

Consequently, the base being supposed to remain the same, the entire friction H will be proportional to the weight P, just as if the degree of smoothness was the same throughout the entire extent of this base; but in general, as we shall have $x_1$ no longer equal to cipher, the direction of the force H will no longer coincide with the horizontal projection of the line GH, so that when the weight P′ draws the weight P on the horizontal plane, the friction will cause the moveable to turn about the vertical, which passes through the centre of gravity G.

462. The weight P being supposed to be laid on a fixed horizontal plane, and the horizontal projection of the point G to coincide with the centre of gravity of the base $b$, if, by any means whatever, there be impressed on this body quantities of motion parallel to the fixed plane, such, however, as not to detach its base from this plane, the moveable will be actuated by two horizontal motions, the one of translation, which will be that of its centre of gravity G, and the other of rotation about the vertical passing through this point. Let us now examine how the friction affects these two different motions.

Let the friction experienced by $d\sigma$ the element of $b$, the direction of which is contrary to that of the velocity of this element, be denoted by $\frac{kP}{b} d\sigma$. Let $r$ be its distance from the axis of rotation, $x$ and $y$ the rectangular coordinates of the centre of gravity of $b$ at the end of any time whatever, referred to fixed axes drawn arbitrarily in the horizontal plane, and let

$\theta$ be the angle which $r$ makes with the production of $x$. The coordinates of $d\sigma$ at the same instant, will be $x + r\cos\theta$ and $y + r\sin\theta$, and if its velocity be denoted by $v$, and the angles which its direction makes with lines drawn parallel to the axes of $x$ and $y$ by $\alpha$ and $\beta$, we shall have

$$\left.\begin{aligned} v\cos\alpha &= \frac{dx}{dt} - r\sin\theta\frac{d\theta}{dt}, \\ v\cos\beta &= \frac{dy}{dt} + r\cos\theta\frac{d\theta}{dt}, \end{aligned}\right\} \quad (1)$$

for the two components of this velocity. Those of the friction will be, at the same time,

$$-\frac{h\text{P}}{b}\cos\alpha d\sigma, \quad -\frac{h\text{P}}{b}\cos\beta d\sigma.$$

Now, since the motion of the centre of gravity is the same, as if, the mass of the moveable being supposed to be concentrated in it, the motive forces of all its points were applied to it parallel to their respective directions, we shall have for the equations of this motion($a'$)

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -\frac{hg}{b}\int\cos\alpha d\sigma, \\ \frac{d^2y}{dt^2} &= -\frac{hg}{b}\int\cos\beta d\sigma, \end{aligned}\right\} \quad (2)$$

in which, as $g$ denotes the gravity, $\frac{\text{P}}{g}$ is equal to the mass M, and the coefficient $h$ is supposed to be constant throughout the extent of $b$.

At the same time, the moveable will turn about the vertical passing through the point G, as if this line was fixed, and the forces which act on this body were not changed. As the moment of the friction of $d\sigma$ is equal to the difference of the moments of its two components, its value will be

$$-\frac{h\text{P}\cos\beta d\sigma}{b}.r\cos\theta + \frac{h\text{P}\cos\alpha d\sigma}{b}r.\sin\theta,$$

the rotation being supposed to take place in the direction in which the angle $\theta$ increases, that is to say, from the axis of the positive $xs$ towards that of the positive $ys$. This being agreed on, if the angular velocity of the moveable at the end of the time $t$ be denoted by $\omega$, and its moment of inertia with respect to the axis of rotation by $\frac{\mathrm{P}k^2}{g}$, we shall have (No. 392)

$$k^2 \frac{d\omega}{dt} = -\frac{hg}{b} \int (\cos\beta \cos\theta - \cos\alpha \sin\theta)\, r d\sigma, \qquad (3)$$

which will be the equation of motion about this axis, and when the integration is performed, we should make $\omega = \frac{d\theta}{dt}$, and extend the integral to $b$ the entire base of the moveable.

As in these equations (2) and (3), the variables $x$, $y$, $\theta$, are not separated, the motions of translation and rotation depend, the one on the other, and, in general, can only be determined by approximation. There are, however, two cases that can be easily resolved, which we now proceed to consider.

463. If G the centre of gravity is at rest, we shall have $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$; and equations (1) will give($b'$)

$$\cos\alpha = -\sin\theta, \quad \cos\beta = \cos\theta, \quad v = r\frac{d\theta}{dt};$$

and, in order that equations (2) may be satisfied, it is necessary that the integrals $\int \sin\theta d\sigma$ and $\int \cos\theta d\sigma$ should be cipher; this depends only on the contour of $b$, and will be the case, for example, when it is symmetrical about the centre of gravity of this base.

Equation (3) will become

$$k^2 \frac{d\omega}{dt} = -\frac{hg}{b} \int r d\sigma;$$

from which we obtain($c'$)

$$\frac{d\omega}{dt} = -\frac{hcg}{bk^2},$$

the constant $\int r d\sigma$ being denoted by $c$. Therefore, the motion of rotation will be uniformly retarded; and if the initial angular velocity be denoted by $\Omega$, we shall have, at any instant whatever,

$$\omega = \Omega - \frac{hcgt}{bk^2};$$

from which it appears, that this motion will cease after the lapse of a time expressed by $\frac{bk^2\Omega}{hcg}$, for which we have $\omega = 0$, and the body will be at rest.

On the other hand, when the velocity of rotation is very slow with respect to that of the motion of translation, equations (1) will give (the square of $\frac{rd\theta}{dt}$ being neglected)

$$v^2 = u^2 - 2\left(\frac{dx}{dt}\sin\theta - \frac{dy}{dt}\cos\theta\right)\frac{rd\theta}{dt},$$

in which $u$ denotes the velocity of the point G, and, consequently,

$$u^2 = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}.$$

By means of this and the same equations (1), we can obtain($d'$)

$$\frac{1}{v} = \frac{1}{u} + \frac{1}{u^3}\left(\frac{dx}{dt}\sin\theta - \frac{dy}{dt}\cos\theta\right)\frac{rd\theta}{dt},$$

$$\cos\alpha = \frac{1}{u}\frac{dx}{dt} - \frac{1}{u^3}\left(\frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta\right)\frac{dy}{dt}\frac{rd\theta}{dt},$$

$$\cos\beta = \frac{1}{u}\frac{dy}{dt} + \frac{1}{u^3}\left(\frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta\right)\frac{dx}{dt}\frac{rd\theta}{dt}.$$

Moreover, as the origin of the polar coordinates $r$ and $\theta$ is at the centre of gravity of the base $b$, we have

$$\int r\sin\theta d\sigma = 0, \quad \int r\cos\theta d\sigma = 0.$$

This being so, if these values of $\cos\alpha$ and $\cos\beta$ be substituted in equations (2), they will become, as $\int d\sigma = b$ ($e'$),

$$\frac{d^2x}{dt^2} = -\frac{hg}{u}\frac{dx}{dt}, \quad \frac{d^2y}{dt^2} = -\frac{hg}{u}\frac{dy}{dt}. \tag{4}$$

For greater simplicity, we shall suppose that the base is symmetrically disposed about its centre of gravity, in which case

$$\int r^2 \cos\theta \sin\theta d\sigma = 0, \quad \int r^2 \sin^2\theta d\sigma = \int r^2 \cos^2\theta d\sigma = b\gamma^2,$$

$\gamma$ being a given line; and if, in equation (3) the values of $\cos\alpha$, $\cos\beta$ be substituted, it will be reduced in this case to

$$k^2 \frac{d^2\theta}{dt^2} = -\frac{hg\gamma^2}{u}\frac{d\theta}{dt}, \tag{5}$$

in which $\frac{d^2\theta}{dt^2}$ is substituted for its value $\frac{d\omega}{dt}$ ($f'$).

The complete integrals of equations (4) are ($g'$)

$$\frac{dx}{dt} = (a - hgt)\cos\varepsilon, \quad \frac{dy}{dt} = (a - hgt)\sin\varepsilon;$$

$a$ and $\varepsilon$ being two arbitrary constants. By means of them we obtain

$$u = a - hgt;$$

from which it appears, that the motion of the point G will be uniformly retarded, and that the initial velocity, and the angle which its direction makes with the axis of $x$, are respectively denoted by $a$ and $\varepsilon$. Equation (5) becomes

$$\frac{d^2\theta}{dt^2} = -\frac{hg\gamma^2}{k^2(a-hgt)}\frac{d\theta}{dt};$$

its complete integral is therefore ($h'$)

$$\frac{d\theta}{dt} = \Omega\left(1 - \frac{hgt}{a}\right)^{\frac{\gamma^2}{k^2}},$$

in which $\Omega$ denotes the arbitrary constant that expresses the initial angular velocity. The values of $u$ and $\frac{d\theta}{dt}$ or $\omega$, will be so much the more accurate, according as the product of $\Omega$ and of the greatest value of $r$ is a smaller fraction of the velo-

city $u$; it appears from an inspection of them, that the motions of translation and rotation terminate together, after the lapse of a time equal to $\frac{a}{hg}$.

When a solid body moves on a fixed plane, which it touches in only one point of its surface, a variety of cases may arise, which it is important to distinguish.

1st. The body may roll, without sliding, on the fixed plane, in such a manner, that the two curves traced, on this plane and on the surface of the moveable, which are the geometric loci of their successive points of contact, may have constantly equal lengths.

2ndly. The moveable may turn on itself, while it constantly touches the fixed plane, in the same point of this plane.

3rdly. The body may slide without turning, in such a manner, that the point of contact may constantly be the same point of its surface.

4thly. Finally, it may slide and turn at the same time.

In the second and third cases, the friction of the moveable against the fixed plane is the same as if the contact was of any extent whatever; its magnitude is proportional to the pressure at the point of contact, and its direction is contrary to that of the velocity at this point. If it be denoted by H, and the pressure by P, we have $H = hP$; the coefficient $h$ being the same as in No. 458. This law follows from the circumstance of the friction being independent of the extent of the contact; it will be, however, necessary to verify it by direct experiment. The force H is what is termed *a friction of the first species*.

In the first case, the friction of the moveable against the fixed plane is termed *a friction of the second species*. It appears from observation that this force is in general very small, and may therefore be neglected.

In the last case, the two kinds of friction obtain at the

same time; that of the second species may be neglected relatively to the friction of the first species, which, at each instant, acts in a direction contrary to that of the velocity of the point of contact, and is always proportional to the pressure at this point.

These results do not obtain, in the case in which the point of contact is the extremity of a point, or when it appertains to a sharp edge; they will, however, have place when the moveable is a cylinder, which touches the fixed plane in a right line; and, as often as its surface has neither points nor sharp edges, they will be sufficient to enable us to form the differential equations of the motions of translation and rotation. The following example will show how they ought to be employed for this object.

464. Let the moveable be a homogeneous sphere, placed on a fixed horizontal plane, and let there be impressed on this body a motion of rotation about a horizontal diameter, and on its centre a horizontal velocity, perpendicular to this diameter. It is evident that the moveable will turn about this diameter, which will be transferred parallel to itself and to the fixed plane, and that the centre of figure and gravity will describe a horizontal line, in the vertical plane perpendicular to the axis of rotation. It is proposed to determine, at any instant whatever, the velocities of these two motions.

Figure 14 represents a section of the moveable, perpendicular to its axis of rotation, and passing through its centre G. The line AKB is the section of the fixed plane; the parallel CGD is the line described by the point G, and the contact at the end of any time $t$, is supposed to take place at the point K. At this instant, let CG, the distance from the fixed point C, be denoted by $x$, the velocity of the point G by $\frac{dx}{dt}$, and the angular velocity of the moveable about its axis of rotation by $\omega$; this last will be regarded as positive or negative, according as the rotation has place in the direction indicated by the

sagitta $s$ or in the contrary direction. If, at the same instant, the absolute velocity of the point K be denoted by $v$, and GK the radius of the sphere by $c$, we shall have

$$v = \frac{dx}{dt} + c\omega.$$

According as this quantity is positive or negative, the point K will advance towards B or towards A; and the friction which has place in this point K in a contrary direction to $v$, will act in the direction KA or KB. When $v = 0$, the body rolls without sliding, and the friction is only of the second species.

This being established, let, as before, the weight of the body be denoted by P, its gravity by $g$, the moment of inertia with respect to the axis of rotation by $\frac{\text{P}k^2}{g}$, and the friction at the point K by $h$P; then if the velocity $v$ be supposed to be positive, and, consequently, the friction to act in the direction KA, the differential equations of the motions of translation and rotation will be

$$\frac{d^2x}{dt^2} = -hg, \quad k^2\frac{d\omega}{dt} = -hcg; \tag{a}$$

for G the centre of gravity should move as if $\frac{\text{P}}{g}$ the mass of the moveable being concentrated in it, the friction was applied to it parallel to its direction; and the moveable should, at the same time, turn about its axis of rotation, as if this axis was fixed, and the point of application of the friction and the direction of this force were not changed($i$). If the moveable was a sphere, we would have also

$$k^2 = \frac{2c^2}{5};$$

and if this body was a solid of revolution, or a right cylinder with a circular base, the preceding equations would likewise obtain, if, in each case, the body turned about its axis of

figure; however, the value of $k^2$ would in these cases be different from what it is in the sphere.

The coefficient $h$ being supposed to be constant, we obtain by integrating equations (a)

$$\frac{dx}{dt} = a - hgt, \quad \omega = \alpha - \frac{hcgt}{k^2}; \qquad \text{(b)}$$

$a$ and $\alpha$ being two arbitrary constants which denote the initial velocities $\frac{dx}{dt}$ and $\omega$. We shall have, at the same time,

$$v = a + c\alpha - \left(1 + \frac{c^2}{k^2}\right) hgt.$$

The constant quantity $a + c\alpha$ is positive by hypothesis; therefore, the velocity $v$ is so likewise, for $\theta$ an interval of time, the value of which is

$$\theta = \frac{(a + c\alpha)k^2}{(c^2 + k^2)gh} = 2\left(\frac{a + c\alpha}{7\,hg}\right),$$

in the case of a sphere; at the end of this time $v = 0$. During this interval, the preceding values of $\frac{dx}{dt}$ and $\omega$ will subsist, and the two motions of the moveable will be uniformly retarded. The value of $\frac{dx}{dt}$ will be cipher after the lapse of a time equal to $\frac{a}{hg}$; if therefore this time be less than $\theta$, which will be the case if

$$a < \frac{2c\alpha}{5},$$

the velocity of $\frac{dx}{dt}$ will become negative beyond $t = \frac{a}{hg}$, and the centre of the sphere will retrograde. This is the case, for example, when a billiard ball is struck in such a manner as to be made to turn rapidly about a horizontal diameter, while at the same time its centre advances with a less velocity, so that the quantities $a$ and $\alpha$ may be both positive, and satisfy the preceding inequality. The motion of translation is very soon destroyed by the friction against the cloth; but, as the motion

of rotation still subsists, the friction continues to act in a direction contrary to this last motion; and it is this force, which, transferred to the centre of gravity, causes it to return towards its point of departure.

If, at the commencement of the motion, the sphere did not turn, in which case we would have $\alpha = 0$, or, more generally, if we had

$$a > \frac{2c\alpha}{5},$$

the velocity $\frac{dx}{dt}$ would not become cipher before the velocity $v$, and the centre G would not retrograde. But in all cases, as K, the point in which the moveable meets the plane, has no longer any velocity after the end of the time $\theta$, $h$P the friction of the first species will disappear, the sphere will continue to roll without sliding, and it will only produce a friction of the second species. The velocities $\frac{dx}{dt}$ and $\omega$ will become constant, or will only decrease very slowly; their values will be those of formulæ (b), when $t = \theta$, namely,

$$\frac{dx}{dt} = \frac{5a - 2c\alpha}{7}, \quad \omega = \frac{2c\alpha - 5a}{7c}.$$

Thus, the general effect of the ordinary friction, or of that of the first species, is to reduce to a state of rest bodies which slide without turning, and to reduce solely to a state of uniformity and equality, in opposite directions, the two motions of bodies which slide and roll at the same time. In a perfect vacuum, the rolling of the moveable which results from these two motions, will subsist indefinitely, and the friction of the second species will maintain the velocity $v$ continually equal to cipher, and cause the two velocities $\frac{dx}{dt}$ and $\omega$ to be always equal and contrary. But the resistance of the air deranges this equality; the friction of the first species has place again, and the combined effect of these two forces eventually reduces the moveable to a state of rest.

# CHAPTER VII.

## OF THE IMPACT OF BODIES OF ANY FORM WHATEVER.

465. THE position and state of a solid body in motion, are completely determined at any given instant, when the coordinates of its centre of gravity, the components of the velocity of this point, the directions of the three principal axes which intersect in this same point, and the components of the angular velocity of rotation about these three axes, are known at this same instant. If this body is met by another moveable, all the circumstances of whose state are equally known, the components of their velocities of rotation and translation will be changed by the impact, but, both the positions of their centres of gravity, and also the directions of their principal axes, will be the same as before the impact; for, although the shock is not instantaneous, still the duration of this phenomenon is always sufficiently short to permit us to neglect the displacement of the different points of the two moveables, while it is taking place, and, consequently, we are justified in considering their centres of gravity and the points of the two moveables which appertain to their principal axes, as sensibly immoveable. Therefore in the problem of the impact of two bodies, the object will be to determine in magnitude and direction, their velocities of translation and rotation after the impact, from knowing the values of these same quantities before the impact, and also the relative forms and positions of the moveables.

We proceed now to give the general solution of this problem, in the two extreme cases, in which the moveables are

soft and destitute of elasticity, and in which, on the other hand, these bodies are perfectly elastic.

466. First, let us suppose that there are only two moveables which touch in one sole point, and that they are entirely free. Let M and M′ denote their masses, G and G′ (fig. 15) their centres of gravity, K their point of contact, HKH′ the normal to their surfaces at this point; likewise, let G$x$, G$y$, G$z$ be the principal axes of M, and G$x'$, G$y'$, G$z'$ those of M′. Let $u, v, w$ be the components of the velocity of G in the direction of the axes G$x$, G$y$, G$z$ immediately before the impact, at the same time let the angular velocity of M about the instantaneous axis of rotation passing through the point G, be denoted by $\omega$, and the three components of this velocity about the same axes G$x$, G$y$, G$z$, by $p, q, r$, so that $\frac{p}{\omega}, \frac{q}{\omega}, \frac{r}{\omega}$ may be the cosines of the angles which the instantaneous axis makes with these lines (No. 407), and also

$$\omega^2 = p^2 + q^2 + r^2.$$

This being agreed on, if $x, y, z$ be the three coordinates of any point M referred to the axes G$x$, G$y$, G$z$, the components of its velocity parallel to these axes, and arising from the motion of rotation of the moveable (No. 418), will be

$$qz - ry, \quad rx - pz, \quad py - qx;$$

consequently, we shall have those of its absolute velocity, by adding to them respectively $u, v, w$, the components of the velocity of the point G, which are therefore relatively to the axes of $x, y, z$,

$$u + qz - ry, \quad v + rx - pz, \quad w + py - qx.$$

Let $u_1, v_1, w_1, p_1, q_1, r_1$ denote what $u, v, w, p, q, r$ become immediately after the impact. Then, as the point, whose coordinates are $x, y, z$, does not sensibly change its position during the impact, the three preceding components will become

$$u_1 + q_1 z - r_1 y, \quad v_1 + r_1 x - p_1 z, \quad w_1 + p_1 y - q_1 x;$$

and as the axes G$x$, G$y$, G$z$, to which they are parallel, are also supposed to be immoveable during the impact, the velocities lost by this point, estimated in the directions of these axes, will be obtained by taking these last quantities from the preceding. If, therefore, the element of the mass M, whose coordinates are $x$, $y$, $z$, be denoted by $dm$, the components parallel to the axes G$x$, G$y$, G$z$, of the quantity of motion lost during the continuance of the impact, will be

$$[u - u_1 + (q - q_1)z - (r - r_1)y]dm,$$
$$[v - v_1 + (r - r_1)x - (p - p_1)z]dm,$$
$$[w - w_1 + (p - p_1)y - (q - q_1)x]dm.$$

In virtue of the general principle of dynamics (No. 353), the quantities of motion thus lost by M and M′ ought to be in equilibrio; and, by what has been observed in No. 265, the equations of equilibrium of these two solid bodies which press, the one against the other, can be formed, by considering each of them by itself, when there is added to the forces relative to M an unknown force N, acting in the direction KH, and to the forces applied to M′, the same force N acting at the point K in the direction KH′.

These forces N, acting in the directions KH and KH′, will be the quantities of motion impressed by the impact on the masses M and M′; and, previously to describing the equations of equilibrium, it may be remarked that the effects of the impact which they will enable us to determine, will be the same for the mass M, for example, as if $\mu$ an arbitrary part of this mass, having its centre of gravity in the line KH, was applied to it, actuated by such a velocity $h$ parallel to KH, that $\mu k =$ N; for it is evident that the resultant of the quantities of motion of $\mu$ will be equal to N, both in magnitude and direction; therefore the percussion exerted on M, in the direction KH, is always equivalent, as has been stated in No. 435, to

equal velocities impressed parallel to this normal KH, on all the points of a part of M, whose centre of gravity exists on this line.

467. This being established, let $a$, $b$, $c$ be the three co-ordinates of K, referred to the axes G$x$, G$y$, G$z$, and $\alpha$, $\beta$, $\gamma$, the angles which the line KH makes with lines drawn parallel to these axes through the point K; the six equations of equilibrium of the quantities of motion lost by all the points of M, will be

$$\text{N}\cos\alpha + \int[u - u_1 + (q - q_1)z - (r - r_1)y]\,dm = 0,$$
$$\text{N}\cos\beta + \int[v - v_1 + (r - r_1)x - (p - p_1)z]\,dm = 0,$$
$$\text{N}\cos\gamma + \int[w - w_1 + (p - p_1)y - (q - q_1)x]\,dm = 0,$$

$$\text{N}(a\cos\beta - b\cos\alpha) + \int[v - v_1 + (r - r_1)x - (p - p_1)z]\,x\,dm - \int[u - u_1 + (q - q_1)z - (r - r_1)y]\,y\,dm = 0,$$
$$\text{N}(c\cos\alpha - a\cos\gamma) + \int[u - u_1 + (q - q_1)z - (r - r_1)y]\,z\,dm - \int[w - w_1 + (p - p_1)y - (q - q_1)x]\,x\,dm = 0,$$
$$\text{N}(b\cos\gamma - c\cos\beta) + \int[w - w_1 + (p - p_1)y - (q - q_1)x]\,z\,dm - \int[v - v_1 + (r - r_1)x - (p - p_1)z]\,z\,dm = 0,$$

in which the integrations are supposed to extend to the entire mass M.

As G is the centre of gravity of M, and G$x$, G$y$, G$z$, are principal axes, we have

$$\int x\,dm = 0, \quad \int y\,dm = 0, \quad \int z\,dm = 0,$$
$$\int yz\,dm = 0, \quad \int zx\,dm = 0, \quad \int xy\,dm = 0.$$

Moreover, let A, B, C, denote the moments of inertia of M with respect to these same axes, so that

$$\text{A} = \int(y^2 + z^2)\,dm, \quad \text{B} = \int(z^2 + x^2)\,dm, \quad \text{C} = \int(x^2 + y^2)\,dm;$$

then as $\int dm = \text{M}$, the six preceding equations of equilibrium will be reduced to

$$\left.\begin{aligned}
&\text{N} \cos \alpha + \text{M}(u - u_1) = 0, \\
&\text{N} \cos \beta + \text{M}(v - v_1) = 0, \\
&\text{N} \cos \gamma + \text{M}(w - w_1) = 0, \\
&\text{N}(a \cos \beta - b \cos \alpha) + \text{C}(r - r_1) = 0, \\
&\text{N}(c \cos \alpha - a \cos \gamma) + \text{B}(q - q_1) = 0, \\
&\text{N}(b \cos \gamma - c \cos \beta) + \text{A}(p - p_1) = 0.
\end{aligned}\right\} \quad (1)$$

With respect to M′ and its principal axes $G'x'$, $G'y'$, $G'z'$, let the quantities which occur in the preceding equations be denoted by the same letters, with a stroke above each of them, so that, for example, $\alpha'$, $\beta'$, $\gamma'$, may be the angles which the line KH′ makes with lines drawn through K parallel to these axes, and $a'$, $b'$, $c'$, may be the coordinates of this point, referred to these same axes. As the magnitude of N must be the same in the case of the two bodies M and M′, the equations of equilibrium of the quantities of motion lost by M′, will be

$$\left.\begin{aligned}
&\text{N} \cos \alpha' + \text{M}'(u' - u_1') = 0, \\
&\text{N} \cos \beta' + \text{M}'(v' - v_1') = 0, \\
&\text{N} \cos \gamma' + \text{M}'(w' - w_1') = 0, \\
&\text{N}(a' \cos \beta' - b' \cos \alpha') + \text{C}'(r' - r_1') = 0, \\
&\text{N}(c' \cos \alpha' - a' \cos \gamma') + \text{B}'(q' - q_1') = 0, \\
&\text{N}(b' \cos \gamma' - c' \cos \beta') + \text{A}'(p' - p_1') = 0.
\end{aligned}\right\} \quad (2)$$

Besides the twelve unknown quantities $u_1, v_1, w_1, u_1', v_1', w_1', p_1, q_1, r_1, p_1', q_1', r_1'$, which these twelve equations (1) and (2) contain, they also involve the unknown quantity N, consequently they are not sufficient to determine these thirteen unknown quantities; a thirteenth equation must therefore be joined to them, which, in the case of bodies destitute of elasticity, may be obtained by the following consideration.

468. If the two moveables were considered to be perfectly hard, and if the compression which they experience during the continuance of the impact, was not taken into account, the solution of the problem would, in point of fact, be indeterminate.

But, if this compression, (however small we may suppose it to be), be taken into account, then it may be conceived to arise from this, namely, that those points in which the two moveables are in contact, have not the same velocity in the direction of the common normal to their surfaces. In consequence of the difference of the normal velocities of these two points, the two bodies lean and gradually press the one against the other; at the same time, this difference diminishes by insensible degrees until it entirely ceases; and when the two bodies are destitute of elasticity, the phenomenon of the impact is completed at the instant in which this difference becomes cipher, and these two bodies retain the form which they assume at the instant of their greatest compression. It is this equality of the normal velocities of the two points of contact, at the end of the impact, which furnishes the thirteenth required equation, and which consequently causes the indetermination, that would be otherwise in the problem, to disappear. Inasmuch as the point K appertains to the body M, its coordinates referred to the axes G$x$, G$y$, G$z$, are $a, b, c$; by substituting them in place of $x, y, z$, in the formulæ of No. 466, we have immediately after the impact $u_1+q_1c-r_1b$, $v_1+r_1a-p_1c$, $w_1+p_1b-q_1a$, for the values of the components of its velocity parallel to these three axes, and as $\alpha, \beta, \gamma$ are the angles, which the line KH makes with the directions of these components, the value of the final velocity of this point in the direction of this line, will be

$$(u_1+q_1c-r_1b)\cos\alpha+(v_1+r_1a-p_1c)\cos\beta+(w_1+p_1b-q_1a)\cos\gamma.$$

If the same point K be considered as constituting a part of the body M′, its velocity after the impact in the direction of KH′, will be

$$(u_1'+q_1'c'-r_1'b')\cos\alpha'+(v_1'+r_1'a'-p_1'c')\cos\beta' \\ +(w_1'+p_1'b'-q_1'a')\cos\gamma'.$$

Now, in order that in the two cases, the normal velocity

of the point K may be the same, and have the same direction, these two last quantities should be equal and affected with contrary signs, or in other words, their sum should be cipher, consequently, we shall have

$$\left.\begin{array}{l}(u_1+q_1c-r_1b)\cos\alpha+(u_1'+q_1'c'-r_1'b')\cos\alpha' \\ +(v_1+r_1a-p_1c)\cos\beta+(v_1'+r_1'a'-p_1'c')\cos\beta' \\ +(w_1+p_1b-q_1a)\cos\gamma+(w_1'+p_1'b'-q_1'a')\cos\gamma'=0.\end{array}\right\}\quad(3)$$

As equations (1), (2), (3), are all of the first degree, with respect to the unknown quantities N, $u_1$, $v_1$, &c., determinate values of these quantities may be obtained by means of these equations, which values will make known the state of the two moveables after the impact, and also the equal and opposite quantities of motion which are impressed on them by the percussion in the direction of the common normal to their surfaces.

469. If now, the two bodies are elastic, there are three successive epochs to be distinguished in the phenomenon of the impact; the first respects the instant at which the two 1° moveables commence to touch at the point K of their surfaces; the second will be that of their greatest compression, when 2° the normal velocities of the point K will be equal and directed 3° the same way for the two bodies; the third will be the termination of the impact, at which instant the two moveables will separate, the one from the other, and will exactly resume their primitive forms, if they be perfectly elastic.

From the first to the second epoch, the phenomenon will be the same as if the bodies were destitute of elasticity. Therefore we can determine by means of the preceding equations, the twelve components $u_1$, $v_1$, &c., of the velocities of translation and rotation of the moveables at the instant of their greatest compression, and N the quantity of motion which will be impressed on each of these bodies in the direction of KH for M, and of KH′ for M′. From the second to the third epoch, these two bodies, in reverting to their original forms,

will receive, along these directions, a new quantity of motion, which, in the case of perfect elasticity, will be also equal to N. Consequently, this second part of the phenomenon ought to be considered as a second percussion identical with the first, but exerted on bodies actuated by the velocities of translation and rotation, which obtain at the end of the first part.

Therefore, by this principle, agreeably to what has been already explained in No. 360, if, at the third epoch, U, V, W, be the components of the velocity of the point G in the direction of the axes $Gx$, $Gy$, $Gz$, and P, Q, R, the components of the angular velocity of M, about the same axes, we shall have, in order to determine these six unknown quantities, these six equations

$$
\begin{aligned}
&\mathrm{N}\cos\alpha + \mathrm{M}(u_1 - \mathrm{U}) = 0,\\
&\mathrm{N}\cos\beta + \mathrm{M}(v_1 - \mathrm{V}) = 0,\\
&\mathrm{N}\cos\gamma + \mathrm{M}(w_1 - \mathrm{W}) = 0,\\
&\mathrm{N}(a\cos\beta - b\cos\alpha) + \mathrm{C}(r_1 - \mathrm{R}) = 0,\\
&\mathrm{N}(c\cos\alpha - a\cos\gamma) + \mathrm{B}(q_1 - \mathrm{Q}) = 0,\\
&\mathrm{N}(b\cos\gamma - c\cos\beta) + \mathrm{A}(p_1 - \mathrm{P}) = 0,
\end{aligned}
$$

which are obtained from equations (1), by substituting U, V, W, P, Q, R, in place of $u_1$, $v_1$, $w_1$, $p_1$, $q_1$, $r_1$, also these last quantities in place of $u$, $v$, $w$, $p$, $q$, $r$, and retaining the quantity N.

If each of these six equations be added to the equation (1) which corresponds to it, the intermediate unknown quantities $u_1, v_1, w_1, p_1, q_1, r_1$, will disappear, and there will result

$$
\left.
\begin{aligned}
&2\mathrm{N}\cos\alpha + \mathrm{M}(u - \mathrm{U}) = 0,\\
&2\mathrm{N}\cos\beta + \mathrm{M}(v - \mathrm{V}) = 0,\\
&2\mathrm{N}\cos\gamma + \mathrm{M}(w - \mathrm{W}) = 0,\\
&2\mathrm{N}(a\cos\beta - b\cos\alpha) + \mathrm{C}(r - \mathrm{R}) = 0,\\
&2\mathrm{N}(c\cos\alpha - a\cos\gamma) + \mathrm{B}(q - \mathrm{Q}) = 0,\\
&2\mathrm{N}(b\cos\gamma - c\cos\beta) + \mathrm{A}(p - \mathrm{P}) = 0.
\end{aligned}
\right\} \quad (4)
$$

These equations (4) are those of the equilibrium of the quantities of motion lost by M during the entire continuance of the

impact, joined to the quantity of motion $2N$, which is impressed on this mass, in the direction KH, during this same percussion. The value of $N$, which can be obtained from equation (3), after having substituted in it the values of the unknown quantities $u_1$, $v_1$, &c., $u'_1$, $v'_1$, &c., which are furnished by equations (1) and (2), should be put instead of it in this equation (4); however, as the general expression of this quantity $N$ is extremely complicated, and as its numerical value can be calculated without any difficulty, in each particular case, we shall not write it down at length. If the two moveables were not perfectly elastic, $N$ will be less in the second part of the impact than in the first, we should then assume for its value in the second part, a certain fraction, such as $f$, of its value deduced from equations (1), (2), (3), and then substitute $(1+f)N$ in place of $2N$ in equations (4). This fraction $f$ will depend on the degree of elasticity of the two moveables, and can only be determined by means of experiments instituted on bodies consisting of the same kind of matter, and in the simplest case, with respect to their primitive form and initial motion. We shall restrict ourselves to the consideration of the case of perfect elasticity, observing, however, at the same time, that the concluding remark made in No. 466, obtains equally, whatever be the degree of elasticity.

With respect to the body $M'$, if $U'$, $V'$, $W'$, be the components of the velocity of $G'$ in the direction of the axes $G'x'$, $G'y'$, $G'z'$, at the end of the impact, and $P'$, $Q'$, $R'$, the components of the angular velocity of $M'$ about these axes, we shall obtain equations similar to equations (4), namely,

$$\left.\begin{array}{l}
2N\cos\alpha' + M'(u' - U') = 0,\\
2N\cos\beta' + M'(v' - V') = 0,\\
2N\cos\gamma' + M'(w' - W') = 0,\\
2N(a'\cos\beta' - b'\cos\alpha') + C'(r' - R') = 0,\\
2N(c'\cos\alpha' - a'\cos\gamma') + B'(q' - Q') = 0,\\
2N(b'\cos\gamma' - c'\cos\beta') + A'(p' - P') = 0;
\end{array}\right\} \quad (5)$$

By means of which these six unknown quantities may be determined.

The complete and general solution of the problem of the impact, in the case of two bodies entirely free, which are either destitute of all elasticity, or perfectly elastic, is obtained in this manner. It may without difficulty be extended to the impact of three or a greater number of moveables; we shall give farther on an example of this.

470. From the preceding equations it appears, that in the impact of the bodies M and M′, the velocities of their centres of gravity G and G′, estimated in a direction parallel to the plane which is a common tangent at K, to their surfaces, is not changed.

In fact, if through the point G, a line be drawn parallel to this plane, making the angles $\lambda, \mu, \nu$, with the axes G$x$, G$y$, G$z$, we shall have, as this line is perpendicular to the parallel to KH drawn through the same point G,

$$\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu = 0;$$

and, therefore, if after having multiplied the three first equations (1) or (4) by $\cos\lambda$, $\cos\mu$, $\cos\nu$, respectively, they be then added together, there will result

$$u\cos\lambda + v\cos\mu + w\cos\nu = u_1\cos\lambda + v_1\cos\mu + w_1\cos\nu$$
$$= \text{U}\cos\lambda + \text{V}\cos\mu + \text{W}\cos\nu,$$

From which it appears, that the velocity of G estimated in *any* direction whatever, parallel to the tangent plane at K, will be the same before and after the impact, whether the bodies be soft or elastic. Therefore, in order to obtain the final velocity of G, it will be sufficient to determine, in each case, the velocity of this point after the impact parallel to the normal KH, and to take the resultant of it and the velocity of G parallel to the tangent plane, which it had previously, and which is not changed during the percussion. The same will be the case relatively to G′ the centre of gravity of M′.

If, after having multiplied the three last equations (1) or (4), by $\cos\gamma$, $\cos\beta$, $\cos\alpha$, they be added together, there results

$$Cr\cos\gamma + Bq\cos\beta + Ap\cos\alpha = Cr_1\cos\gamma + Bq_1\cos\beta + Ap_1\cos\alpha$$
$$= CR\cos\gamma + BQ\cos\beta + AP\cos\alpha.$$

Now, these three equal quantities are the moments relative to the axis KH (No. 409), of the quantities of motion, with which all the points of M are actuated, before the impact, at the instant of its greatest compression, and at the end of the impact; hence it follows, that the magnitude of this moment for the moveable M, is not at all affected by the impact, and the same is the case, for the moveable M′, and this obtains whether the bodies be soft or elastic.

2

471. We now proceed to apply the general equations which have been obtained to several examples.

First, we shall suppose that KH, the normal at the point of contact of the two moveables, passes through G the centre of gravity of M, so that it may coincide with the line KGH (fig. 16). It is evident from a consideration of what the letters $\alpha$, $\beta$, $\gamma$, $a$, $b$, $c$ signify, that in this case we have

$$a\cos\beta = b\cos\alpha,$$
$$c\cos\alpha = a\cos\gamma,$$
$$b\cos\gamma = c\cos\beta;$$

and then the three last equations (1) and (4) will give

$$p_1 = p, \quad q_1 = q, \quad r_1 = r, \quad P = p, \quad Q = q, \quad R = r;$$

from which it appears, that the direction of the instantaneous axis of M, and the velocity of rotation immediately before and after the impact, are the same. Therefore, whenever the normal at the point of contact passes through the centre of gravity of one of the two moveables, the impact does not at all affect the rotatory motion of this body, it only influences the motion of translation, and this is the case whether the bodies be soft or elastic.

If the same normal passes also through G′ the centre of gravity of M′, in which case we have likewise

$$a' \cos\beta' = b' \cos a',$$
$$c' \cos a' = a' \cos\gamma',$$
$$b' \cos\gamma' = c' \cos\beta',$$

the velocities of rotation will disappear from equation (3), which will be reduced to

$$u_1 \cos a + v_1 \cos\beta + u_1 \cos\gamma + u_1' \cos a'$$
$$+ v_1' \cos\beta' + w_1' \cos\gamma' = 0\,;$$

but from the three first equations (1) and (2), we obtain($a$)

$$\left.\begin{aligned} \text{N} &= \text{M}\,[(u_1 - u)\cos a + (v_1 - v)\cos\beta + (w_1 - w)\cos\gamma], \\ \text{N} &= \text{M}'[(u_1' - u')\cos a' + (v_1' - v')\cos\beta' + (w_1' - w')\cos\gamma']\,; \end{aligned}\right\}\ (6)$$

and by dividing each of these equations by M and M′ respectively, and then adding them to the preceding, there results

$$\frac{\text{N}}{\text{M}} + \frac{\text{N}}{\text{M}'} + u\cos a + v\cos\beta + w\cos\gamma + u'\cos a'$$
$$+ v'\cos\beta' + w'\cos\gamma' = 0.$$

Now, if GL and G′L′ are the directions of G and G′ before the impact, and $\theta$ and $\theta'$ their initial velocities, we shall have

$$\theta \cos \text{HGL} = u\cos a + v\cos\beta + w\cos\gamma,$$
$$\theta' \cos \text{H}'\text{G}'\text{L}' = u'\cos a' + v'\cos\beta' + w'\cos\gamma',$$

as is evident from a consideration of what is denoted by the letters $u, v, w, a, \beta, \gamma, u', v', w', a', \beta', \gamma'$. Consequently, we shall have($b$)

$$\text{N} = -\frac{\text{MM}'\,(\theta\cos\text{HGL} + \theta'\cos\text{H}'\text{G}'\text{L}')}{\text{M} + \text{M}'},$$

for the value of N, which must be essentially positive, for if it was negative there would evidently be no impact between the two moveables M and M′.

In like manner, if $\mathrm{G}l$ and $\mathrm{G}'l'$ be the directions of G and G′ at the instant of the greatest compression of the two moveables, and $\theta_1$ and $\theta_1'$ their velocities at the same instant, we shall also have

$$\theta_1 \cos \mathrm{HG}l = u_1 \cos \alpha + v_1 \cos \beta + w_1 \cos \gamma,$$
$$\theta_1' \cos \mathrm{H'G'}l' = u_1' \cos \alpha' + v_1' \cos \beta' + w_1' \cos \gamma';$$

and from these several equations we obtain(*c*),

$$\left.\begin{aligned} \theta_1 \cos \mathrm{HG}l &= \frac{\mathrm{M}\theta \cos \mathrm{HGL} - \mathrm{M}'\theta' \cos \mathrm{H'G'L'}}{\mathrm{M} + \mathrm{M}'}, \\ \theta_1' \cos \mathrm{H'G'}l' &= \frac{\mathrm{M}'\theta' \cos \mathrm{H'G'L'} - \mathrm{M}\theta \cos \mathrm{HGL}}{\mathrm{M} + \mathrm{M}'}, \end{aligned}\right\} \quad (7)$$

for the components of the velocities of G and G′ in the directions GH and GH′, at the instant in question. They are (as appears from inspection) equal and contrary; hence it follows, that at the instant of the greatest compression the velocities of the centres of gravity of the two moveables, estimated in the direction of the normal at the point of contact K, are equal. In the case of soft bodies, this normal velocity will be that which obtains after the impact. When the two moveables are perfectly elastic, we shall have

$$\theta_1 \cos \mathrm{HG}l = \mathrm{U} \cos \alpha + \mathrm{V} \cos \beta + \mathrm{W} \cos \gamma,$$
$$\theta_1' \cos \mathrm{H'G'}l' = \mathrm{U}' \cos \alpha' + \mathrm{V}' \cos \beta' + \mathrm{W}' \cos \gamma';$$

the velocities $\theta_1$ and $\theta_1'$, and the directions $\mathrm{G}l$ and $\mathrm{G}'l'$ being supposed to be those which have place at the end of the impact. Hence, in virtue of the three first equations (4) and (5), and of the value found above for N, we shall have(*d*)

$$\left.\begin{aligned} \theta_1 \cos \mathrm{HG}l &= \frac{(\mathrm{M} - \mathrm{M}')\,\theta \cos \mathrm{HGL} - 2\,\mathrm{M}'\theta' \cos \mathrm{H'G'L}}{\mathrm{M} + \mathrm{M}'}, \\ \theta_1' \cos \mathrm{H'G'}l' &= \frac{(\mathrm{M}' - \mathrm{M})\,\theta' \cos \mathrm{H'G'L'} - 2\,\mathrm{M}\theta \cos \mathrm{HGL}}{\mathrm{M} + \mathrm{M}'}, \end{aligned}\right\} \quad (8)$$

which will be the components of the final velocities of G and G′ in the directions GH and G′H′.

472. In the particular case, in which the points G and G′ move, before the impact, in the direction of the normal HKH′, their velocities estimated in a direction perpendicular to this line will be cipher, and they will be likewise so after the impact; so that we shall have

$$\cos \text{HGL} = \pm 1, \quad \cos \text{HG}l = \pm 1,$$
$$\cos \text{H}'\text{G}'\text{L}' = \pm 1, \quad \cos \text{HG}l' = \pm 1,$$

the sign depends on their directions, the velocities $\theta, \theta', \theta_1, \theta_1'$, being always regarded as positive.

If G and G′ move in the same direction before the impact, as for example from H′ towards H, the angle HGL will be cipher, and the angle H′G′L′ equal to two right angles; therefore, in virtue of equations (7), we shall have

$$\cos \text{HG}l = 1, \quad \cos \text{H}'\text{G}'l' = -1, \quad \theta_1 = \theta_1' = \frac{\text{M}\theta + \text{M}'\theta'}{\text{M} + \text{M}'}.$$

If, on the contrary, G and G′ move in opposite directions before the impact, in such a manner that the point G proceeds from H towards H′, and the point G′ from H′ towards H, we shall have $\cos \text{HGL} = \cos \text{H}'\text{G}'\text{L}' = -1$, and equations (7) will give

$$\cos \text{HG}l = \mp 1, \quad \cos \text{H}'\text{G}'l' = \pm 1, \quad \theta_1 = \theta_1' = \pm \frac{\text{M}\theta - \text{M}'\theta'}{\text{M} + \text{M}'};$$

in which, the superior or inferior sign should be taken, according as the difference $\text{M}\theta - \text{M}'\theta'$ is positive or negative. Formulæ (8) may, in the same manner, be applied to these two hypotheses.

These results coincide with those obtained in Nos. 361 and 362, relatively to the impact of homogeneous spherical bodies; but it appears, from what has been just established, that they are independent of the form of these bodies, and of their motion of rotation, and that they only suppose that the centres of gravity of the two moveables move, before the impact, along the normal to the point of contact.

473. If we suppose $M' = M$, equations (8) become

$$\theta_1 \cos HGl = -\theta' \cos H'G'L', \quad \theta_1' \cos H'G'l' = -\theta \cos HGL,$$

from whence it follows, that in the impact of two perfectly elastic bodies whose masses are equal, the centres of gravity of the two moveables interchange their velocities estimated in a direction parallel to the normal at the point of contact, when this normal, which is common to the two surfaces at this point, passes at the same time through these two centres.

If the point $G'$ is at rest before the impact, in which case $\theta' = 0$, and if, moreover, the mass $M$, on account of its density, is so small that it may be neglected with respect to $M'$, we shall have, in virtue of equations (7) (*e*).

$$\theta_1 \cos HGl = 0, \quad \theta_1' \cos H'G'l' = 0\,;$$

so that when the bodies are soft, their centres of gravity $G$ and $G'$ will not, in this case, have any normal velocity after the impact. But if, on the contrary, the bodies are perfectly elastic, we shall have, in virtue of equations (8),

$$\theta_1' \cos H'G'l' = 0, \quad \theta_1 \cos HGl = -\theta \cos HGL\,;$$

from which it appears, that the centre of gravity $G'$ will not be actuated by any velocity, and that $G$ will acquire, after the impact, a normal velocity equal and contrary to that which it had before.

It is easy to infer, that the centre of gravity $G$ will be reflected, making with the normal at the point of contact of the two moveables, the angle of reflexion equal to the angle of incidence. In fact, let there be assumed on the production of $GL$ (fig. 17) a part such as $gG$ to represent the velocity of $G$ which moves from $g$ towards $G$, before the impact; and let $GH$ be the normal to the point of contact of the two moveables, which passes by hypothesis through the point $G$; if the velocity $gG$ be resolved into two others, the one $aG$ acting in the direction of this normal, the other $bG$ parallel to the tangential

plane, the second will not be affected by the impact, and the first will be changed into a velocity $cG$ equal and contrary to $aG$. Consequently, if the rectangle $Gbdc$ be completed, and if the diagonal $Gd$ be prolonged, so that $Gl$ may be equal to $Gd$, the velocity of the point G after the impact will be represented in magnitude and direction by $Gl$; hence the angle of reflexion $HGl$ will be equal to the angle of incidence $HGg$, and, moreover, the magnitude of the velocity of the moveable before and after the impact, will be the same. This case is that of an elastic body that meets with a fixed obstacle, which is itself also endowed with a perfect elasticity.

474. When the moveables are homogeneous spheres, the condition that the normal $HKH'$ (fig. 16) passes through their centres of gravity G and G′, is always satisfied. Consequently, if these bodies are perfectly elastic, when they impinge they will interchange their velocities in the direction of the line drawn from one centre to the other, and they will retain without any change, their velocities perpendicular to this line; and when they strike a fixed obstacle, which is itself perfectly elastic, they will be reflected, making the angle of incidence equal to the angle of reflexion. It is on these principles that the game of billiards is founded; but it is necessary to observe, that not only the balls and sides of the tables are supposed to be perfectly elastic, but likewise, that the fact of the velocity of the moveables, when estimated either in a direction parallel to their common tangential plane, or in a direction parallel to the sides which they meet, not being affected by the impact, is only true on the supposition that the friction arising from their rotation, and from the sliding of one surface on the other, is not taken into account. We proceed to show, for example, that the angle of reflexion depends on the rotation of the ball, and that it is not equal to the angle of incidence, when the friction of the ball against the side is taken into account. The same is the case in rocket firing, when a bullet hops on the earth; the friction of this projectile against the

ground and its velocity of rotation likewise affect the angle of reflexion, which may, on this account, be different from the angle of incidence.

This question will enable us to explain in what manner the friction should be taken into account in the impact of bodies, and thus complete what has been stated, in the second paragraph of the preceding chapter, respecting this kind of resistances. We now therefore proceed to detail the principles which are to guide us in this delicate matter; we have been conducted to them by analogy, but they, as well as the consequences which follow from them, should be confirmed by direct experiments.

475. When the equations of equilibrium of the quantities of motion lost in the impact, by the two masses M and M', were formed, the quantities of motion produced by the weights of these bodies during the continuance of the percussion, were not taken into account, because as this duration was very short, these quantities, which are proportional to it, are likewise very small, and may be neglected relatively to those which the moveables receive from their mutual impact. But this is not the case, as has been already remarked (No. [illegible]), with respect to the friction which takes place during the impact, when the surfaces of the two moveables in contact slide the one against the other. Although no observations have been made on the intensity of this friction, it may be supposed by induction, that it follows the general laws of the friction of bodies subjected to pressures properly so called, since the percussion is only a pressure of very great intensity, acting for a very short interval of time. It may therefore be admitted, that during the continuance of the impact, the friction, at each instant, is proportional to the magnitude of the normal pressure, which at this instant causes the two moveables to press against one another, and that, in the case of each moveable, it acts in a direction contrary to that of the relative velocity with which this moveable slides on the other, being at the same

time independent of the magnitude of this velocity; also that when the friction is changed into a simple rolling of one body on the other, it disappears, or may be neglected.

Now, the entire quantity of motion impressed on M in the direction of the normal KH (fig. 15), has been denoted by N, when these two moveables are destitute of elasticity, and by 2N, when they are perfectly elastic. If, therefore, during the continuance of the impact, the surface of M slides in one and the same direction on that of M′, and if Q be the quantity of motion impressed on M by the friction, estimated in a sense opposite to that direction, we shall have $Q = hN$, in the case of bodies destitute of elasticity, and $Q = 2hN$ in the case of perfectly elastic bodies; $h$ being a coefficient which depends on the nature of the surfaces of M and M′ at the point of contact K, and for which that value may be assumed which has been determined by experiment, relatively to ordinary pressures (No. 459). If the sliding takes place in one direction during one part of the impact, and in the opposite direction during the other part, we shall have $Q = h(N' - N'')$, in which N′ and N″ denote the quantities of motion produced by the percussion during these respective parts of the impact, so that their sum $N' + N''$ may be equal to N or 2N, N′ being supposed $> N''$. If, at the end of the first part, the sliding is changed into a simple rolling, we should assume $Q = hN'$, the friction of the second species, which has place during the second part, being neglected.

If Q′ denotes what Q becomes relatively to M′, it is evident that, in all cases, Q′ will be a quantity of motion equal and directly opposed to Q, for the normal pressure that M′ exercises on M during the continuance of the impact, is of the same magnitude as that of M on M′, and the relative velocity of the sliding of M′ on M, is always equal and contrary to that of M on M′.

It follows from this, that in order to obtain the equation of equilibrium of the quantities of motion lost during the im-

pact by the body M, the effect of the friction being taken into account, it will be sufficient to join to the quantity of motion N or 2 N, impressed on the normal in the direction of the normal KH, another quantity of motion Q, acting in a direction contrary to KH, and expressed in the manner just stated above; and that, in order, at the same time, to obtain the equations relative to M′, there should be joined to the quantity of motion N or 2N, acting in the direction KH′, a quantity of motion Q′ equal and contrary to Q. This is what we now proceed to do in the case of the impact of a homogeneous spherical projectile against a fixed obstacle.

476. For greater clearness, we shall suppose that the fixed obstacle that is struck by the sphere M is a horizontal plane, and that previously to the impact, this sphere turns about a horizontal axis, which is perpendicular to GH the direction of its centre of gravity. Figure 18 represents a section of the fixed plane and of M made by a vertical plane passing through G. As every thing is similar on each side of this section, the point G will not deviate from this last plane during the impact, M will continue to turn about the diameter perpendicular to this same plane, and the point of contact K will slide, during this percussion, along AB, the intersection of this vertical plane and of the fixed plane.

Let $a$ be the horizontal velocity of G, immediately before the impact, acting in the direction GD, $b$ its vertical velocity acting in the direction GK, $\gamma$ the angle of incidence HGK, in which case we have

$$\text{tang}\,\gamma = \frac{a}{b}.$$

As the moveable M, and also the obstacle which it strikes, are supposed to be perfectly elastic, it will first lose, during its compression, its quantity of vertical motion denoted by M$b$; then in returning to its primitive figure, it will resume an equal and contrary quantity of motion, so that after the impact, G the centre of gravity will be actuated by a vertical

velocity $b$ acting in the direction of GE the production of GK. If therefore its horizontal velocity at this epoch directed along GD, or in the opposite direction, according as it is positive or negative, be denoted by $a'$; if GH′ be the direction of this point, and if the angle of reflexion EGH′ be denoted by $\gamma'$, we shall have

$$\tan\gamma' = \frac{a'}{b}.$$

The line GH′ will be situated to the left of the vertical EK, like the line GH, or to the right of EK, according as the quantity $a'$ is positive or negative. In order that the angle of reflexion may be equal to the angle of incidence, this velocity $a'$ should be positive and equal to $a$. When these velocities are not equal, the sign of $\gamma' - \gamma$, the difference of these two angles, will be always the same as that of $a' - a$; and the point G will retrograde when the velocity $a'$ is negative.

In like manner, let the angular velocity of rotation of M, before the impact be denoted by $\alpha$; we shall consider it as positive or negative, according as it has place in the direction indicated by the sagitta $s$, or in the contrary direction. Let $\alpha'$ be what this velocity becomes after the impact. The object of the problem will be to determine $a'$ and $\alpha'$ when $a$ and $\alpha$ are known. Its solution gives rise to several distinct cases, according as the absolute velocity of the point K is positive or negative, that is to say, according as it is directed along KA or KB, during a part of the continuance of the impact, or during its entire continuance; these cases we proceed to examine successively in the following number.

477. In the first place, let us suppose that the velocity of the point K is positive, or directed along KA, during the entire continuance of the impact; then if GK the radius of M be denoted by $c$, this velocity will be equal to $a + c\alpha$ at the commencement, and to $a' + c\alpha'$ at the end of the impact, so that these two quantities should be positive. As the entire quantity of motion impressed on M in the direction GK, while the

moveable is compressed, and also while it reverts to its primitive figure, is equal to $2\text{M}b$; by No. 475 the quantity of motion arising from the friction, and acting in the direction KB, will be $2h\text{M}b$; consequently, $a'$ the horizontal velocity of the centre of gravity G after the impact will be the same, as if the mass M being condensed into it, the quantities of motion $\text{M}a$ and $2h\text{M}b$ were applied to it, in directions contrary the one to the other; this gives

$$a' = a - 2hb.$$

Now since $\frac{2}{5}\text{M}c^2$ is the moment of inertia of M with respect to its axis of rotation, and $2h\text{M}bc$ is the moment of the friction directed along KB with respect to the same axis, it is easy to perceive that $\alpha - \alpha'$ the diminution of the angular velocity of rotation will be determined by the equation $(f)$

$$\tfrac{2}{5}\text{M}c^2(\alpha - \alpha') = 2h\text{M}bc;$$

from which we obtain

$$\alpha' = \alpha - \frac{5hb}{c}.$$

From these values of $a'$ and $\alpha'$, there will result

$$a' + c\alpha' = a + c\alpha - 7hb;$$

and, as in the case we are considering, $a' + c\alpha'$ must be positive, the quantity $a + c\alpha$, which is also positive, must be greater than $7hb$. When this condition obtains, we shall have

$$\text{tang}\,\gamma' = \frac{a}{b} - 2h = \text{tang}\,\gamma - 2h,$$

by means of which the angle of reflexion $\gamma'$ can be determined, when $\gamma$ the angle of incidence and the coefficient $h$ are known.

If the absolute velocity of the point K is constantly negative, or directed along KB, the friction will act in the direction KA, and it will be sufficient to change the signs of the terms

multiplied by $h$, in the formulæ of the preceding case, which will become

$$a' = a + 2hb, \quad \alpha' = \alpha + \frac{5hb}{c}, \quad \text{tang}\,\gamma' = \text{tang}\,\gamma + 2h.$$

In order that this case may obtain, the direction of the initial velocity of the point K must be contrary to that of G, this implies, that the direction of the primitive rotation is contrary to that which is indicated by the sagitta $s$. Moreover, as

$$a' + c\alpha' = a + c\alpha + 7\,hb,$$

it is necessary that this negative quantity $a + c\alpha$ should exceed the positive term $7\,hb.(g)$ In this second case, the angle of reflexion will be greater than the angle of incidence, while in the first case, $\gamma'$ was less than $\gamma$.

If, the velocity of the point K being positive at the commencement of the impact, it becomes at a certain point of its duration, cipher, and if it continues to be nothing unto the end of the impact, the total effect of the friction should, by No. 475, be taken equal to $h\text{M}(b + b')$, $b'$ denoting the vertical velocity of G at the instant in question, which should be regarded as negative or positive, according as this instant occurs during the time the moveable is compressed, or while it is reverting to its primitive figure. From hence we infer, as in the first case,

$$a' = a - h(b + b'), \quad \alpha' = \alpha - \frac{5h(b+b')}{2c},$$

and, consequently,

$$\text{tang}\,\gamma' = \text{tang}\,\gamma - h\frac{(b+b')}{b}.$$

If the velocity of the point K is cipher from the commencement of the impact, we shall have $b' = -b$; and as the friction is cipher, or of the second species, during the entire continuance of the percussion, it will not affect the values of $a'$, $\alpha'$, tang $\gamma'$. If, on the contrary, the velocity of the point K does

not become cipher until the end of the impact, we shall have $b = b'$, and these formulæ will coincide with those of the first case. In general, the value of $b'$ will be unknown, and all that is determined respecting it is, that it cannot surpass $\pm b$; but as the final velocity of the point K is supposed to be cipher, we must have

$$a' + ca' = a + ca - \tfrac{7}{2} h (b + b') = 0;$$

from which we obtain

$$h(b + b') = \frac{2(a + ca)}{7};$$

and, consequently ($h$),

$$a' = -ca' = \frac{5a - 2ca}{7}, \quad \text{tang}\,\gamma' = \text{tang}\,\gamma - \frac{2(a + ca)}{7b}.$$

If, the velocity of the point K being positive in one part of the impact, it becomes negative in the following part, and if $b'$ be the vertical velocity of G at the instant of this change of sign, which velocity may be either positive or negative, the quantities of motion impressed on M in the direction of KG, during these two parts of the percussion, will be M$(b + b')$ and M$(b - b')$ respectively. Therefore, by No. 475, $h$M$(b + b')$, $-h$M$(b - b')$ will be the entire quantity of motion produced by the friction in the direction KB or KA, according as it is positive or negative; and as this quantity is reducible to $2h$M$b'$, it follows that the formulæ relative to this fourth case may be deduced from those of the first, by substituting $b'$ in place of $b$. If the velocity of K was first negative, then in order that it should afterwards become positive, the sign of $h$ should be changed, as in the second case. But as the quantity $b'$ is not given, the diminution or increase of the angle of reflexion cannot be known by means of these formulæ; all we know respecting them is, that both the one and the other is less than in the first or second case.

The question would be still more complicated, if the pro-

jectile turned about an axis which was not perpendicular, as has been supposed, to the vertical plane in which the point G moves before the impact. The friction would then cause this point to deviate from its plane during the percussion; and not only would the angle of reflexion differ from the angle of incidence, but likewise these two angles would no longer be comprised in the same vertical plane.

478. Now in order to show, independently of friction, the influence of the impact on the motion of rotation, let a simple example be taken, in which the normal at the point of contact of the two moveables, which may always be regarded as the direction of the impact, does not pass through the centre of gravity of one of these two bodies.

If during the impact the instantaneous axis of rotation of M be supposed to coincide with one of the principal axes, which intersect at its centre of gravity, for example, with the axis G$z$ (fig. 15), we shall then have $p = 0$ and $q = 0$. Likewise, let us suppose that the point K and the common normal to the two surfaces at this point, are comprised in the plane of the axes G$x$ and G$y$; in consequence of this, the two quantities $c$ and $\cos\gamma$ will be cipher. By making $p = 0$, $q = 0$, $c = 0$, $\cos\gamma = 0$, in the two last equations (1) or (4), it follows that $p_1 = 0$, and $q_1 = 0$, or $P = 0$, and $Q = 0$; so that in the two cases, namely, when the bodies are soft, and also when they are elastic, the axis of rotation will still coincide after the impact with the axis G$z$, and the impact will only change the velocity of rotation without at all affecting the instantaneous axis, agreeably to what has been already observed in No. 437.

If the body M′ be a homogeneous sphere, the direction of the impact will pass through its centre of gravity; consequently, we shall have, as in No. 471,

$$a'\cos\beta' = b'\cos\alpha', \quad c'\cos\alpha' = a'\cos\gamma', \quad b'\cos\gamma' = c'\cos\beta',$$

hence, the suppositions just stated being taken into account, equation (3) will be reduced to

$$(a\cos\beta - b\cos\alpha)r_1 + u_1\cos\alpha + v_1\cos\beta$$
$$+ u_1'\cos\alpha_1 + v_1'\cos\beta_1 + w_1'\cos\gamma' = 0;$$

and, by combining it with equation (6), we obtain($i$)

$$\frac{\text{N}}{\text{M}} + \frac{\text{N}}{\text{M}'} + (a\cos\beta - b\cos\alpha)r_1 + u\cos\alpha + v\cos\beta$$
$$+ u'\cos\alpha' + v'\cos\beta' + w'\cos\gamma' = 0.$$

Through the point K, let lines be drawn parallel to the directions of $\theta$ and $\theta'$, the velocities of G and G$'$ before the impact; then if $\delta$ and $\delta'$ be the angles which these parallels make with KH, we shall have

$$u\cos\alpha + v\cos\beta = \theta\cos\delta,$$
$$u'\cos\alpha' + v'\cos\beta' + w'\cos\gamma' = -\theta'\cos\delta',$$

which will, therefore, be the components of $\theta$ and $\theta'$ along this part of the normal to the point K; and if $r_1$ be eliminated by means of the fourth equation (1), the preceding equation will become

$$\frac{\text{N}}{\text{M}} + \frac{\text{N}}{\text{M}'} + \frac{(a\cos\beta - b\cos\alpha)^2\text{N}}{\text{C}} + (a\cos\beta - b\cos\alpha)r$$
$$+ \theta\cos\delta - \theta'\cos\delta' = 0;$$

from which we deduce

$$\text{N} = \frac{\text{MM}'\text{C}\,[b\cos\alpha - a\cos\beta)r + \theta'\cos\delta' - \theta\cos\delta]}{(\text{M}+\text{M}')\,\text{C} + \text{MM}'\,(b\cos\alpha - a\cos\beta)^2}.$$

By means of this value of N, the three first equations (1) or (4), according as the moveables are soft or elastic, will make known $u_1, v_1, w_1$, or U, V, W, the three components of the velocity of G after the impact; and the three first equations (2) or (5), will, in like manner, determine the final velocity of G$'$. With respect to the value of $r'$ or R, it may be deduced from the fourth equation (1) or (4).

The quantity N must, as has been already remarked, be

always positive; for when its value is negative, there cannot be any impact between the two moveables. The denominator of this value is positive, and so is the factor of its numerator, namely MM′C. $\theta$ and $\theta'$ the factors of its two other terms, are also positive, but the quantities $a$, $b$, $\cos \alpha$, $\cos \beta$, $\cos \delta$, $\cos \delta'$, may be either positive or negative; and their signs will be given in each particular case. As $p=0$, and $q=0$, $r$ will be, abstracting from the sign, the velocity of rotation before the impact. In order to know the sign with which it should be affected in the value of N, let a point be assumed on the axis G$x$, at the unit of distance from the point G; the velocity of this point before the impact, estimated in a direction parallel to the axis G$y$, will be $v+r$ (No. 466); hence it follows, that its part $r$ must be positive or negative, according as the direction of the primitive motion of rotation is from the axis G$x$ towards the axis G$y$, or from the axis G$y$ towards the axis G$x$, that is to say, in the direction of the sagitta $s$, or in the opposite direction. After the impact, the rotation will have place in the first or second direction, according as the value of $r_1$ or R is positive or negative.

479. Hitherto we have supposed, that the two bodies M and M′ are entirely free; but if they are retained by a fixed point or axis, the determination of their motions after the impact will always depend on the same principles, and will merely differ in the number of equations which should be considered.

For example, if the moveable M is retained by a fixed point G, the three first equations of No. 467 will be no longer necessary for the equilibrium of N and of the quantities of motion which are lost, during the impact, by all the points of this body. This fixed point G will not be always, as in the preceding cases, the centre of gravity of M, and, consequently, the integrals $\int xdm$, $\int ydm$, $\int zdm$ will be no longer cipher; but the six quantities $u$, $v$, $w$, $u_1$, $v_1$, $w_1$, will be zero; and by assuming that G$x$, G$y$, G$z$, are the three principal axes of M

which intersect in this point G, the three last equations of equilibrium will be then reduced to

$$\begin{aligned}
&\text{N}\,(a\cos\beta - b\cos\alpha) + \text{C}\,(r - r_1) = 0,\\
&\text{N}\,(c\cos\alpha - a\cos\gamma) + \text{B}\,(q - q_1) = 0,\\
&\text{N}\,(b\cos\gamma - c\cos\beta) + \text{A}\,(p - p_1) = 0,
\end{aligned}$$

as in the number cited. If to these equations there be joined the six equations (2) relative to the body M′, which is assumed to continue entirely free, and also equation (3), in which we should make $u_1 = 0$, $v_1 = 0$, $w_1 = 0$, we shall then have the ten equations which are necessary in order to determine the value of N, and the motions of the two bodies after the impact, when they are destitute of elasticity. If they are perfectly elastic, the three last equations (4) should be substituted for the three preceding equations, and equations (5) should be employed in place of equations (2). If the body M is retained by a fixed axis G$z$, which is not a principal axis, the fourth equation of No. 467 will be the only one necessary in order to determine the equilibrium of N and of the quantities of motion lost by M. As the axis of rotation then coincides with G$z$, before and after the impact, we shall have $p = 0$, $q = 0$, $p_1 = 0$, $q_1 = 0$; and as the three components of the velocity of G are also cipher, this equation will be reduced to

$$\text{N}\,(a\cos\beta - b\cos\alpha) + \text{C}\,(r - r_1) = 0;$$

C being always the moment of inertia with respect to the axis G$z$. When the two bodies are perfectly elastic, it should be replaced by

$$2\text{N}\,(a\cos\beta - b\cos\alpha) + \text{C}\,(r - \text{R});$$

and by joining to it equation (3), and those which refer to the body M′, we shall have all the equations which are required for the determination of N and the motions of the two moveables after the impact.

480. If in place of two bodies only, three or a greater number impinge on each other simultaneously, the equations of

equilibrium of the quantities of motion which are lost in the impact, by each of these bodies, should be formed by considering it by itself, after having joined to the quantities of motion lost by all its points, other unknown forces N, N′, N″, &c., applied to the points of contact of this body with all the others, and drawn from each of these points internally, in the direction of the respective normals to its surface at these points. When all the moveables are considered, the number of these unknown forces will be the same as that of the points of contact of these bodies; for they will represent equal and contrary quantities of motion, for every two of the moveables which touch in each point. But, at the instant of the greatest compression, that is to say, at the end of the impact of bodies destitute of elasticity, equation (3) will obtain for each point of contact; hence it follows that we shall have always a sufficient number of equations, to enable us to determine the state of all the moveables, and the values of N, N′, N″, &c., at this instant. When the moveables are perfectly elastic, the solution of the problem can be obtained for each of them separately, by considerations similar to those employed in No. 469.

481. In order to give a simple example of this general solution, let M, M′, $\mu$, denote the masses of three homogeneous spheres, whose centres are G, G′, C, (fig. 19). If the sphere $\mu$, which is supposed to be at rest before the impact, be struck simultaneously by M and M′, which touch it at the points K and K′, then though M and M′ should be actuated by a motion of rotation before the impact, it will not be affected by this impact; and as $\mu$ does not acquire any during this percussion, we have only to determine the velocities of G, G′, C, after the impact, in magnitude and direction, by means of the velocities and directions of G and G′, before this impact.

Let therefore $a$, $b$, $c$ denote the components of the velocity of G before the impact, parallel to the three fixed rectangular axes, $ox$, $oy$, $oz$, and let $a'$, $b'$, $c'$ be the components of G′ parallel to the same axes. Let $u$, $v$, $w$, $u'$, $v'$, $w'$, represent the

values of these six components at the instant of greatest compression, and $u_1, v_1, w_1$, the components of the velocity of C in the direction of the same axes, at this instant. Likewise, let $\alpha, \beta, \gamma$, be the angles comprised between the radius KC, and lines drawn through the point K parallel to the axes $ox$, $oy$, $oz$, and $\alpha', \beta', \gamma'$ the angles, which the radius K′C makes with lines drawn through the point K′ parallel to these axes. Let N denote the quantity of motion communicated at the instant in question to $\mu$ by M in the direction of KC, or to M by $\mu$ in the direction KG, and N′ that which is communicated to $\mu$ by M′ in the direction K′C, or to M′ by $\mu$ in the direction K′G′. The nine equations of equilibrium, of the quantities of motion which are lost, and of the forces N and N′, which it will be sufficient for us to consider, will be

$$\left.\begin{aligned}
&M(a-u) - N\cos\alpha = 0,\\
&M(b-v) - N\cos\beta = 0,\\
&M(c-w) - N\cos\gamma = 0,\\
&M'(a'-u') - N'\cos\alpha' = 0,\\
&M'(b'-v') - N'\cos\beta' = 0,\\
&M'(c'-w') - N'\cos\gamma' = 0,\\
&N\cos\alpha + N'\cos\alpha' - \mu u_1 = 0,\\
&N\cos\beta + N'\cos\beta' - \mu v_1 = 0,\\
&N\cos\gamma + N\cos\gamma' - \mu w_1 = 0;
\end{aligned}\right\} \quad (a)$$

with respect to which it should be remarked, that KG and K′G′ are the productions of KC and K′C.

Equation (3) applied to the points K and K′ will give, at the same time,

$$\left.\begin{aligned}
&u_1\cos\alpha + v_1\cos\beta + w_1\cos\gamma = u\cos\alpha + v\cos\beta + w\cos\gamma,\\
&u_1\cos\alpha' + v_1\cos\beta' + w_1\cos\gamma' = u'\cos\alpha' + v'\cos\beta' + w'\cos\gamma';
\end{aligned}\right\} \quad (b)$$

and, in this manner, we shall have the eleven equations which are necessary and sufficient to determine the eleven unknown quantities N, N′, $u$, $v$, &c.

If we make

$$\cos a \cos a' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma' = \cos\delta,$$
$$a \cos a + b \cos\beta + c \cos\gamma = k,$$
$$a' \cos a' + b' \cos\beta' + c' \cos\gamma' = k',$$

$\delta$ will be the angle GCG', and $k$ and $k'$ will represent the primitive velocities of G and G' in the directions GK and G'K'.

And since

$$\cos^2 a + \cos^2\beta + \cos^2\gamma = 1, \quad \cos^2 a' + \cos^2\beta' + \cos^2\gamma' = 1,$$

equations (a) will give

$$u \cos a + v \cos\beta + w \cos\gamma = k - \frac{N}{M},$$
$$u' \cos a' + v' \cos\beta' + w' \cos\gamma' = k - \frac{N'}{M'},$$
$$u_1 \cos a + v_1 \cos\beta + w_1 \cos\gamma = \frac{N + N' \cos\delta}{\mu},$$
$$u_1 \cos a' + v_1 \cos\beta' + w_1 \cos\gamma' = \frac{N' + N \cos\delta}{\mu};$$

by means of which equations (b) will become

$$M\mu k = N(M + \mu) + N'M \cos\delta,$$
$$M'\mu k' = N'(M' + \mu) + NM' \cos\delta;$$

hence we obtain ($k$)

$$N = \frac{k(M' + \mu)M\mu - k'MM'\mu \cos\delta}{(M + \mu)(M' + \mu) - MM' \cos^2\delta},$$
$$N' = \frac{k'(M + \mu)(M'\mu - kMM'\mu \cos\delta}{(M + \mu)(M' + \mu) - MM' \cos^2\delta};$$

values which should be always positive. By substituting them in equations (a), the values of $u$, $v$, &c., the nine components of the velocities of G, G', C, which have place at the end of the impact, may be determined when the bodies are destitute of elasticity.

If, on the contrary, they are perfectly elastic, and if U, V, W

be the components of the velocity of G, and U′, V′, W′, those of the velocity of G′, and $U_1$ $V_1$ $W_1$ those of the velocity of C at the end of the impact, we can obtain, by considerations similar to those already employed in No. 469, the following nine equations:

$$M(u - U) - N\cos\alpha = 0,$$
$$M(v - V) - N\cos\beta = 0,$$
$$M(w - W) - N\cos\gamma = 0,$$
$$M'(u' - U') - N'\cos\alpha' = 0,$$
$$M'(v' - V') - N'\cos\beta' = 0,$$
$$M'(w' - W') - N'\cos\gamma' = 0,$$
$$N\cos\alpha + N'\cos\alpha' + \mu(u_1 - U_1) = 0,$$
$$N\cos\beta + N'\cos\beta' + \mu(v_1 - V_1) = 0,$$
$$N\cos\gamma + N'\cos\gamma' + \mu(w_1 - W_1) = 0.$$

If each of these equations be added to the equation which corresponds to it amongst equations (a), there results

$$M(a - U) - 2N\cos\alpha = 0,$$
$$M(b - V) - 2N\cos\beta = 0,$$
$$M(c - W) - 2N\cos\gamma = 0,$$
$$M'(a' - U') - 2N'\cos\alpha' = 0,$$
$$M'(b' - V') - 2N'\cos\beta' = 0,$$
$$M'(c' - W') - 2N\cos\gamma' = 0,$$
$$2N\cos\alpha + 2N'\cos\alpha' - \mu U_1 = 0,$$
$$2N\cos\beta + 2N'\cos\beta' - \mu V_1 = 0,$$
$$2N\cos\gamma + 2N'\cos\gamma' - \mu W_1 = 0;$$

and in order to obtain immediately from these last equations, the values of the nine unknown quantities U, V, W, U′, V′, W′, $U_1$, $V_1$, $W_1$, it is only requisite to substitute the preceding values of N and N′ in them.

The final velocities of the points G, G′, C will be still the same, whether the impacts of M and M′ on $\mu$, instead of being simultaneous, follow one another at such a very short interval of time, that these three points may not be sensibly displaced during this very short time. The very short durations of the two impacts, whether simultaneous or successive, may likewise be unequal, and the instant of the greatest compression at the points K and K′ may not be the same.

## CHAPTER VIII.

### EXAMPLES OF THE MOTION OF A FLEXIBLE STRING.

### I. *Vibrations of a Flexible String.*

482. LET AMB (fig. 20) a perfectly flexible string, very little extensible, homogeneous, and of the same thickness throughout, be stretched in the direction of its length, by a force equivalent to a given weight $w$, and let it be attached at its two extremities to the fixed points A and B. As its weight is neglected relatively to $w$, it can be considered in its state of equilibrium as rectilinear($a$). This being the case, if it be made to deviate ever so little from this direction, and if small velocities be impressed on all its points, this string will oscillate on each side of the line AMB; and the object of this chapter is to determine its position and the velocities of its different points, at any instant whatever. At the end of $t$ any time whatever, let us suppose that the string assumes the form of the curve AM′B, which may be either plane or one of double curvature, and let M′ be the position that M, a given point on this string assumes. Let P be the projection of M′ on the line AMB, and

$$AM = x, \quad AP = x + u;$$

likewise let $y$ and $z$ be the two other coordinates of M′, perpendicular to each other and to the axis AB.

As the displacements of the points of the string are very small, the variables $u$, $y$, $z$ will be likewise very small, and the object of our investigation will be to determine their values in functions of $x$ and $t$.

Let the differential element of the curve AM′B at the point

M′, be denoted by $ds$, and the density of the string at this point by $\varepsilon$, the area of the section perpendicular to its length at this same point multiplied by $\varepsilon$, or $\varepsilon ds$, is equal to the element of its mass. In the state of equilibrium, the elements of this mass are proportional to the lengths, as the string is homogeneous and of a constant thickness; hence the length of the element at the point M being $dx$, its mass will be $\frac{pdx}{gl}(b)$, $p$ and $l$ denoting the weight and length of the entire string, and $g$ the gravity; hence as the mass of this element does not change during the motion, we shall have constantly

$$\varepsilon ds = \frac{pdx}{gl}.$$

If this element $\varepsilon ds$ was solicited by a given motive force, whose components parallel to the axes of the coordinates were $X\varepsilon ds$, $Y\varepsilon ds$, $Z\varepsilon ds$, the components of the motive force lost, during the instant $dt$ in the direction of these axes, would be

$$\left(X - \frac{d^2u}{dt^2}\right)\varepsilon ds, \quad \left(Y - \frac{d^2y}{dt^2}\right)\varepsilon ds, \quad \left(Z - \frac{d^2z}{dt^2}\right)\varepsilon ds;$$

consequently, in order to obtain the equations of equilibrium of these forces, which will be those of the motion of the string,

$$X - \frac{d^2u}{dt^2}, \quad Y - \frac{d^2y}{dt^2}, \quad Z - \frac{d^2z}{dt^2},$$

should be put in place of X, Y, Z, in equations (1) of No. 298, and its preceding value should be substituted for $\varepsilon ds$. Now, as by hypothesis, the quantities X, Y, Z, are cipher, there results

$$\left.\begin{aligned} d\,.\,\frac{T d\,(x+u)}{ds} &= \frac{p}{gl}\frac{d^2u}{dt^2}\,dx, \\ d\,.\,T\frac{dy}{ds} &= \frac{p}{gl}\frac{d^2y}{dt^2}\,dx, \\ d\,.\,T\frac{dz}{ds} &= \frac{p}{gl}\frac{d^2z}{dt^2}\,dx; \end{aligned}\right\} \quad (1)$$

T being the tension of the element $\epsilon ds$, and $x + u$ the abscissa of the point M′ to which these equations refer. They can only be integrated, when reduced to a linear form, by the consideration of the small extent of the vibrations of the string.

483. As $dx$ the element of the string in the state of equilibrium becomes $ds$ in the state of motion, and as $w$ and T are the measures of the respective tensions which it experiences in these two states, $\text{T} - w$ their difference must be proportional to the ratio of its extension $ds - dx$ to its primitive length $dx$ (No. 288), therefore, we shall have

$$\text{T} - w = q\frac{(ds - dx)}{dx};$$

$q$ being a given constant weight, which will depend on the material and thickness of the string. Besides, we have

$$ds^2 = (dx + du)^2 + dy^2 + dz^2;$$

and if not only the points of the curve AM′B, but also the directions of its tangents deviate little from the line AMB, the quantities $\frac{dz}{ds}$ and $\frac{dy}{ds}$ will be very small fractions; therefore, if their squares be neglected, we shall have

$$ds = dx + du, \quad \text{T} = w + q\frac{du}{dx};$$

and if the products $\frac{du}{dx}\frac{dy}{ds}$ and $\frac{du}{dx}\frac{dz}{ds}$ be likewise neglected, equations (1) will become($c$)

$$\frac{d^2u}{dt^2} = a^2\frac{d^2u}{dx^2}, \quad \frac{d^2y}{dt^2} = a'^2\frac{d^2y}{dx^2}, \quad \frac{d^2z}{dt^2} = a'^2\frac{d^2z}{dx^2}, \qquad (2)$$

in which, for conciseness, we make

$$\frac{glq}{p} = a^2, \quad \frac{glw}{p} = a'^2.$$

As the variables $u$, $y$, $z$, are separated in these equations (2), it follows, that the vibrations of the string parallel to the

axes of $x$, $y$, $z$, will be independent the one of the other, and will coexist together without mutually influencing each other. Moreover, it appears that the *transversal* vibrations will be the same in the direction of the axis of $y$ and of the axis of $z$, so that it will be sufficient to consider one of them, the first, for example. With respect to the *longitudinal* vibrations, it also appears from comparing the first of equations (2) with either of the two last, that they follow the same laws as the others, from which they only differ in the magnitude of the coefficient $a^2$, which surpasses $a^2$ in the ratio of $q$ to $w$.

484. The complete integral of the following equation of partial differences of the second order,

$$\frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2},$$

is

$$y = f(x + at) + \mathrm{F}(x - at); \qquad (3)$$

$f$ and F denoting two arbitrary functions. In fact, we have, whatever be the nature of the function $\psi$,

$$\frac{d\psi(x \pm at)}{dt} = \pm a\frac{d\psi(x \pm at)}{dx},$$

$$\frac{d^2\psi(x \pm at)}{dt^2} = a^2\frac{d^2\psi(x \pm at)}{dx^2};$$

hence we infer,

$$\frac{d^2y}{dt^2} = a^2\frac{d^2f(x + at)}{dx^2} + a^2\frac{d^2\mathrm{F}(x - at)}{dx^2};$$

and as we have also

$$\frac{d^2y}{dx^2} = \frac{d^2f(x + at)}{dx^2} + \frac{d^2\mathrm{F}(x - at)}{dx^2},$$

these values render the given equation identical.

If the time $t$ is reckoned from the commencement of the motion, and if $a$ or $\sqrt{\frac{glw}{p}}$ be regarded as a positive quan-

quantity, $x+at$ will be positive during the entire continuance of the motion, and $x-at$ will be either a negative quantity, or a positive quantity less than $l$. If therefore, $\zeta$ be a positive variable, it will be sufficient, in order to be able to apply formula (3), to know the values of $f\zeta$ and $\mathrm{F}(-\zeta)$ from $\zeta=0$ to $\zeta=\infty$, and those of $\mathrm{F}\zeta$ from $\zeta=0$ to $\zeta=l$. Now these values of $f\zeta$ and $\mathrm{F}(\pm\zeta)$ may be determined, as we proceed to show, by the condition of the immobility of the points A and B during the motion, combined with the initial state of the string.

485. If, at the commencement of the motion,

$$y=\phi x, \quad \frac{dy}{dt}=\phi' x;$$

these two functions $\phi x$ and $\phi' x$ will be cipher when $x=0$ and when $x=l$, and from $x=0$ to $x=l$, they will be given by the initial figure of the cord, and the velocities impressed, at this epoch, on its different points. If $t$ be made $=0$ in formula (3), and in its differential relative to $t$, we shall have

$$\phi x=fx+\mathrm{F}x, \quad \phi' x=a\frac{dfx}{dx}-a\frac{d\mathrm{F}x}{dx};$$

and if we make

$$\frac{1}{a}\int \phi' x dx=\Phi x,$$

and substitute $\zeta$ for $x$, we shall have(e)

$$f\zeta=\tfrac{1}{2}\phi\zeta+\tfrac{1}{2}\Phi\zeta, \quad \mathrm{F}\zeta=\tfrac{1}{2}\phi\zeta-\tfrac{1}{2}\Phi\zeta. \qquad (4)$$

The function $\Phi\zeta$ will contain an arbitrary constant, but it is evident that it will disappear in formula (3), which is composed of the sum of the values of $f\zeta$ and $\mathrm{F}\zeta$, relative to two different values of $\zeta$. It is not necessary therefore to take this constant into account, and we may assume, for greater clearness, that the function $\Phi\zeta$ vanishes when $\zeta=0$. By means of equations (4) the values of $f\zeta$ and $\mathrm{F}\zeta$ will be known, but only from $\zeta=0$ to $\zeta=l$, since the functions $\phi\zeta$ and $\Phi\zeta$ are only given in this interval.

As the points A and B are fixed, the values of $y$, when $x=0$, and when $x=l$, must be constantly cipher. If $at$ be supposed equal to $\zeta$, we shall therefore have by equation (3),

$$f\zeta + \mathrm{F}(-\zeta) = 0, \quad f(l+\zeta) + \mathrm{F}(l-\zeta) = 0, \qquad (5)$$

for all values of the positive variable $\zeta$.

In virtue of the first of these two equations, the values of $\mathrm{F}(-\zeta)$ will be equal and of a contrary sign to those of $f(\zeta)$. If $l+\zeta$ be substituted for $\zeta$ in the second equation (5), and if it be then taken from the first, there results

$$f(2l+\zeta) = f\zeta;$$

by means of which, $f\zeta$ will be known from $\zeta = 0$ to $\zeta = \infty$, when this function shall have been determined from $\zeta = 0$ to $\zeta = 2l$. Finally, if $\zeta \angle l$, we obtain by substituting $l - \zeta$ for $\zeta$ in the second equation (5),

$$f(2l-\zeta) = -\mathrm{F}\zeta.$$

Consequently, the values of $f(2l-\zeta)$ from $\zeta = 0$ to $\zeta = l$, or what is the same thing, those of $f\zeta$ from $\zeta = l$ to $\zeta = 2l$, will be known, when the values of $\mathrm{F}\zeta$, from $\zeta = 0$ to $\zeta = l$, are given. Hence, as the values of $f\zeta$ and $\mathrm{F}\zeta$ are given by equations (4), from $\zeta = 0$ to $\zeta = l$, equations (5) will determine those of $f(l+\zeta)$ and of $\mathrm{F}(-\zeta)$, from $\zeta = 0$ to $\zeta = \infty$ ($f$). Therefore all the values of these two functions, on which those of $y$ depend, will be known for all points of the string, and at any instant whatever of the motion. The corresponding values of $\frac{df\zeta}{d\zeta}$, $\frac{d\mathrm{F}\zeta}{d\zeta}$, and, consequently, those of $\frac{dy}{dt}$ will be likewise known; and the values of $z$ and $\frac{dz}{dt}$ may be obtained in the same manner. Consequently, the figure of the string, and the transversal velocities of all its points at any instant whatever, will be known; and this completely solves the problem, as far as the motion of the string perpendicularly to its natural direction, is concerned.

There is nothing in the question, by means of which the values of $f(-\zeta)$, or those of $F\zeta$, can be determined, when $\zeta$ is greater than $l$; consequently, these parts of the two arbitrary functions, the knowledge of which is not necessary in the application of formula (3), will continue altogether indeterminate.

486. In order to know the value of $y$ which will result from equations (3), (4), (5), we shall consider successively the part of this value which arises from the initial figure of the string, or of the function $\phi x$, and that which arises from the initial velocities of its different points, or form the function $\phi' x$.

1st. At the commencement of the motion, let ACB (fig. 21) be the given projection of the string on the plane of the axes of $x$ and $y$, so that if on AB, a part AD $= x$ be taken, DC the corresponding ordinate may be equal to $\phi x$. On the production of AB, let there be traced a curve BC′A′, equal to ACB, but inversely posited, so that if BD′ be assumed equal to BD, the ordinate D′C′ may be equal to DC, and affected with an opposite sign. On the two productions of AA′, let the curve ACBC′A be repeated indefinitely, so that A′C″B′C‴A″ may be the position which ACBC′A′ would assume, if this curve slided parallel to the axis of $x$, until A coincided with A′, and A′ with A″, and that $\text{AC}_{,}\text{B}_{,}\text{C}_{,,}\text{A}_{,}$ may be the position of A′C′BCA, when it slides in such a manner that A′ may coincide with A, and A with $\text{A}_{,}$; and let the same be supposed to take place with respect to the productions beyond A″ and $\text{A}_{,}$. This being done, if there be taken two abscissæ,

$$\text{AE} = x + at, \quad \text{AE}' = x - at,$$

of which the second may be either positive or negative, and the corresponding ordinates EF and E′F′, which may be either positive or negative, be erected, their semisum

$$\tfrac{1}{2}(\text{EF} + \text{E}'\text{F}'),$$

will be the part of $y$ depending on the initial figure of the cor

2ndly. Let us suppose that the ordinates of the curve ACB, instead of representing the primitive displacements of the points of the string, denote now their initial velocities divided by $a$, so that if AD be taken equal to $x$, then DC may be equal to $\frac{1}{a}\phi' x$. Let another curve ACH (fig. 22) be so traced that to the abscissa $AD = x$ the ordinate $DC = \frac{1}{a}\int \phi' x dx = \Phi x$ may correspond. Then as the integral commences with $x$, and as the function $\phi' x$ is also cipher when $x = 0$, this curve will touch the axis of $x$ at the point A. If $AB$ be taken $= l$, and if BH be the corresponding ordinate, we shall have

$$BH = \frac{1}{a}\int_0^l \phi' x dx,$$

and because $\phi' x = 0$ when $x = l$, the tangent at H will be parallel to the axis of the abscissæ ($g$). Let there be traced the curve HC′A′ equal to ACH, and so placed that if BD′ be taken equal to BD, we may have $D'C' = DC$; then on the two productions of AA′, let the curve ACHC′A′ be repeated indefinitely, as in the preceding construction; and this being done, if there be taken the two abscissæ

$$AK = x + at, \quad AK' = x - at,$$

the second of which may be either positive or negative, and if there be raised the corresponding ordinates KL and K′L′, which may also be positive or negative, we shall have

$$\tfrac{1}{2}(KL - K'L'),$$

for the part of $y$, which results from the initial velocities of the points of the string.

Therefore the complete value of $y$ will be

$$y = \tfrac{1}{2}(EF + E'F') + \tfrac{1}{2}(KL - K'L'); \qquad (a)$$

and a similar construction will give the corresponding value of $\frac{dy}{dt}$. In fact, we have

$$\frac{d\,.\,\text{EF}}{dt} = a\frac{d\,.\,\text{EF}}{dx}, \quad \frac{d\,.\,\text{E}'\text{F}'}{dt} = -a\frac{d\,.\,\text{E}'\text{F}'}{dx},$$

$$\frac{d\,.\,\text{KL}}{dt} = a\frac{d\,.\,\text{KL}}{dx}, \quad \frac{d\,.\,\text{K}'\text{L}'}{dt} = -a\frac{d\,.\,\text{K}'\text{L}'}{dx};$$

therefore we shall have

$$\frac{dy}{dt} = \frac{a}{2}\left(\frac{d\,.\,\text{EF}}{dx} - \frac{d\,.\text{E}'\text{F}'}{dx}\right), \quad + \frac{a}{2}\left(\frac{d\,.\,\text{KL}}{dx} + \frac{d\,.\,\text{K}'\text{L}'}{dx}\right).$$

Now, if through the points F, F′, L, L′ (fig. 21 and 22), the tangents $\text{F}f$, $\text{F}'f'$, $\text{L}l$, $\text{L}'l'$, be drawn, and also the lines $\text{F}x$, $\text{F}'x$, $\text{L}x$, $\text{L}'x$, parallel to the axis of $x$, and in the direction of the positive values of $x$, we shall also have

$$\frac{d\,.\,\text{EF}}{dx} = \text{tang}\, x\text{F}f, \quad \frac{d\,.\,\text{E}'\text{F}'}{dx} = \text{tang}\, x\text{F}'f',$$

$$\frac{d\,.\,\text{KL}}{dx} = \text{tang}\, x\text{L}l, \quad \frac{d\,.\,\text{K}'\text{L}'}{dx} = \text{tang}\, x\text{L}'l';$$

hence there will result,

$$\left.\begin{aligned} \frac{dy}{dt} &= \frac{a}{2}(\text{tang}\, x\text{F}f - \text{tang}\, x\text{F}'f'), \\ &+ \frac{a}{2}(\text{tang}\, x\text{L}l + \text{tang}\, x\text{L}'l'); \end{aligned}\right\} \quad (b)$$

in this formula the angles may be either positive or negative, but they will be always acute, (which is indeed indicated by the figure), for each of the points F, F′, L, L′. The values of $z$ and $\frac{dz}{dt}$ may be constructed in a similar manner.

487. It appears from the construction of the curves represented by figures 21 and 22, that when the product $at$ is increased by $2l$, the ordinate $y$ and the velocity $\frac{dy}{dt}$, expressed by formulæ (a) and (b) resume the values which they had before this addition of $2l$. The same is the case with respect to the values of $z$ and $\frac{dz}{dt}$. Consequently, at the end of an inter-

val of time equal to $\frac{2l}{a}$, the string reverts to the same state relatively to its form and the transversal velocities of all its points. Therefore in a vacuum, if the points A and B be firmly fixed, the string will perform an indefinite series of small oscillations, and the duration of each oscillation, which the string takes in going and returning, is equal to $\frac{2l}{a}$. But the resistance of the air and the communication of a part of the motion of the string to its extreme points A and B, gradually diminish the amplitudes of its oscillations, and at length eventually destroy them, without however sensibly affecting the isochronism; this is a result similar to that furnished by the motion of the pendulum in the air (No. 190); and it has been adverted to here, as a consequence of analysis, that has been confirmed by observation($h$). Hence if T denote the duration of an entire oscillation or vibration of the string, and if $n$ be the number of vibrations which take place in the unit of time, we shall have

$$\text{T} = \frac{2l}{a} = 2\sqrt{\frac{pl}{gw}}, \quad n = \tfrac{1}{2}\sqrt{\frac{gw}{pl}}.$$

The greater the number of vibrations performed in a given time, the higher will be the *tone*. It is therefore determined by the number $n$, which is evidently independent of the magnitudes of the amplitudes supposed to be very small. For the same string, this number is proportional to the square root of the tension $w$; for two strings of equal thickness, and consisting of the same materials, the weight $p$ is proportional to the length $l$, and, when the tension is given, the number $n$ is in the inverse ratio of this length; finally, for two strings of the same length, and equally stretched, $n$ is in the inverse ratio of the square roots of their weights. These different laws have long since been confirmed by experiment. Nevertheless there are cases to which we shall shortly advert, in which the string, in consequence of its initial state, is divided into equal parts

connected by points which remain immoveable during the entire continuance of the motion; this raises the tone proportionally to the number of these aliquot parts.

If the points of the string are not actuated by any initial velocity, we shall have simply

$$y = \tfrac{1}{2}(\text{EF} + \text{E}'\text{F}'), \quad \frac{dy}{dt} = \frac{a}{2}(\text{tang}\, x\text{F}f - \text{tang}\, x\text{F}'f').$$

It is evident from a consideration of the form of the curve represented by the figure 21, that when $at$ is any multiple of $l$, the velocity $\frac{dy}{dt}$ will be cipher, and the string will resume the same figure, but situated in positions, which are alternately the reverse one of the other. ACB (fig. 23) being its figure when $t = 0$, it will be likewise its figure and position, when $at$ is an even multiple of $l$; but if $at$ be an odd multiple of $l$, then the cord will assume the inverse position AC′B, which is such, that if we make AD′ $=$ BD, we shall have D′C′ $= -$ DC. In these two extreme positions, ACB and A′C′B′ the transversal velocities of all the points of the cord will be cipher; and the cord will take $\frac{1}{2}$ T, the time of a semivibration, to pass from the one to the other.

488. In general, the parts of the lines represented by figures 21 and 22 are not the analytical productions the one of the other; these lines form *discontinuous* curves, that is to say, curves, all whose points are not subjected to the same equation between the abscissa and ordinate; but, at the points of junction of $A_1$, $B_1$, A, B, A′, B′, &c. (fig. 21), $A_1$, $H_1$, A, H, A′, H′, &c. (fig. 22), of two different portions, the tangent is always common to the two adjacent parts. The curve relative to the initial form of the string, and that which represents the law of the velocities impressed on all its points, may likewise be discontinuous curves; provided that in each of the points in which their form changes, the tangent continues nevertheless the same for the two adjacent parts. This restriction is founded on this, that by their nature, the accele-

rating force of a material point, and the velocity with which it is actuated, are always finite, real and measurable, so that in the problems of dynamics, the functions of the time which express the velocities and accelerating forces of the different points of a moveable, can never become infinite. In the present case, the condition relative to the velocity is satisfied; for the transversal velocities are expressed by formula (b), by means of tangents of certain angles, multiplied by the constant quantity $a$; and by hypothesis, these angles never attain to 90°, but, on the contrary, are always very small. With respect to the accelerating forces, they would become infinite in the points where two portions of the curve intersect under a finite angle, and these forces would increase without limit, near to similar points of junction. In fact, let $m$ and $m'$ (fig. 20) be two points of the string, very near to M′, whose distances from this point we shall suppose infinitely small; and let the forces, which at any instant whatever, act on the portion $m$M′$m'$ of the string be considered, that is to say, the tensions which have place at its extremities, and which act in the direction of $mh$ and $m'h'$, the parts of the tangents at $m$ and $m'$. If these tensions be denoted by H and H′, and the mass of $m$M′$m'$ by $\mu$, then in order that the accelerating force of this small mass, resolved parallel to AB, may not be infinite when $\mu$ is infinitely small, the difference H − H′ must be very small, and at least proportional to $\mu$. Moreover, the components of H and H′ perpendicular to AB and parallel to the axis of $y$, will be $\text{H}\left(\frac{dy}{dx}\right)$ and $\text{H}'\left[\frac{dy}{dx}\right]$, in which $\frac{dy}{dx}$ is substituted for $\frac{dy}{ds}$, as in No. 483, and $\left(\frac{dy}{dx}\right)$, $\left[\frac{dy}{dx}\right]$ denote the values of $\frac{dy}{dx}$, at the points $m$ and $m'$ respectively. Hence the value of the motive force which draws $\mu$ towards AB, will be

$$\text{H}\left(\frac{dy}{dx}\right) - \text{H}'\left[\frac{dy}{dx}\right].$$

Therefore, in order that the corresponding accelerating force may not be extremely great, and become infinite, when the mass $\mu$ is infinitely small, it will be necessary that this difference should be also very small, and at least proportional to $\mu$; and as the quantities H and H′ differ already very little from each other, the same must be the case with respect to $\left(\frac{dy}{dx}\right)$ and $\left[\frac{dy}{dx}\right]$, the difference of which should be infinitely small, when the points $m$ and $m'$ are infinitely near to M′. Therefore, in no point of the string, and at no instant, can the tangents $mh$ and $m'h'$, at two points infinitely near to each other, intersect under a finite angle; which was required to be demonstrated.

This conclusion will likewise obtain, when the string is composed of two parts consisting of different materials; at their point of junction, the ordinate $y$ and its differential coefficient $\frac{dy}{dx}$ must have constantly the same value for these two parts; this, like the constant position of the extreme points, will furnish the equations that are indispensable for the determination of the arbitrary functions, and without which the solution of the problem would be incomplete; however, for further information on this point, the reader is referred to the *Journal de l'Ecole Polytechnique*, 18th Number, page 442.

489. D'Alembert was the first who resolved the problem of vibrating strings; the solution which he gave was that which has been detailed in the preceding numbers, and which is founded on the integration, in a finite form, of the equation $\frac{d^2y}{dt^2} = a^2\frac{d^2x}{dx^2}$; but this question may be also solved in another manner by means of formula (a) of Nov. 323. Whatever may be the nature of the given functions $\phi x$ and $\phi' x$, provided that they vanish when $x = 0$ and when $x = l$, we have, by the formula just cited,

$$\left.\begin{aligned}\phi x &= \frac{2}{l}\Sigma\left(\int_0^l \sin\frac{i\pi x'}{l}\phi x' dx'\right)\sin\frac{i\pi x}{l},\\ \phi' x &= \frac{2}{l}\Sigma\left(\int_0^l \sin\frac{i\pi x'}{l}\phi' x' dx'\right)\sin\frac{i\pi x}{l},\end{aligned}\right\} \qquad \text{(a)}$$

for all values of $x$, from $x = 0$ to $x = l$ inclusively, that is to say, for the entire length of the string; $i$ being, as in No. 323, a positive integer number, and the characteristics $\Sigma$ indicating sums which embrace all values of $i$, from $i = 1$, to $i = \infty$. On the other hand, it is easy to shew, that all expressions, such as

$$y = (\text{A} \sin a\, at + \text{B} \cos a\, at) \sin (ax + \beta), \qquad \text{(b)}$$

satisfy the given equation($h$)

$$\frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2}; \qquad \text{(c)}$$

A, B, $a$, $\beta$ being arbitrary constants.

Hence if we assume

$$\left.\begin{aligned}y &= \frac{2}{l}\Sigma\left(\int_0^l \sin\frac{i\pi x'}{l}\phi x' dx'\right)\sin\frac{i\pi x}{l}\cos\frac{i\pi at}{l}\\ &+ \frac{2}{\pi a}\Sigma\left(\int_0^l \sin\frac{i\pi x'}{l}\phi' x' dx'\right)\frac{1}{i}\sin\frac{i\pi x}{l}\sin\frac{i\pi at}{l},\end{aligned}\right\} \qquad \text{(d)}$$

this value of $y$ will satisfy all the conditions of the problem, and, consequently, will contain its solution. In fact, each of the terms of the sums $\Sigma$ satisfy, separately, equation (c); consequently, as this equation is linear, these sums will likewise satisfy it. If in formula (d), we make $x = 0$, or $x = l$, we have $y = 0$, whatever be the value of $t$; this satisfies the condition of the extreme points of the string being fixed. Finally, formula (d) gives ($i$)

$$\left.\begin{aligned}\frac{dy}{dt} &= \frac{2}{l}\Sigma\left(\int_0^l \sin\frac{i\pi x'}{l}\phi' x' dx'\right)\sin\frac{i\pi x}{l}\cos\frac{i\pi at}{l}\\ &- \frac{2\pi a}{l^2}\Sigma\left(\int_0^l \sin\frac{i\pi x'}{l}\phi x' dx'\right) i \sin\frac{i\pi x}{l}\sin\frac{i\pi at}{l};\end{aligned}\right\} \qquad \text{(e)}$$

and if in these values of $y$ and $\frac{dy}{dt}$, we make $t = 0$, they will become respectively $\phi x$ and $\phi' x$, in virtue of equation (a); this satisfies the initial state of the string in all its generality.

We are indebted to Lagrange for this other solution of the problem, who has also demonstrated that it coincides with that of D'Alembert.

Previously to the time of Lagrange, D'Bernoulli had already solved the problem of vibrating strings, by assuming for $y$, a value composed of terms comprised in formula (b), and which are subjected to become cipher, when $x = 0$ and when $x = l$, that is to say, by means of the expression,

$$\left.\begin{aligned} y = &\left(A \sin \frac{\pi a t}{l} + B \cos \frac{\pi a t}{l}\right) \sin \frac{\pi x}{l} \\ &+ \left(A' \sin \frac{2\pi a t}{l} + B' \cos \frac{2\pi a t}{l}\right) \sin \frac{2\pi x}{l} \\ &+ \left(A'' \sin \frac{3\pi a t}{l} + B'' \cos \frac{3\pi a t}{l}\right) \sin \frac{3\pi x}{l} \\ &+ \&c. \end{aligned}\right\} \quad (f)$$

in which $A$, $A'$, $A''$, &c., $B$, $B'$, $B''$, &c., are arbitrary constants. In order that this solution may be complete, these coefficients should be determined by means of an initial state of the string which is arbitrarily given; but this (as far at least as the analysis was concerned) constituted the principal difficulty of the question; otherwise, this formula (f) enables us to determine the different modes of transversal vibrations of musical strings, and the laws of these vibrations.

490. Formulæ (d) and (e) indicate the laws of motion of a vibrating string which were stated in No. 487; they likewise shew, that the tone may sometimes rise, as was observed in that number, and $n$ the number of vibrations performed in the unit of time, become a multiple of its general value, without the tension of the cord undergoing any change. In fact, if we suppose, that the values of $\phi x'$ and $\phi' x'$ are such that

$$\int_0^l \phi x' \sin \frac{i\pi x'}{l} dx' = 0, \quad \int_0^l \phi' x' \sin \frac{i\pi x'}{l} dx' = 0, \qquad \text{(g)}$$

for all values of $i$, which are not multiples of a given number $m$, conditions which can be satisfied in an infinite variety of different ways; formulæ (d) and (e) will only contain sines and cosines of the multiples of $\frac{m\pi at}{l}$; consequently, the state and position of the cord will become the same, whenever $at$ is increased by a multiple of $\frac{2l}{ma}$; and from the value of $a$ given in No. 483, it is evident that the number $n$, on which the elevation of the tone depends, (No. 487) will be

$$n = \frac{m}{2}\sqrt{\frac{gw}{pl}},$$

that is to say, it will be increased in the ratio of $m$ to unity.

In this case, formula (d) will only contain the sines of the multiples of $\frac{m\pi x}{l}$; we shall therefore have $y=0$, for N, N′, N″, &c., the equidistant points of the cord (fig. 24), which correspond to $x = \frac{l}{m}, = \frac{2l}{m}, = \frac{3l}{m}$, &c.; so that these points, the number of which is $m-1$, will remain immoveable, during the entire continuance of the motion, like the extreme points A and B. On this account, the points N, N′, N″, &c., are termed *nodes of vibration*. At the commencement of the motion, they will not be actuated by any velocity, and will not, therefore, deviate from the line AB. ACN, NC′N′, N′C″N″, &c., the parts of the string which are situated alternately on one side or the other of AB, will vibrate as detached strings whose lengths AN, NN′, N′N″, &c., are equal to $\frac{l}{m}$, and of which the isochronous and simultaneous vibrations will be performed in a time equal to $\frac{2l}{ma}$.

The simplest manner in which the conditions expressed by equations (g) can be satisfied, is by taking, for example,

$$\phi x = h \sin \frac{m\pi x}{l}, \quad \phi' x = 0;$$

$h$ being a given constant. This implies, that the points of the string are not actuated by any initial velocities, and that at the commencement of the motion, it consisted of $m$ equal parts, which are situated alternately on one side or the other of AB. Each of these parts of the curve is termed a *trochoid*, the length of which is $\frac{l}{m}$ and the height $h(k)$. In this case, the value of $y$ in formula (d) is reduced to the first term of the second member, which corresponds to $i = m$. If the integration be performed relative to $x'$, we have simply($l$),

$$y = h \sin \frac{m\pi x}{l} \cos \frac{m\pi at}{l};$$

therefore, during the entire continuance of the motion, the figure of the cord is composed of a number $m$ of trochoids, of a constant breadth and variable height; and when $at$ is an odd multiple of $\frac{l}{2m}$, it coincides with the line AB. This particular solution of the problem of vibrating strings was the one given by Taylor, before the general solution was known.

491. All that has been stated relatively to transversal vibrations, may be immediately applied to longitudinal vibrations. For, in order to have at any instant whatever, the expression of the variable $u$ of No. 482, it is only necessary to substitute in that which has been found for $y$, the constant $a$ of No. 483 in place of $a$. Then we should take for $\phi x$ the displacement of the point M (fig. 20), at the commencement of the motion, in the direction of the length of the string, that is to say, the initial value of MP; and $\phi' x$ will express the initial velocity of the point M, which will be in the direction

MB or MA, according as it is positive or negative. These functions $\phi x$ and $\phi' x$ will be given arbitrarily from $x=0$ to $x=l$; and if they change their form in this interval, it is necessary that for those values of $x$ in which this takes place, each of these functions, and its differential coefficient, should notwithstanding have the same values in the two adjacent parts of the string.

It follows from this, that if T be the entire duration of a longitudinal vibration, that is to say, the interval between two identical states of the string, and $n'$ the number of vibrations in the unit of time, we shall have (No. 487),

$$T' = \frac{2l}{a} = 2\sqrt{\frac{pl}{gq}}, \quad n' = \tfrac{1}{2}\sqrt{\frac{gq}{pl}}.$$

This number $n'$, and the tone which it determines, do not depend on the tension $w$; however it appears from observation that the longitudinal tone is raised a little when the tension is increased; which must be attributed to the circumstance, that while the length of the string comprised between the points A and B remains the same, its weight $p$ diminishes according as it is more stretched.

492. It appears from a comparison of this number $n'$, with that of the transversal vibrations of the same string, that

$$n' = n\sqrt{\frac{g}{w}};$$

so that, when every thing else is the same, the tone produced by the longitudinal vibrations will be acuter than that which corresponds to the transversal vibrations in the ratio of $\sqrt{g}$ to $\sqrt{w}$.

The weight $q$ is the tension which should be employed in order to double the natural length of the string, on the supposition that the law of its extension is constant. In fact, if for a given tension $\Delta$, the length of any part of the string is increased in the ratio of $1+\delta$ to 1, the element adjacent to

the point M, which experiences successively, the tensions $w$ and T, in the state of equilibrium, and in the state of motion, will be increased in the ratio of $1 + \frac{\delta w}{\Delta}$ and of $1 + \frac{\delta T}{\Delta}$ to unity respectively; therefore, the lengths $dx$ and $ds$ in these two states, will be to each other as $\Delta + \delta w$ is to $\Delta + \delta T$, so that we shall have

$$\frac{ds}{dx} = \frac{\Delta + \delta T}{\Delta + \delta w};$$

hence we obtain

$$\frac{ds - dx}{dx} = \frac{\delta(T - w)}{\Delta},$$

the square of the fraction $\delta$ being neglected. Therefore, from the values of $ds - dx$ and of $T - w$ given in No. 483, we shall have

$$q = \frac{\Delta}{\delta};$$

consequently, $q$ will be the tension which corresponds to $\delta = 1$, or which will double the length of the cord, if its lengthening always increases uniformly($l$).

As $w$ the tension of a musical string, is always considerably less than what is requisite in order to double the length, it follows that the ratio $\sqrt{\frac{q}{w}}$, which is equal to $\frac{n'}{n}$, is always very great; it may be determined *à priori*, from knowing the increase of length produced by the tension $w$, and measured directly; for if this increase of length be denoted by $\gamma$, we shall have,

$$w = \frac{\gamma \Delta}{\delta l},$$

since $\delta l$ is that which corresponds to the tension $\Delta$; and by substituting this value of $w$ and that of $q$ in the expression of $\frac{n'}{n}$, there results,

$$\frac{n'}{n} = \sqrt{\frac{l}{\gamma}};$$

hence we obtain, conversely,

$$\gamma = \left(\frac{n}{n'}\right)^2 l,$$

for the value of $\gamma$, the increase of length, when that of $\frac{n}{n'}$ is known.

This simple relation between the number of longitudinal vibrations and that of transversal vibrations of the same string, has been verified by an experiment made by M. Cagniard-Latour on a very long string, the transversal vibrations of which were visible and sufficiently slow to permit him to reckon them.

## II. *Longitudinal Vibrations of an Elastic Rod.*

493. We shall suppose that this rod is homogeneous, and, that in its natural state, it is either of a prismatic or a cylindrical form; figure 25 then represents a section made through the mean filament AB, that is to say, through the line which passes through the centres of gravity of all the sections of the rod perpendicular to its length (No. 314). If, for example, the rod is a cylinder with a circular base, AB is its axis of figure; its diameter is very small, and, in all cases, the dimensions of the normal sections are very small relatively to the length of this line; but they are, however, sufficiently great to enable the rod to resist the flexion, so that it may be what has been termed an elastic rod in No. 306. In the longitudinal motion of this rod, which we propose to consider first, all the points that belong to the same normal section will have, at each instant, the same velocity parallel to AB; so that it will be sufficient to determine the motion of any *point* whatever of this line, such as M.

Let a fixed point C be taken on this line, and let $x$ denote the distance CM in the natural state of the rod; this distance will be positive or negative according as M appertains to the

part CB or to the part CA of AB. In the state of motion, let M′ be the position which M assumes at the end of the time $t$, then if $\text{MM}' = u$, it should be considered as positive or negative, according as this displacement has place on the side of B or on that of A, hence we shall always have $\text{CM}' = x + u$. It is proposed to determine the value of $u$ in a function of $x$ and $t$.

Let $p$ denote the weight of the rod, $l$ its length AB, and $g$ the gravity. In the natural state of the rod, the mass of the element which corresponds to the point M, and whose length is $dx$, is $\frac{pdx}{gl}$, this mass will not undergo any change during the motion; and if the element is solicited by an accelerating force such as X, acting in the direction M′B or M′A, according as it is positive or negative, its force lost during the instant $dt$ will be

$$\frac{pdx}{gl}\left(\text{X} - \frac{d^2u}{dt^2}\right).$$

Let T denote the tension of the same element acting at its extremity M′, and which will be positive or negative, according as it is directed from within the rod outwards, or from without inwards; $\text{T} + \frac{d\text{T}}{dx}dx$ will express the tension, which will act at the other extremity, in an opposite direction from that of T; consequently, it will be drawn in the direction M′B, by a force equal to $\frac{d\text{T}}{dx}dx$; and, in order that this force and the preceding may be in equilibrio, we should have

$$\frac{d\text{T}}{dx} + \frac{p}{gl}\left(\text{X} - \frac{d^2u}{dt^2}\right) = 0;$$

which agrees with equation (a) of No. 316.

It is necessary besides, that at the two extremities A and B, the value of T should be equal to a particular force, which will act in the direction of AB at the extremity A, and in that of the production of AB, at the extremity B.

494. As the natural length of the element under consideration is $dx$, and as its length becomes $dx + du$, when it is subjected to the tension T, we shall have

$$\text{T} = q\frac{du}{dx};$$

$q$ denoting a constant coefficient, the value of which, given by observation, will be($m$)

$$q = \frac{\Delta}{\delta},$$

if $\delta l$ denotes the entire lengthening of the rod, when it is subjected to a constant and given tension $\Delta$ (No. 492). If no given force acts on the points of the rod, then we should make $\text{X} = 0$ in the equation of the motion, and, by substituting for T its value, there will result

$$\frac{d^2u}{dt^2} = a^2\frac{d^2u}{dx^2}, \tag{1}$$

in which, for conciseness, we make

$$\frac{glq}{p} = a^2.$$

Moreover, if $v$ be the velocity of the point M′, and $s$ the dilatation of the rod at this same point, we shall have

$$v = \frac{du}{dt}, \quad s = \frac{du}{dx}, \quad \text{T} = qs.$$

When the value of $s$ is negative, this dilatation will be changed into a contraction; and the tension T will act in the direction M′A, or in the direction MB′, according as the rod is dilated or contracted.

Hence then the state of the rod, at any instant whatever, will be known, when $u$ shall have been determined in a function of $x$ and $t$; but, in order to obtain its value, equation (1) should be combined with those which refer to the initial state of the rod and to its extremities. Now, when $t = 0$, we shall suppose that

$$u = \phi x, \quad v = \phi' x;$$

so that $\phi x$ and $\phi' x$ may be functions arbitrarily given from $x = 0$ to $x = l$, c being taken as the initial position of A. Moreover, it is necessary, that at each *fixed* extremity of the rod, $u$ should be equal to cipher during the entire continuance of the motion; and T will then express the pressure which this fixed point will have to sustain. In like manner it is requisite that at each *free* extremity, which is not solicited by any given force, we should have $T = 0$, or $\frac{du}{dx} = 0$, for all values of $t$.

495. This system of equations may be resolved in the same manner as those which refer to vibrating strings, either by setting out from the integral under a finite form of equation (1), or by formulæ similar to those of No. 489. The following are the results which correspond to the different hypotheses that may be made on the state of the extremities of the rod.

1st. If the two points A and B are fixed, the given functions $\phi x$ and $\phi' x$ must be cipher, when $x = 0$ and when $x = l$, and we shall have, as in the number cited,

$$u = \frac{2}{l} \Sigma \left( \int_0^l \sin \frac{i\pi x'}{l} \phi x' dx' \right) \sin \frac{i\pi x}{l} \cos \frac{i\pi at}{l}$$

$$+ \frac{2}{\pi a} \Sigma \left( \int_0^l \sin \frac{i\pi x'}{l} \phi' x' dx' \right) \frac{1}{i} \sin \frac{i\pi x}{l} \sin \frac{i\pi at}{l}.$$

As often as $at$ is increased by $2l$, this value of $u$, and those of $v$ and $s$ which may be deduced from it, and consequently the state of the rod, become the same as before; hence, if T denote the time of an entire vibration, and $n$ the number of vibrations in the unit of time, we shall have

$$T = \frac{2l}{a} = 2\sqrt{\frac{pl}{gq}}, \quad n = \tfrac{1}{2}\sqrt{\frac{gq}{pl}};$$

so that the tone will be the same, as if the rod was a flexible string vibrating longitudinally.

2ndly. If the point A is fixed and the point B entirely free, the functions $\phi x$ and $\phi' x$ must be cipher when $x = 0$, and we must also have $\frac{d\phi x}{dx} = 0$ when $x = l$; the expression for $u$ will in this case be

$$u = \frac{2}{l}\Sigma\left(\int_0^l \sin\frac{(2i-1)\pi x'}{2l}\phi x' dx'\right)\sin\frac{(2i-1)\pi x}{2l}\cos\frac{(2i-1)\pi at}{2l}$$
$$+\frac{4}{\pi a}\Sigma\left(\int_0^l \sin\frac{(2i-1)\pi x'}{2l}\phi' x' dx'\right)\frac{1}{2i-1}\sin\frac{(2i-1)\pi x}{2l}\sin\frac{(2i-1)\pi at}{2l},$$

in which the sums indicated by $\Sigma$, extend to all the values of the integer number $i$, from $i = 0$ to $i = \infty$. In fact, all the terms of this value of $u$ satisfy equation (1)($n$); they fulfil, whatever be the value of $t$, the conditions $u = 0$ when $x = 0$, and $\frac{du}{dx} = 0$ when $x = l$, which answers to this second case; and, for $t = 0$, we obtain from it

$$u = \phi x = \frac{2}{l}\Sigma\left(\int_0^l \sin\frac{(2i-1)\pi x'}{2l}\phi x' dx'\right)\sin\frac{(2i-1)\pi x}{2l},$$
$$\frac{du}{dt} = \phi' x = \frac{2}{l}\Sigma\left(\int_0^l \sin\frac{(2i-1)\pi x'}{2l}\phi' x' dx'\right)\sin\frac{(2i-1)\pi x}{2l};$$

which we know is in fact the case, in virtue of equation (7) of No. 326.

The value of $u$, and those of $s$ and $v$ which are deduced from it, will become the same as often as $at$ is increased by any multiple whatever of $4l$; consequently, if $T'$ denotes the duration of an entire vibration of the rod, or the interval comprised between two consecutive returns of the rod to same state, we shall have

$$T' = \frac{4l}{a}.$$

This duration is therefore double of that which has place in the first case, and the number of vibrations in the unit of time will be only half. Consequently, the longitudinal tone of a rod fixed at one end and free at its other extremity, is an

*octave* below the tone of the same rod, when fixed at its two ends, which is in fact, confirmed by experiment.

3rdly. Finally, if the rod be free at both ends, the values of $\frac{d\phi x}{dx}$ must be cipher for $x=0$ and $x=l$, and we shall have in this case

$$u=\frac{1}{l}\int_0^l \phi x'dx'+\frac{2}{l}\Sigma\left(\int_0^l \cos\frac{i\pi x'}{l}\phi x'dx'\right)\cos\frac{i\pi x}{l}\cos\frac{i\pi at}{l}$$

$$+\frac{t}{l}\int_0^l \phi'x'dx'+\frac{2}{\pi a}\Sigma\left(\int_0^l \cos\frac{i\pi x'}{l}\phi'x'dx'\right)\frac{1}{i}\cos\frac{i\pi x}{l}\sin\frac{i\pi at}{l};$$

in which the sums indicated by $\Sigma$, extend as before, to all values of the integer $i$, from $i=1$ to $i=\infty$.

This value of $u$ satisfies, in fact, equation (1), and also the condition $\frac{du}{dx}=0$, for $x=0$ and $x=l$, which ought to have place in this third case, whatever be the value of $t$. For $t=0$, it gives

$$u=\phi x=\frac{1}{l}\int_0^l \phi x'dx'+\frac{2}{l}\Sigma\left(\int_0^l \cos\frac{i\pi x'}{l}\phi x'dx'\right)\cos\frac{i\pi x}{l},$$

$$\frac{du}{dt}=\phi'x=\frac{1}{l}\int_0^l \phi'x'dx'+\frac{2}{l}\Sigma\left(\int_0^l \cos\frac{i\pi x'}{l}\phi'x'dx'\right)\cos\frac{i\pi x}{l};$$

which agrees with formula (8) of No. 326(*o*).

When $\int_0^l \phi'x'dx'$ does not vanish, the rod has, independently of its vibrations, a uniform progressive motion, the common velocity of all whose points is equal to this integral divided by $l$. If it be supposed cipher, the rod will revert to the same state, for all values of $t$ which differ by a multiple of $\frac{2l}{a}$; so that the duration of each of its vibrations, and their number in the unit of time, will be the same as in the first case. It follows, therefore, that the tone of a rod fixed at its two ends is in *unison* with that of the same rod entirely free, which likewise agrees with experiment.

It is to be observed, that in what precedes, it is only the

*fundamental*, or lowest tone, of an elastic rod, that has been considered. The remark made in No. 490, on the nodes of the vibrations, and on the elevations of tone which correspond to them, may be easily extended to the motion of this rod, in each of the cases that have been examined.

496. When the rod, the longitudinal motion of which has been considered, extends indefinitely on each side of the point C, it will be no longer necessary to take into account what occurs at its two extremities, and the values of the velocity $v$, and of the dilatation $s$, relative to any point and instant whatever, may be immediately deduced from the integral of equations (1) under a finite form, in which it will be sufficient to determine the two arbitrary functions, by means of the initial values of $v$ and $s$, which will be given in functions of $x$.

This integral is

$$u = \phi(x + at) + \psi(x - at);$$

in which $\phi$ and $\psi$ indicate the two arbitrary functions.

We obtain from it, at any instant whatever,

$$\frac{du}{dt} = v = a\left(\frac{d\phi(x+at)}{dx} - \frac{d\psi(x-at)}{dx}\right),$$

$$\frac{du}{dx} = s = \frac{d\phi(x+at)}{dx} + \frac{d\psi(x-at)}{dx}.$$

When $t = 0$, we suppose that

$$v = fx, \quad s = Fx.$$

In the case under consideration, these two functions will be given for all positive and negative values of the variable; by making $t = 0$ in the preceding formulæ, we shall have

$$a\frac{d\phi x}{dx} - a\frac{d\psi x}{dx} = fx, \quad \frac{d\phi x}{dx} + \frac{d\psi x}{dx} = Fx,$$

hence we obtain

$$\frac{d\phi x}{dx} = \tfrac{1}{2}Fx + \frac{1}{2a}fx, \quad \frac{d\psi x}{dx} = \tfrac{1}{2}Fx - \frac{1}{2a}fx,$$

and, consequently,

$$\frac{d\phi\,(x+at)}{dx} = \tfrac{1}{2}\,\mathrm{F}\,(x+at) + \frac{1}{2a}f(x+at),$$

$$\frac{d\psi\,(x-at)}{dx} = \tfrac{1}{2}\,\mathrm{F}\,(x-at) - \frac{1}{2a}f(x-at).$$

Therefore, whatever be the values of $t$ and $x$, we shall have

$$\left.\begin{aligned} v &= \tfrac{1}{2}f(x+at) + \tfrac{1}{2}f(x-at) \\ &\quad + \frac{a}{2}\,\mathrm{F}\,(x+at) - \frac{a}{2}\,\mathrm{F}\,(x-at), \\ s &= \frac{1}{2a}f(x+at) - \frac{1}{2a}f(x-at) \\ &\quad + \tfrac{1}{2}\,\mathrm{F}\,(x+at) - \tfrac{1}{2}\,\mathrm{F}\,(x+at)\,; \end{aligned}\right\} \quad (2)$$

these formulæ enable us to determine the state of the rod at any instant whatever; and thus the problem is completely solved.

497. By means of these equations (2), the laws of the propagation of sonorous waves along an elastic rod can be obtained, and in general, along an homogeneous solid bar of indefinite length, whose sections, perpendicular to its length, are every where equal, and of small extent.

If the sound issues from the point C, the bar will be agitated, at the commencement of the motion, through an inconsiderable extent on each side of this point. If the length of the primitive agitation be denoted by $2a$, the functions $fx$ and $\mathrm{F}x$ will be cipher from $x = a$ to $x = \infty$, and from $x = -a$ to $x = -\infty$; they will be given arbitrarily and independently of each other, for all values of $x$ comprised between $\pm a$; and likewise the functions $f(x+at)$, $f(x-at)$, $\mathrm{F}\,(x+at)$, $\mathrm{F}\,(x-at)$, will not have values different from zero, except when $x+at$ or $x-at$, the quantity contained under the sign $f$ or $\mathrm{F}$, is greater than $-a$, and less than $a$, regard being had to the signs, and the quantity $a$ being always considered as positive.

From this it appears that when $at$ surpasses $2\alpha$, we shall have $v = 0$ and $s = 0$ for all points contained in the extent of the primitive agitation; so that the motion of this part of the bar will only continue for a time equal to $\frac{2\alpha}{a}$. We shall have for any point M situated beyond the sphere of this agitation, and on the side of the positive $xs$.

$$x > \alpha, \quad f(x + at) = 0, \quad \mathrm{F}(x + at) = 0;$$

and equations (2) will be reduced to

$$v = \tfrac{1}{2} f(x - at) - \frac{a}{2}\mathrm{F}(x - at);$$

$$s = -\frac{1}{2a} f(x - at) + \tfrac{1}{2}\mathrm{F}(x - at);$$

hence there results

$$v = -as.$$

As long as $x > at + \alpha$, these values of $v$ and $s$ will be cipher; and they will become so again when $x \angle at - \alpha$; after the lapse of a portion of time equal to $\frac{x - \alpha}{a}$, the agitation will reach the point M, its duration will be $\frac{2\alpha}{a}(p)$; and the length of the part of the bar which will be agitated all at once, will be $2\alpha$. The same results obtain on the side of the negative values of $x$.

Thus, on each side of the primitive agitation, *a sonorous wave* will be produced, the extent of which will be constant and equal to that of this agitation, and it will be propagated with a uniform velocity equal to $a$. The velocities with which the points of the bar are successively actuated, will not vary with their distance from the place of the primitive agitation; so that the intensity of sound, which depends on the magnitude of these velocities, will be constant, and will not become feebler according as it is propagated farther; this arises from the circumstance of the propagation taking place in a cylindrical or prismatic bar.

In the extent of a sonorous wave, the velocity will not be, as in all the points of the primitive agitation, independent of the corresponding dilatation; for the one will be proportional to the other, in virtue of the equation $v = -as$, from which it appears that $v$ the velocity of M any point whatever, is a fraction of the velocity of propagation, expressed by $s$ the dilatation which corresponds to the same point, and that the proper motion of M will take place in the contrary direction to that of the propagation, or in the same direction, according as there is a dilatation or condensation at this point.

It is of importance to observe that it is in consequence of this relation between $v$ and $s$, each sonorous wave that is produced is not divided into two others, but is propagated in one sole direction. In like manner, if this relation exists in the entire extent of the *primitive* agitation, the motion will only be propagated on one side. Thus, if we suppose that $fx = -a\text{F}x$, equations (2) will be reduced to

$$v = f(x - at), \quad s = -\frac{1}{a}f(x - at);$$

therefore, for negative values of $x$, and which are, abstracting from the sign, greater than $-a$, we shall have $v = 0$, and $s = 0$; so that the motion will not be propagated beyond the extent of the primitive agitation on the side of the negative values of $x$. This will likewise be the case on the side of the positive values of $x$, when $fx$ is supposed to be equal to $a\text{F}x$.

It appears from No. 495, that $a$, the velocity of the propagation of sound in an indefinite bar, may be inferred from the duration of the longitudinal vibrations of an elastic rod consisting of the same materials, and having a given length. If this rod is supposed to be fixed or free at its two extremities, the value of $a$ will be equal to twice its length divided by the duration of each of its vibrations, which duration is obtained from their number in the unit of time, and, consequently, from the lowest tone of the rod; if the rod was fixed at only one extremity, the result of this division should be doubled.

498. If the bar, instead of extending indefinitely in the direction of the positive values of $x$, is terminated at a point B, situated beyond the extent of the primitive agitation, the sound, after having reached B, will be reflected towards the point C; and there will be an *echo* at this point B, whether it be supposed to be fixed, or entirely free.

If the distance CB, which will be greater than $a$, be denoted by $c$, and if B is a fixed point, we must have always $v = 0$, when $x = c$. Now, this condition can be satisfied by substituting for formulæ (2) the following,

$$v = \tfrac{1}{2}f(x + at) + f(x - at) - \tfrac{1}{2}f(2c - x - at)$$
$$+ \frac{a}{2}\,\text{F}(x + at) - \frac{a}{2}\,\text{F}(x - at) + \frac{a}{2}\,\text{F}(2c - x - at),$$

$$s = \frac{1}{2a}f(x + at) - \frac{1}{2a}f(x - at) - \frac{1}{2a}f(2c - x - at)$$
$$+ \tfrac{1}{2}\,\text{F}(x + at) + \tfrac{1}{2}\,\text{F}(x - at) + \tfrac{1}{2}\,\text{F}(2c - x - at),$$

which expressions continue to represent the initial state of the bar; and the value of $u$ which may be obtained by means of the equations

$$\frac{du}{dt} = v, \quad \frac{du}{dx} = s,$$

still continues to satisfy equation (1).

In fact, as the variable $x$ cannot be greater than $c$ for any point of the bar, and as $c$ surpasses $a$, we have $2c - x > a$, and, consequently, $f(2c - x) = 0$ and $\text{F}(2c - x) = 0$; hence there results $v = fx$ and $s = \text{F}x$, when $t = 0$. And since $c > a$, we have also $f(c + at) = 0$ and $\text{F}(c + at) = 0$; consequently, we have $v = 0$ when $x = c$ whatever be the value of $t$. Finally, we have identically $\frac{dv}{dx} = \frac{ds}{dt}$; and the value of $u(q)$, the complete differential of which is $vdt + sdx$, will be the sum of a function of $x - at$ and of a function of $x + at$, which will, consequently, satisfy equation (1). This being established, for a point M such that $x > a$, the quantities $f(x + at)$ and

$F(x + at)$ will be cipher, and the preceding values of $v$ and $s$ will be reduced to

$$v = v' + v_1, \quad s = -\frac{v'}{a} + \frac{v_1}{a},$$

by making, for conciseness,

$$\tfrac{1}{2}f(x - at) - \frac{a}{2}F(x - at) = v',$$

$$\frac{a}{2}F(2c - x - at) - \tfrac{1}{2}f(2c - x - at) = v_1.$$

The quantity $v'$ ceases to be cipher when $at > x - a$; it becomes so again for $at = x + a$; the time continuing to increase, $v_1$ ceases to be cipher for $at = 2c - x - a$, and will become so again for $at = 2c - x + a$; hence it follows that the point M will experience two agitations separated the one from the other by an interval of time equal to $\dfrac{2(c - x + a)}{a}$. The first will be the direct, and the second the reflected sound; the intensity of each of them will be the same, and they will be propagated with the same velocity $a$; and as, agreeably to the direction of the propagation, the one corresponds to $v'$, and the other to $- v_1$, it is evident that the same relation exists, for each of them, between the proper velocity of the point M, and the positive or negative dilatation with which it is accompanied. The same results are obtained when the point B is supposed to be entirely free, in which case we must have constantly $s = 0$ for $x = c$.

These laws of the propagation or reflexion of sound in a solid bar, obtain equally in the case of air contained in a very narrow, cylindrical, or prismatic canal. Those of the longitudinal vibrations of an elastic rod, which have been explained in No. 495, are likewise applicable to the vibrations of air contained in a tube of a given length, open or closed at its extremities, that is to say, to the sounds of *flutes*, in which, however, we should always except the modifications which

may arise at the embouchure. For further details on this subject, the reader is referred to the author's memoir on the Motion of Elastic Fluids in Cylindrical Tubes, and on the Theory of Wind Instruments, in the second volume of the *Memoires of the Academy of Sciences.*

## III. *Longitudinal Impact of Elastic Rods.* [illegible]

499. The formulæ of 495 are applicable to the impact of two or more elastic rods, consisting of the same materials, and having the same normal section, in which also the mean filaments move in the same right line. For this purpose, they can be considered as constituting, during the entire continuance of the contact of these bodies, one elastic, cylindrical, or prismatic rod, the state of which being variable from one instant to another, can be determined by these formulæ throughout its entire length, except for an extent of insensible magnitude, on each side of the points of junction.

In fact, if only two rods whose mean filaments are AE and FB (fig. 26) be considered, then when, in consequence of the differences of their velocities, they approach each other, EF the distance of their extremities E and F will become insensible, and will no longer exceed the radius of activity of the molecular forces, so that the extreme molecules of one of the two rods will commence to act on those of the other, and conversely; this mutual action will subsist, with varying intensity, as long as the distance EF is less than the radius of activity; the total force may be repulsive or attractive, and it is in this action at insensible distances, of the extreme points of the two bodies, that the phenomenon of the impact really consists. Now, as the law of molecular action in a function of the distance is unknown to us, we cannot determine the value of EF in a function of the time, no more than the variations of the velocities which the extreme points of the two rods experience in virtue of this force; so that if $e$ and $f$ be points of

AE and FB, situated at distances from E and F, which are insensible, and less than the radius of molecular activity, the velocities of the material points that belong to the slices whose thicknesses are $e$E and F$f$, will be unknown during the entire continuance of the shock. But beyond $e$ and $f$, and in the entire extent of A$e$ and $f$B, equation (1) of No. 494 will obtain, and the state of these two parts of the entire rod will be determined, at any instant whatever, by means of the integral of this equation, according as the two ends R and B are supposed to be fixed or moveable, that is to say, by means of the different formulæ of No. 495, in which it is only necessary to substitute suitable values for the arbitrary functions $\phi x$ and $\phi' x$.

500. As the molecular forces vary very rapidly with the distance, it follows that the unknown velocities of the extreme points of the two rods will also vary, so that at any instant whatever, the velocities of the points E and F may differ considerably from those of the points $e$ and $f$, although the distances $e$E and $f$F be insensible. The same will be the case with respect to the velocities of the points $e$ and $f$, compared with one another, and determined by means of their initial values, for they will be unequal, and may even have different signs; but it may be demonstrated as in No. 488, that T the tension, whether positive or negative, must be sensibly the same in these points $e$ and $f$, otherwise the accelerating force of the mass of insensible magnitude, comprised between the normal sections in these same points, should become extremely great and almost infinite. We shall suppose, that before the impact, each of the two rods is actuated by the same velocity throughout its entire extent; in this state, T the tension will be cipher for all the points of the two moveables; therefore at the commencement of the impact, that is to say, when the distance EF is equal to the radius of molecular activity, we shall have $T = 0$ at the points $e$ and $f$, as in all others. The tension, which is always equal for these two extreme

points, will then cease to be cipher; and it will appear, by a determination of its value, that it becomes again cipher after a certain interval of time. Now, if at this epoch, the velocities of the points $e$ and $f$ are such that the two rods may separate, that is to say, if the rods move with these velocities in opposite directions, or if when they move in the same direction, the velocity of that which precedes is the greater, the two rods must, in point of fact, separate, and the impact will terminate. But if, at the epoch in question, the velocities of $e$ and $f$ do not satisfy one of these two conditions, the impact will, so to speak, recommence; and the tension, which at the points $e$ and $f$ is equal, will reappear; it will become cipher again at the end of a new interval of time; and so on, it will continue to move in this manner, so that the two rods will not separate, but will vibrate as a single rod, whose length is AB.

Thus, the condition which is necessary and sufficient in order that the impact may terminate, and that one of the two rods may separate from the other, is the concurrence of these two circumstances: 1st, it is necessary that the tension should be cipher at the points $e$ and $f$, in order that the two rods may not press the one against the other; 2ndly, it is also necessary that, at the same time, the rods should move in opposite directions, or, if they move in the same direction, the velocity of the point which precedes should be the greater.

With respect to the two extremities A and B, we shall suppose, first, that each of them is entirely free, and secondly, that only one is free and the other fixed.

501. Let AE and FB, the lengths of the two rods, be denoted by $c$ and $c'$, and the entire distance AB by $l$, so that, the insensible distance EF being neglected, $c + c'$ may be equal to $l$ during the entire continuance of the impact. Let M be any point whatever belonging either to AE or FB; and immediately before the impact, let $x$ denote the distance of the point M from a fixed point taken on the line AB, which will be the position of the point A at this instant. Let $x + u$ be the distance

of the same point M from this fixed point, at the end of the time $t$, reckoned from this epoch; we shall have

$$\frac{d^2u}{dt^2} = a\frac{d^2u}{dx^2}; \tag{1}$$

$a$ being a constant quantity, which denotes the velocity of the propagation of sound in that description of matter of which the rods are formed (No. 497).

At the same time, $v$ the velocity of the point M, and T the tension at this point, which may be either positive or negative, will be respectively expressed as follows,

$$v = \frac{du}{dt}, \quad \mathrm{T} = q\frac{du}{dx},$$

$q$ denoting a given constant. The dilatation which accompanies the velocity $v$ may be deduced from T, and will have for its value $\frac{1}{q}\mathrm{T}$.

These three equations will obtain for all values of $x$, from $x = 0$ to $x = l$, except those which belong to points situated between $e$ and $f$, and which consequently differ from $c$ by an insensible quantity either more or less.

502. In the case of $t = 0$, we shall have $u = 0$ throughout the entire extent of AB; consequently, the term depending on $\phi x$ should be suppressed in the formulæ of 495. We shall first examine the case in which the two extremities A and B are entirely free.

Let $h$ be the velocity common to all the points of AE, at the instant when the impact commences, which velocity we shall suppose to be positive, or directed from A to B. In like manner, let $h'$ denote the velocity of the points of FB, at the same instant, which will be positive or negative, according as the two rods move in the same or opposite directions. These constant quantities $h$ and $h'$ will be given, and their difference $h-h'$ must be a positive quantity, in order that the shock may

have place. In the expression of $u$ relative to the third case of No. 495, $\phi' x$ should be assumed equal to $h$, from $x = 0$ to a value of $x$ ever so little less than $c$, and equal to $h'$, from a value of $x$ ever so little greater than $c$, to $x = l$, or, what is nearly the same thing, $x = c + c'$. We shall then have, without any appreciable error ($q$),

$$\int_0^l \phi' x' dx' = hc + h'c',$$

$$\int_0^l \phi' x' \cos \frac{i\pi x'}{l} dx' = \frac{l}{i\pi}(h - h') \sin \frac{i\pi c}{l};$$

and, since $\phi x' = 0$, the expression for $u$ will become

$$u = (hc + h'c')\frac{t}{l} + \frac{2l}{\pi^2 a}(h - h') \Sigma \frac{1}{i^2} \sin \frac{i\pi c}{l} \cos \frac{i\pi x}{l} \sin \frac{i\pi at}{l};$$

in which the sum $\Sigma$ extends to all integer and positive values of $i$, from $i = 1$, to $i = \infty$.

Therefore, in this first case, we shall have ($r$)

$$\left.\begin{aligned} v &= \frac{1}{l}(hc + h'c') + \frac{2}{\pi}(h - h') \Sigma \frac{1}{i} \sin \frac{i\pi c}{l} \cos \frac{i\pi x}{l} \cos \frac{i\pi at}{l}, \\ \mathrm{T} &= \frac{2q}{\pi a}(h - h') \Sigma \frac{1}{i} \sin \frac{i\pi c}{l} \cos \frac{i\pi at}{l}; \end{aligned}\right\} \quad (2)$$

and if $m$ and $m'$ denote the masses of the two rods, which are respectively proportional to their lengths $c$ and $c'$, the first term of this value of $v$ is $\dfrac{mh + m'h'}{m + m'}$, the velocity of their centre of gravity. If the two velocities $h$ and $h'$ are equal and affected with the same sign, we shall have constantly $v = h$ and $\mathrm{T} = 0$; and, in fact, the two rods move the one after the other, with a common velocity, and without compressing each other.

The periodic and convergent series that these formulæ contain, are comprised among those of which the sums may be exactly determined. For all given values of $x$ and $t$, these

sums may, without difficulty, be deduced from the known formula

$$\tfrac{1}{2}\theta = \sin\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta - \tfrac{1}{4}\sin 4\theta + \text{\&c.}, \qquad (3)$$

in which $\theta$ is a variable contained between the limits $\pm\pi$ exclusively. Consequently, the values of the velocity $v$ and of the tension T, may be calculated at each instant, and for every point of A$e$ and B$f$; and this is, in fact, the complete solution of the problem.

There are several ways of arriving at formula (3). For example, it may be obtained by differentiating equation (8) of No. 326, with respect to $x$, $x^2$ being previously substituted for $\phi x$; this gives

$$x = -\frac{1}{l}\Sigma\left(\int_0^l x'^2 \cos\frac{i\pi x'}{l}.\frac{i\pi dx'}{l}\right)\sin\frac{i\pi x}{l};$$

an equation which obtains for all values of $x$ less than $l$, and in which the sum $\Sigma$ extends to all values of the integer number $i$, from $i=1$ to $i=\infty$. We shall have, by performing the integration in the usual manner,

$$\int_0^l x'^2 \cos\frac{i\pi x'}{l}.\frac{i\pi dx'}{l} = \frac{2l^2}{i\pi}\cos i\pi;$$

consequently, we shall have

$$\frac{\pi x}{2l} = -\Sigma\frac{\cos i\pi}{i}.\sin\frac{i\pi x}{l};$$

a result which coincides with equation (3), by making (s) $\frac{\pi x}{l} = \theta$.

503. In virtue of the second equation (2), the variable T is cipher, not only when $t=0$, but also when $t$ is any multiple whatever of $\frac{l}{a}$; it is also cipher, whatever be the value of $t$, at the two extremities A and B, in which case we have $x=0$ and $x=l$.

If $t$ is cipher, or an even multiple of $\frac{l}{a}$, the first equation (2) gives (t)

$$v = \frac{1}{l}(hc + h'c')$$
$$+ \frac{1}{\pi}(h - h')\left[\Sigma\frac{1}{i}\sin\frac{i\pi(c-x)}{l} + \Sigma\frac{1}{i}\sin\frac{i\pi(c+x)}{l}\right],$$

or, what comes to the same thing, because $c + c' = l$, and $\cos i\pi = (-1)^i$,

$$\left.\begin{array}{l} v = \dfrac{1}{l}(hc + h'c') \\ -\dfrac{1}{\pi}(h-h')\left[\Sigma\dfrac{(-1)^i}{i}\sin\dfrac{i\pi(c'+x)}{l} + \Sigma\dfrac{(-1)^i}{i}\sin\dfrac{i\pi(c'-x)}{l}.\right] \end{array}\right\} \quad (4)$$

Now, if in formula (3) we take $\frac{\pi(c'-x)}{l}$ for $\theta$, there will result

$$\Sigma\frac{(-1)^i}{i}\sin\frac{i\pi(c'-x)}{l} = -\frac{\pi(c'-x)}{2l}.$$

If $x \angle c$, that is to say, if the point M appertains to A$e$, we may likewise assume for $\theta$ the quantity $\frac{\pi(c'+x)}{l}$, which will be less than $\pi$; hence we shall have

$$\Sigma\frac{(-1)^i}{i}\sin\frac{i\pi(c'+x)}{l} = -\frac{\pi(c'+x)}{2l};$$

in consequence of these values, equation (4) will be reduced to $v = h$. If, on the contrary, the point M appertains to $f$B, we shall have $x > c$ and $2l - c' - x \angle l$; we can therefore take

$$\theta = \frac{\pi(2l - c' - x)}{l};$$

and since

$$\sin\frac{i\pi(c'+x)}{l} = -\sin\frac{i\pi(2l - c' - x)}{l},$$

formula (3) will give

$$\Sigma\frac{(-1)^i}{i}\sin\frac{i\pi(c'+x)}{l} = \frac{\pi(2l - c' - x)}{2l};$$

by means of this value, and of that of $\Sigma \frac{(-1)^i}{i} \sin \frac{i\pi(c'-x)}{l}$, equation (4) will be reduced to $v = h'(u)$.

Therefore, $h$ and $h'$ the initial velocities of A$e$ and $f$B, the two parts of the entire rod, are by this means verified. Moreover, it appears that they obtain not only when $t = 0$, but likewise whenever $t$ is an even multiple of $\frac{l}{a}$; and since at these epochs, T is cipher for the entire rod, it follows that for all these values of $t$, the two parts of the rod will be in the same state as at the commencement of the impact.

It should be remarked, that the first equation (2) fails, when it is applied to the *initial* velocity of the point E; for if $t$ is supposed equal to cipher, when $x$ is exactly equal to $c$, there will result,

$$v = \frac{1}{l}(hc + h'c') + \frac{2}{\pi}(h - h')\,\Sigma \frac{(-1)^i}{i} \sin \frac{i\pi c}{l}.$$

But, by equation (3), we have($v$)

$$\Sigma \frac{(-1)^i}{i} \sin \frac{i\pi c}{l} = -\frac{\pi c}{l};$$

consequently, we shall have $v = h'$, which will not be true except when $h = h'$, in which case the state of the part corresponding to $e$E cannot differ from the rest of the rod. But it was already stated that, in the general case, this part, and that which belongs to F$f$, are not comprised in the equations of motion.

If $t$ is an odd multiple of $\frac{l}{a}$, the first equation (2) gives at once($x$)

$$v = \frac{1}{l}(hc + h'c')$$

$$+ \frac{1}{\pi}(h - h')\left[\Sigma \frac{(-1)^i}{i} \sin \frac{i\pi(c-x)}{l} + \Sigma \frac{(-1)^i}{i} \sin \frac{i\pi(c+x)}{l}\right].$$

Now it appears, by comparing this value of $v$ with formula (4), that the one may be obtained from the other by

merely changing the letters $h$ and $h'$, $c$ and $c'$; hence it follows, without any new calculation, that when $t$ is an odd multiple of $\frac{l}{a}$, the points which belong to a value of $x$ less than $c'$, will be actuated with the velocity of $h'$, and those which refer to $x > c$, with the velocity $h$; that is to say, if G is a point so situated, that AG $= c'$ and GB $= c$, and if $g$ and $g'$ be assumed at insensible distances on each side of G, the part A$g$ will move with the velocity $h'$, and the part $g'$B with the velocity $h$.

504. It results from this discussion, that if $c = c'$, the part A$e$ will be actuated with the velocity $h'$ at the end of $t$ a portion of time equal to $\frac{l}{a}$, and the part $f$B, with the velocity $h$ at the end of the same time; and as at this instant, T the tension is every where equal to cipher, and since, by hypothesis, we have $h > h'$, it follows, that the rods will separate from each other (No. 500); so that, in this case, the duration of the impact will be $\frac{l}{a}$, and the two moveables being perfectly elastic and equal in mass, will, after the impact, exchange the velocities which they had before the impact.

Conversely, if the lengths $c$ and $c'$ are different, the impact will not terminate, and the two elastic rods cannot separate; for the epochs at which the tension is cipher will always coincide with those in which the two extremities E and F, or more accurately, the two points $e$ and $f$, have a common velocity equal either to $h$ or $h'$. But if $c' > c$, in which case the point G will belong to $f$B, and if the elastic rod be supposed to be cut in this point, so that the part FB may itself be supposed to consist of two parts FG and GB, which are actuated by the same velocity $h'$ before the impact, the part GB will separate from FG at the end of $t$, a portion of time $= \frac{l}{a}$. In fact, at this instant, the tension T will be cipher, and $h$ and $h'$ the velocities of the points $g$ and $g'$ will be such, that the parts AG and GB may separate

from one another. After the impact, the duration of which will be equal to $\frac{l}{a}$, as in the case of $c'=c$, the part GD will move with the velocity $h$, and the parts AE and FG, with the common velocity $h'$. The same thing will also obtain, if the three parts AE, FG, GB, were themselves cut and divided into other portions either equal or unequal, provided that before the impact, all the portions of AE had the same velocity $h$, and all the portions of FG and GD a common velocity $h'$. Thus, for example, if we suppose that a prismatic or cylindrical homogeneous rod is cut into $n+n'$ equal parts; and if the $n$ first parts actuated with the velocity $h$, impinge on the series of the $n'$ other parts, supposed to be at rest before this percussion, then if $n$ surpasses $n'$, no part will separate, and they will be all transferred in the direction of the impact by oscillating in this direction, and producing a sound corresponding to the entire length of the rod, supposed to be free at its two extremities; but if $n'>n$, a number $n$ of the anterior parts will be detached from the others, which will move with a common velocity equal to $h$, and the $n'$ other parts will remain at rest and in juxta position. This result, which may by analogy be applied to a series of spheres, is applicable to the phenomenon discussed in No. 363.

505. Let us now consider the case in which the point A is fixed, and let us suppose that before the impact, the part AE is at rest, and that all the points of the part FB are actuated by a common velocity, the direction of which we shall suppose to be negative, and denoted by $-k$. It is necessary then to employ the expression for $u$ relative to the second case of No. 495, in which we shall make $\phi' x=0$, from $x=0$ to $x=c$, and $\phi' x=-k$ from $x=c$ to $x=c+c'=l$. Since $\phi x=0$, for all values of $x$, there will result($y$)

$$u=-\frac{8lk}{\pi^2 a}\Sigma\frac{1}{(2i-1)^2}\cos\frac{(2i-1)}{2l}\pi c\sin\frac{(2i-1)\pi x}{2l}\sin\frac{(2i-1)\pi at}{2l};$$

hence we obtain

$$\left.\begin{aligned} v &= -\frac{4k}{\pi}\Sigma\frac{1}{2i-1}\cos\frac{(2i-1)\pi c}{2l}\sin\frac{(2i-1)\pi x}{2l}\cos\frac{(2i-1)\pi at}{2l}, \\ \text{T} &= -\frac{4qk}{\pi a}\Sigma\frac{1}{2i-1}\cos\frac{(2i-1)\pi c}{2l}\cos\frac{(2i-1)\pi x}{2l}\sin\frac{(2i-1)\pi at}{2l}; \end{aligned}\right\} \quad (5)$$

in these formulæ, the series are such, that the sums may be determined, consequently, the velocity and tension may be exactly obtained at each instant for any given point of A$e$, or $f$B. We shall employ, for this purpose, the known formula($z$)

$$\frac{\pi}{4} = \cos\theta - \tfrac{1}{3}\cos 3\theta + \tfrac{1}{5}\cos 5\theta + \&c., \qquad (6)$$

which obtains for all values of $\theta$ comprised between $\pm\frac{\pi}{2}$, exclusively, and which may, for example, be deduced from formula (7) of No. 326, by substituting $x$ for $\phi x$, then differentiating the result with respect to $x$, and finally making $\frac{\pi x}{2l} = \theta$.

506. In virtue of the second equation (5), T the tension is nothing in all points of the two rods, when $t$ is cipher or any odd multiple whatever of $\frac{2l}{a}$.

If $t$ is cipher, or an even multiple of $\frac{2l}{a}$, the first equation (5) gives

$$v = -\frac{2k}{\pi}\left[\Sigma\frac{1}{2i-1}\sin\frac{(2i-1)\pi(x+c)}{2l} + \sin\frac{(2i-1)\pi(x-c)}{2l}\right],$$

or, what comes to the same thing, because $c + c' = l$ and $\sin\frac{(2i-1)\pi}{2} = -(-1)^i(a')$,

$$\left.\begin{aligned} v = \frac{2k}{\pi}\Big[&\Sigma\frac{(-1)^i}{2i-1}\cos\frac{(2i-1)\pi(x-c')}{2l} \\ &-\Sigma\frac{(-1)^i}{2i-1}\cos\frac{(2i-1)\pi(x+c')}{2l}\Big]. \end{aligned}\right\} \quad (7)$$

Now, it appears from the initial state of the two rods, that this formula, must in the case of $t=0$, be reduced to $v=0$, for $x \angle c$, and to $v=-k$ for $x>c$; which it is easy at once to verify.

By taking $\frac{\pi(x-c')}{2l}$ equal to $\theta$ in equation (6), there results

$$\Sigma\frac{(-1)^i}{2i-1}\cos\frac{(2i-1)\pi(x-c')}{2l}=-\frac{\pi}{4}.$$

We may also assume $\frac{\pi(x+c')}{2l}$ for $\theta$, when $x \angle c$, this gives

$$\Sigma\frac{(-1)^i}{2i-1}\cos\frac{(2i-1)\pi(x+c')}{2l}=-\frac{\pi}{4};$$

and these formulæ, in fact, reduce equation (7) to $v=0$. When $x>c$, we shall also have $2l-x-c' \angle l$; therefore, by assuming

$$\theta=\frac{\pi(2l-x-c')}{2l},$$

and observing that $(b')$

$$\cos\frac{(2i-1)\pi(x+c')}{2l}=-\cos\frac{(2i-1)\pi(2l-x-c')}{2l},$$

formula (6) will give

$$\Sigma\frac{(-1)^i}{2i-1}\cos\frac{(2i-1)\pi(x+c')}{2l}=\frac{\pi}{4};$$

by means of which, and of the preceding value of $\Sigma\frac{(-1)^i}{2i-1}\cos\frac{(2i-1)\pi(x-c')}{2l}$, equation (7) will be reduced to $v=-k$, as we know it should. When $t$ is an odd multiple of $\frac{2l}{a}$, the value of $v$, furnished by the first equation (5), is equal and of a contrary sign to that which obtains when $t$ is cipher, or an even multiple of $\frac{2l}{a}$; it follows, therefore, that at the end

of a portion of time equal to $\frac{2l}{a}$, the velocities of each of the points of A$c$ are cipher, and each of the points of $f$B is actuated by a velocity which is positive and equal to $k$; and as at this instant T the tension is throughout equal to cipher, it follows that the rod FB will be detached from the rod AE, and it will be reflected back with a velocity equal and contrary to that which it had before the impact.

Thus, the impact of the rod FB against the rod AE, which presses at A against a fixed obstacle, will continue for a portion of time equal to $\frac{2l}{a}$, this accords with what has been stated in No. 362, relative to the reflexion of a perfectly elastic body. It may be also remarked that at the middle of the impact, that is to say, at the end of a portion of time equal to $\frac{l}{a}$, we shall have $v = 0$, by the first equation (5), for all values of $x$; so that at this instant, the impinging rod FB will have lost all its velocity, and likewise the rod AE will not have acquired any motion. At the same instant, we shall have, in virtue of the second equation (5),

$$\text{T} = \frac{2qk}{\pi a}\Big[\Sigma\frac{(-1)^i}{2i-1}\cos\frac{(2i-1)\pi(x-c)}{2l} + \Sigma\frac{(-1)^i}{2i-1}\cos\frac{(2i-1)\pi(x+c)}{2l}\Big];$$

hence by the same process as in the case of equation (7), we infer $\text{T} = -\frac{qk}{a}$ or $\text{T} = 0$, according as $x \angle c$ or $x > c$. Therefore, at the middle of the impact, the tension is cipher throughout the entire extent of the impinging rod; but the struck rod is uniformly condensed; and it is the pressure which it exercises in the direction AE, or from within outwards, that causes the striking rod to rebound.

## IV. *Digression on the Integrals of Equations of partial Differences.*

507. With respect to equations of partial differences, those of an order higher than the first are not integrable, except in very few cases, under a finite form, even when the equations are linear. Therefore in order to solve those problems which lead to such equations, we are, for the most part, obliged to recur to their integrals expressed in series; and it is then necessary that we should be assured, in each case, that the series employed has all the generality that is suitable to the given equation of partial differences, and that it contains a sufficient number of arbitrary functions to enable us to express the complete integral of this equation. Now, there is no general rule on this subject: this number may be less than that which indicates the order of the given equation, or of the highest partial differences that it contains; it changes with the quantity according to the powers of which the series is expanded; and it may even occur that all the arbitrary functions disappear, and that the series only contains an infinite number of arbitrary constants, while at the same time it does not cease to express the complete integral. It is these various circumstances which we now proceed to examine, first in general, and afterwards more particularly, in what respects those linear equations to which we are led in the greater number of mechanical and physical problems.

508. Let $u$ be a function of any number whatever of independent variables such as $t, x, y, z$, &c. Let us suppose that this function satisfies $L = 0$, a given equation of partial differences. Whatever may be the value of $u$, it can be always conceived to be developed into a series arranged according to the powers of one of the variables $t, x, y, z$, &c., or more generally, of another quantity $\theta$ depending on one or more of these variables. Let, therefore,

$$u = \text{P}\theta^{\alpha} + \text{Q}\theta^{\beta} + \text{R}\theta^{\gamma} + \&c.; \tag{a}$$

$\alpha, \beta, \gamma$, &c., P, Q, R, &c., being indeterminate exponents and coefficients. If this value of $u$ be substituted in the equation $\text{L} = 0$, and if L be then developed according to the powers of $\theta$, and the coefficients of all the terms of this development be put separately equal to cipher, there will result a series of equations, each of which will contain one independent variable less than $\text{L} = 0$ $(a)$; and if the most general values of $\alpha, \beta, \gamma$, &c., P, Q, R, &c., which can satisfy this equation were obtained, series (a) would be also the most general value of $u$ which will satisfy the equation $\text{L} = 0$. Thus according to the quantity $\theta$ that shall have been selected, these expressions of $u$ in a series will be different, but all of them will be, under equivalent forms, the complete integral of $\text{L} = 0$; so that if this integral can be expressed in a finite form, each of these series will be a different development of it, and may always be deduced from it. However, when we have succeeded, by means of the other conditions of the problem which will have conducted us to $\text{L} = 0$, in determining all the arbitrary quantities which are contained in series (a), it is necessary that it should be convergent, in order that we may be able to make use of it; and if it becomes divergent for these values of $\theta$, this quantity should be changed, and the series (a) replaced by another, arranged according to the powers of a different variable.

This being established, if there be assumed for $\text{L} = 0$, linear equations of different orders, it is evident that the coefficients P, Q, R, &c., determined in the most general manner, may, notwithstanding, contain unequal numbers of arbitrary functions, according as the series (a) is arranged with respect to the powers of such or such a variable $\theta$; and it is likewise evident, as has been stated above, that a case may occur in which all the arbitrary functions would disappear from this series, which would then only contain an infinite number of arbitrary constants, and which, notwithstanding, will be also

the complete integral of the equation $\text{L} = 0$. The characteristic property of this singular form of the complete integral, without any arbitrary function, of a linear equation of partial differences, consists in this, that all the terms of the series which represents it, may be determined, independently the one of the other, and thus satisfy separately the given equation, so that the general value of $u$ is the sum of an infinite number of particular values of this function.

509. Let there be taken, for example, the very simple linear equation of partial differences of the second order,

$$\frac{du}{dt} = a\frac{d^2u}{dx^2}, \tag{b}$$

in which $a$ is a given constant quantity. If the value of $u$ be developed according to the powers of $t$, there results for the most general series which satisfies this equation,

$$u = \phi x + \frac{at}{1}\frac{d^2\phi x}{dx^2} + \frac{a^2t^2}{1.2}\frac{d^4\phi x}{dx^4} + \frac{a^3t^3}{1.2.3}\frac{d^6\phi x}{dx^6} + \&c.\,; \tag{c}$$

in which $\phi x$ is an arbitrary function. Under this form, the complete integral of equation (b) requires only one arbitrary function, which represents the value of $u$ when $t = 0$. But if the general value of $u$ be developed according to the powers of $x$, there results

$$\left.\begin{aligned} u = \psi t &+ \frac{x^2}{1.2}.\frac{d\psi t}{adt} + \frac{x^4}{1.2.3.4}\frac{d^2\psi t}{a^2dt^2} + \&c. \\ + x\Psi t &+ \frac{x^3}{1.2.3}.\frac{d\Psi t}{adt} + \frac{x^5}{1.2.3.4.5}\frac{d^2\Psi t}{a^2dt^2} + \&c.\,; \end{aligned}\right\} \tag{d}$$

in which $\psi t$ and $\Psi t$ are arbitrary functions, that express the values of $u$ and $\frac{du}{dx}$, when $x = 0$. Consequently, under this other form, the complete integral of equation (b) contains two arbitrary functions.

These two series can be obtained by the method of indeterminate coefficients and exponents ($b$), by making, succes-

sively, $\theta = t$, and $\theta = x$ in series (a). They may be also deduced from Taylor's theorem, for by this theorem, we have

$$u = \text{U} + (t-a)\,\text{U}' + \frac{(t-a)^2}{1.2}\,\text{U}'' + \frac{(t-a)^3}{1.2.3}\,\text{U}''' + \&\text{c.};$$

in which we suppose that $a$ denotes a particular value of $t$, and that for this value

$$u = \text{U}, \quad \frac{du}{dt} = \text{U}', \quad \frac{d^2u}{dt^2} = \text{U}'', \quad \frac{d^3u}{dt^3} = \text{U}''', \ \&\text{c.}$$

Now, by means of equation (b) and its successive differentials with respect to $t$, we obtain

$$\text{U}' = \frac{ad^2\text{U}}{dx^2}, \quad \text{U}'' = \frac{ad^2\text{U}'}{dx^2} = \frac{a^2d^4\text{U}}{dx^4}, \quad \text{U}''' = \frac{ad^2\text{U}''}{dx^2} = \frac{a^3d^6\text{U}}{dx^6}, \ \&\text{c.}$$

The quantity U will therefore alone remain arbitrary, and we shall have

$$u = \text{U} + a\,(t-a)\,\frac{d^2\text{U}}{dx^2} + \frac{a^2\,(t-a)^2}{1.2}\,\frac{d^4\text{U}}{dx^4} + \&\text{c.};$$

which will coincide with series (c), if the constant $a$ be made equal to cipher, that is to say, when the series is developed according to the powers of $t$, and U is made equal to $\phi x$. Series (d) may be obtained in a similar manner($c$). These two series (c) and (d) may also be transformed, the one into the other; in fact, if $\phi x$ be developed according to the powers of $x$, so that

$$\phi x = \text{A} + \frac{\text{B}x}{1} + \frac{\text{C}x^2}{1.2} + \frac{\text{D}x^3}{1.2.3} + \frac{\text{E}x^4}{1.2.3.4} + \frac{\text{F}x^5}{1.2.3.4.5} + \&\text{c.};$$

in which A, B, C, D, E, F, &c., denote arbitrary constants, we have

$$\frac{d^2\phi x}{dx^2} = \text{C} + \text{D}x + \frac{\text{E}x^2}{1.2} + \frac{\text{F}x^3}{1.2.3} + \&\text{c.};$$

$$\frac{d^4\phi x}{dx^4} = \text{E} + \text{F}x + \&\text{c.};$$

&c.;

by means of which, series (c) will become

$$u = \text{A} + \text{C}at + \frac{\text{E}a^2t^2}{1.2} + \&c.$$

$$+ (\text{B} + \text{D}at + \frac{\text{F}a^2t^2}{1.2} + \&c.)\, x$$

$$+ (\text{C} + \text{E}at + \&c.)\frac{x^2}{1.2}$$

$$+ (\text{D} + \text{F}at + \&c.)\frac{x^3}{1.2.3}$$

$$+ \&c.$$

Now, if we make

$$\text{A} + \text{C}at + \frac{\text{E}a^2t^2}{1.2} + \&c. = \psi t,$$

$$\text{B} + \text{D}at + \frac{\text{F}a^2t^2}{1.2} + \&c. = \Psi t,$$

$\psi t$ and $\Psi t$ will be arbitrary functions independent of each other; we can obtain from thence

$$\text{C} + \text{E}at + \&c. = \frac{d\psi t}{adt},$$

$$\text{D} + \text{F}at + \&c. = \frac{d\Psi t}{adt},$$

$$\&c.;$$

and thus it appears, that the preceding value of $u$ will coincide with series (d). This series (d) may($d$) be in a similar way transformed into series (c).

510. Now, if as usual, the base of the Naperian system of logarithms be denoted by $e$, and if we assume $\theta = c^x$, series (a) will become

$$u = \text{P}e^{\alpha x} + \text{Q}e^{\beta x} + \text{R}e^{\gamma x} + \&c.;$$

the coefficients P, Q, R, &c., will be functions of $t$, and the exponents $\alpha, \beta, \gamma$, &c., will be constant quantities. Therefore, we shall have

$$\frac{du}{dt} = \frac{dP}{dt}e^{\alpha x} + \frac{dQ}{dt}e^{\beta x} + \frac{dR}{dt}e^{\gamma x} + \&c.,$$

$$\frac{d^2u}{dx^2} = \alpha^2 P e^{\alpha x} + \beta^2 Q e^{\beta x} + \gamma^2 R e^{\gamma x} + \&c.$$

If these values be substituted in equation (b), and if the coefficients of the similar terms in the two members be then put equal, there will result

$$\frac{dP}{dt} = a\alpha^2 P, \quad \frac{dQ}{dt} = a\beta^2 Q, \quad \frac{dR}{dt} = a\gamma^2 R, \ \&c.;$$

consequently, the exponents will remain arbitrary, and we shall have $(e)$

$$P = Ae^{a\alpha^2 t}, \quad Q = Be^{a\beta^2 t}, \quad R = Ce^{a\gamma^2 t}, \ \&c.,$$

in which A, B, C, &c., denote arbitrary constants. Hence we shall obtain

$$u = Ae^{a\alpha^2 t}e^{\alpha x} + Be^{a\beta^2 t}e^{\beta x} + Ce^{a\gamma^2 t}e^{\gamma x} + \&c., \qquad \text{(e)}$$

for the complete integral of equation (b), arranged according to the powers of the exponential $e^x$; which series is also the development of this integral, arranged according to the powers of $e^t$. Now, it is evident, that this series (e) does not contain explicitly any arbitrary function; but that it only comprehends two infinite series of arbitrary constants, namely, A, B, C, &c., $\alpha$, $\beta$, $\gamma$, &c., and that each of its terms separately satisfies equation (b).

If this expression of $u$ be developed according to the powers of $t$, we obtain

$$\begin{aligned} u = {} & Ae^{\alpha x} + Be^{\beta x} + Ce^{\gamma x} + \&c. \\ & + (A\alpha^2 e^{\alpha x} + B\beta^2 e^{\beta x} + C\gamma^2 e^{\gamma x} + \&c.)\, at \\ & + (A\alpha^4 e^{\alpha x} + B\beta^4 e^{\beta x} + C\gamma^4 e^{\gamma x} + \&c.)\, \frac{a^2t^2}{1.2} \\ & + \&c.; \end{aligned}$$

and if we make

$$Ae^{\alpha x} + Be^{\beta x} + Ce^{\gamma x} + \&c. = \phi x;$$

$\phi x$ will be an arbitrary function, and we shall have

$$A\alpha^2 e^{\alpha x} + B\beta^2 e^{\beta x} + C\gamma^2 e^{\gamma x} + \&c. = \frac{d^2\phi x}{dx^2},$$

$$A\alpha^4 e^{\alpha x} + B\beta^4 e^{\beta x} + C\gamma^4 e^{\gamma x} + \&c. = \frac{d^4\phi x}{dx^4};$$

&c.;

and series (a) will coincide with series (c). In the same manner we may make series (e) to coincide with series (d), by developing the first according to the powers of $x$, in order to render it comparable with the second.

511. Each of the two series (c) and (e) may be expressed under a finite form, by means of the same definite integral. In the first place, we have evidently($f$)

$$\int_{-\infty}^{\infty} e^{-\omega^2} \omega^{2n-1}\, d\omega = 0,$$

$n$ being any positive integral number whatever.

Likewise, let

$$\int_{-\infty}^{\infty} e^{-\omega^2}\, d\omega = h;$$

then if $g$ denotes an arbitrary constant, and if $\omega\sqrt{g}$ and $\sqrt{g}\,d\omega$ be substituted for $\omega$ and $d\omega$, the limits of this integral will not be changed, and we shall have

$$\int_{-\infty}^{\infty} e^{-g\omega^2}\, d\omega = \frac{h}{\sqrt{g}};$$

hence we obtain, by differentiating $n$ times in succession, with respect to $g$, and then making $g = 1$($g$),

$$\int_{-\infty}^{\infty} e^{-\omega^2} \omega^{2n}\, d\omega = \frac{1.3.5\ldots.2n-1}{2^n}\, h.$$

By means of these values, formula (c) may be written as follows($h$):

$$u = \frac{1}{h}\int_{-\infty}^{\infty} \Big(\phi x + 2\,\omega\sqrt{at}\,.\frac{d\phi x}{dx} + \frac{4\omega^2 at}{1.2}\frac{d^2\phi x}{dx^2}$$

$$+ \frac{8\,\omega^3 at\sqrt{at}}{1.2.3}\frac{d^3\phi x}{dx^3} + \frac{16\,\omega^4 a^2 t^2}{1.2.3.4}\frac{d^4\phi x}{dx^4} + \&c.\Big)\, e^{-\omega^2}\, d\omega;$$

and, by Taylor's theorem, this may be reduced to

$$u = \frac{1}{k}\int_{-\infty}^{\infty} e^{-\omega^2}\phi\,(x + 2\omega\sqrt{at})\,d\omega. \qquad \text{(f)}$$

Whatever be the value of the constant $a$, the limits of the integral $\int_{-\infty}^{\infty} e^{-\omega^2}d\omega$ will not be changed, by substituting $\omega - a\sqrt{at}$ in place of $\omega$; consequently, we shall have

$$\int_{-\infty}^{\infty} e^{-\omega^2 + 2\omega a\sqrt{at} - a^2 at}\,d\omega = k;$$

hence there results

$$e^{a^2 at} = \frac{1}{k}\int_{-\infty}^{\infty} e^{-\omega^2}\,e^{2\omega a\sqrt{at}}\,d\omega.$$

The other exponentials which occur in series (e), may be expressed in the same manner, so that it will by this means become

$$u = \frac{1}{k}\int_{-\infty}^{\infty}\left[\text{A}e^{a(x+2\omega\sqrt{at})} + \text{B}e^{\beta(x+2\omega\sqrt{at})} + \text{C}e^{\gamma(x+2\omega\sqrt{at})} + \&\text{c.}\right]e^{-\omega^2}d\omega.$$

Now, if we make as above,

$$\text{A}e^{ax} + \text{B}e^{\beta x} + \text{C}e^{\gamma x} + \&\text{c.} = \phi x,$$

we shall have, at the same time,

$$\text{A}e^{a(x+2\omega\sqrt{at})} + \text{B}e^{\beta(x+2\omega\sqrt{at})} + \text{C}e^{\gamma(x+2\omega\sqrt{at})} + \&\text{c.}$$
$$= \phi\,(x + 2\omega\sqrt{at})$$

by means of which the preceding value of $u$ coincides with formula (f).

This equation (f) is the integral, under a finite form, of equation (b); it only contains, as appears, one arbitrary function, which can be determined at once by means of the value of $u$ relative to $t = 0$. However, this form of the integral implies, that this value of $u$, which will be that of $\phi x$, increases with the variable in a less ratio than $e^{x^2}$, and that the product $e^{-x^2}\phi x$ vanishes for $x = \pm\infty$, otherwise, the quantity comprised under the sign $\int$ would increase indefi-

nitely with $\omega$, for all values of $t$ different from zero, and the definite integral, the limits of which are $\omega = \pm\infty$, would have generally an infinite value, which is inadmissible.

If we make

$$\frac{d\phi x}{dx} = \phi' x, \quad \frac{d^2\phi x}{dx^2} = \phi'' x,$$

we may deduce from equation (f),

$$\frac{du}{dt} = \frac{a}{k}\int_{-\infty}^{\infty} e^{-\omega^2}\phi'(x + 2\omega\sqrt{at})\frac{\omega d\omega}{\sqrt{at}},$$

$$a\frac{d^2u}{dx^2} = \frac{a}{k}\int_{-\infty}^{\infty} e^{-\omega^2}\phi''(x + 2\omega\sqrt{at})\,d\omega;$$

if these equations are integrated by parts, and if the product of $e^{-\omega^2}$ and $\phi'(x + 2\omega\sqrt{at})$ vanishes at the two limits, there results $(i)$

$$\int_{-\infty}^{\infty} e^{-\omega^2}\phi'(x+2\omega\sqrt{at})\frac{\omega d\omega}{\sqrt{at}} = \int_{-\infty}^{\infty} e^{-\omega^2}\phi''(x+2\omega\sqrt{at})d\omega;$$

from which it appears that the value of $\frac{du}{dt}$ coincides with that of $\frac{ad^2u}{dx^2}$, and, consequently, it satisfies equation (b).

Series (d) would also lead us to an integral under a finite form of this equation, but it would not be so simple as formula (f), and would contain two arbitrary constants.

512. The known value of the quantity $k$, which occurs in the preceding formulæ is $\sqrt{\pi}$. It may be obtained, by employing successively two different variables under the sign $\int$, so that since

$$k = \int_{-\infty}^{\infty} e^{-x^2}dx, \quad k = \int_{-\infty}^{\infty} e^{-y^2}dy;$$

we have

$$k^2 = \int_{-\infty}^{\infty} e^{-x^2}dx\int_{-\infty}^{\infty} e^{-y^2}dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}dx\,dy,$$

because the two variables $x$ and $y$ are independent of each other. If, therefore, we make

there will result

$$x^2 + y^2 = r^2, \quad e^{-x^2-y^2} = z,$$

$$k^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\,dx\,dy;$$

and if $x$, $y$, $z$, be considered as the coordinates of a surface, $k^2$ will be the volume terminated by this surface of revolution, and extended indefinitely about its axis of figure, which will be the axis of the ordinates $z$. Now, the value of this volume may be obtained by decomposing it into an infinite number of cylindrical slices, whose common axis will be this line. The volume of one of these infinitely slender slices, the radii of the interior and exterior surfaces of which are $r$ and $r + dr$, will be equal to the product of its base $2\pi r dr$ multiplied by its height $z$ or $e^{-r^2}$; consequently, the entire volume may be evidently obtained by integrating from $r = 0$ to $r = \infty$; hence we shall have($k$)

$$k^2 = 2\pi\int_0^{\infty} e^{-r^2} r\,dr = \pi,$$

and $k = \sqrt{\pi}$, which it was proposed to establish.

If $\phi x = \cos x$, and if $a^2$ be substituted in place of $at$ in equation (c), then we shall have

$$u = \left(1 - \frac{a^2}{1} + \frac{a^4}{1.2} - \frac{a^6}{1.2.3} + \&c.\right)\cos x,$$

or, what comes to the same thing,

$$u = e^{-a^2}\cos x.$$

Equation (f) becomes, at the same time,

$$u = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\omega^2}\cos(x + 2a\omega)\,d\omega;$$

but we have evidently

$$\int_{-\infty}^{\infty} e^{-\omega^2}\cos 2a\omega d\omega = 2\int_0^{\infty} e^{-\omega^2}\cos 2a\omega d\omega,$$

$$\int_{-\infty}^{\infty} e^{-\omega^2}\sin 2a\omega d\omega = 0;$$

hence there results($l$)

$$u = \frac{2\cos x}{\sqrt{\pi}} \int_0^\infty e^{-\omega^2} \cos 2a\omega d\omega,$$

and by putting this value equal to one of the preceding, we obtain($m$)

$$\int_0^\infty e^{-\omega^2} \cos 2a\omega d\omega = \frac{\sqrt{\pi}}{2} e^{-a^2}.$$

As we can assign an imaginary value to the constant $a$ in this equation, if $a\sqrt{-1}$ be substituted for $a$, we shall have($n$)

$$\int_0^\infty e^{-\omega^2} (e^{2a\omega} + e^{-2a\omega}) d\omega = \sqrt{\pi} e^{a^2}.$$

It is frequently necessary to employ these formulæ, which here naturally present themselves to our consideration; however, they may also be obtained by other means.

513. The equations of partial differences to which we are led in the greater number of physical and mechanical problems, are linear relatively to the unknown $u$, of the first or second order with respect to the time, and generally, it contains four independent variables, of which $u$ is a function, to wit, the time, which is denoted by $t$, and $x, y, z$, the three coordinates of any point whatever of the system in question. If the last term which is independent of $u$, and which can be always made to disappear, be excepted, they do not contain this variable $t$ explicitly, that is to say, in these equations, the coefficients are only functions of $x, y, z$. Now, if $\text{L} = 0$ is one of these equations, without the last term, and if we assume $\theta = e^t$, series (a) will become

$$u = \text{P}e^{\alpha t} + \text{Q}e^{\beta t} + \text{R}e^{\gamma t} + \&c.; \tag{g}$$

and if in L, this series (g) be substituted in place of $u$, it is easy to perceive, that there will result

$$\text{L} = (\text{M}\alpha^2 + \text{N}\alpha + \text{O})e^{\alpha t} + (\text{M}'\beta^2 + \text{N}'\beta + \text{O}')e^{\beta t} + (\text{M}''\gamma^2 + \text{N}''\gamma + \text{O}'')e^{\gamma t} + \&c.,$$

in which M, N, O are quantities that only contain the unknown P. M′, N′, O′ may be deduced from these by substituting Q in place of P, and then M″, N″, O″, by the substitution of R in place of Q, and so on. All the quantities M, M′, M″, &c., will be cipher, when the equation $L = 0$, is only of the first order with respect to $t$. In all cases, this given equation $L = 0$ will be decomposable into the following ($o$),

$$\left.\begin{aligned} &Ma^2 + Na + O = 0, \\ &M'\beta^2 + N'\beta + O' = 0, \\ &M''\gamma^2 + N''\gamma + O'' = 0, \\ &\&c. \end{aligned}\right\} \quad (h)$$

Consequently, the exponents $a, \beta, \gamma$, &c., will be arbitrary constants; the coefficients P, Q, R, &c., may be determined independently the one of the other, by means of these equations (h), which are all similar; and all the terms of series (g), that is to say, of the complete integral of the equation $L = 0$, arranged according to the powers of the exponential $e^t$, will be particular integrals of this same equation.

Equations (h) will be, like $L = 0$, linear relative to the unknown which each of them contains. If L contains only one of the three variables $x, y, z$, they will be simply differential equations; and then series (g) will only contain arbitrary constants, to wit, $a, \beta, \gamma$, &c., and the constants which will be introduced by the integration of equations (h). When they are equations of partial differences, they may be frequently treated as the equation $L = 0$, and their complete integrals can be expressed in series of particular integrals.

514. Series (g) may be made to assume another form, by changing the exponentials into sines and cosines. In fact, if $\lambda, \mu, \nu$, &c., be other arbitrary constants, and $p, q, r$, &c., $p', q', r'$, &c., other unknown quantities, then if in this series $\pm\lambda\sqrt{-1}$, $\pm\mu\sqrt{-1}$, $\pm\nu\sqrt{-1}$, &c., be substituted in place of $a, \beta, \gamma$, &c., and if P, Q, R, &c., be replaced by

$\frac{p}{2} \pm \frac{p'}{2}\sqrt{-1}, \frac{q}{2} \pm \frac{q'}{2}\sqrt{-1}, \frac{r}{2} \pm \frac{r'}{2}\sqrt{-1}$, &c.; we shall obtain, by taking the sum of the values of $u$, which correspond to the two signs of $\sqrt{-1}$ (p)

$$\left.\begin{aligned} u = p\cos\lambda t + q\cos\mu t + r\cos\nu t + \&c. \\ + p'\sin\lambda t + q'\sin\mu t + r'\sin\nu t + \&c. \end{aligned}\right\} \quad \text{(i)}$$

In order that the integral of $L = 0$, which can be expressed either by this last formula or by series (g), may be as general as possible, it is requisite that the constants $\lambda$, $\mu$, $\nu$, &c., and also $\alpha$, $\beta$, $\gamma$, &c., may be real or imaginary; but there are problems in which the determinate values of $\lambda, \mu, \nu$, &c. are all real, and others in which none of the values of $\alpha$, $\beta$, $\gamma$, &c. will be imaginary($q$). In the first case, we should employ formula (i), and in the second, formula (g); and even when equation $L = 0$ is integrable under a finite form, it frequently occurs in mechanical and physical problems, that the expression of its integral, by means of one or other of these two series, will be better adapted than the integral under a finite form, to indicate all the circumstances of the phenomenon in question. Questions respecting the small oscillations of the points of an elastic body or of a fluid, which is made to deviate very little from its state of equilibrium, are those in which it is suitable to employ the unknown quantities under the form of the series (i).

When the general values of P, Q, R, &c., and consequently those of $p$, $q$, $r$, &c., $p'$, $q'$, $r'$, &c., contain only arbitrary constants, formula (i) may be written more briefly in the following manner,

$$u = \Sigma p\cos\lambda t + \Sigma p'\sin\lambda t;$$

in which the characteristics $\Sigma$ indicate sums that extend to all possible values, whether real or imaginary, of $\lambda$ and of the other arbitrary constants contained in $p$ and $p'$. We may, if we please, suppose that these values increase by infinitely

small degrees, and thus replace the sums $\Sigma$ by integrals; but there is no advantage in expressing in this other equivalent form the value of $u$; and it is preferable to retain the preceding.

515. Independently of the equations of partial differences which respect all the points of the system, there are always in physical and mechanical problems, other equations which only obtain for the extreme points; such are, for example, in the problem of the longitudinal vibrations of an elastic rod, the equations relative to the two extremities of this rod, when they are supposed to be entirely free, or when one or both of them are supposed to be fixed. These particular equations will enable us, in each case, to determine the values of a part of the arbitrary quantities, which series (g) or series (i) contains; with respect to those values of these quantities, which continue still undetermined after taking into account all equations of this kind, they will depend on the initial state of the system. M. Poisson, in order to obtain their values, always pursued, in a great number of different problems, one uniform process, which he considered to be applicable to all cases, whether the question presents only one unknown, namely $u$, and leads only to one equation $L = 0$, or whether it is necessary to determine several unknown quantities depending on an equal number of equations of simultaneous linear partial differences. This general process has also the advantage of furnishing, in each example, a demonstration of the reality of the constants $\alpha, \beta, \gamma$, &c., or of the constants $\lambda, \mu, \nu$, &c., which depend on transcendental equations that are frequently very complicated, and the nature of whose roots it will be frequently difficult to determine otherwise. The example which will be given in the following paragraph of the application of this method, will be sufficient to explain it, and to show how it may be employed in other problems. The longitudinal vibrations of elastic rods in the three cases of No. 495, may be determined, without any difficulty, by means of this method, which will lead us, in the most direct manner, to the same formulæ as in this number. The

longitudinal impact of two or more elastic rods, consisting of different materials, may be given, as an example of a question depending on several equations of partial differences; with respect to this question, which has been resolved in the preceding paragraph for the particular case of homogeneity, the general solution is here suppressed solely on account of the extreme length of the formulæ which it involves.

516. Let the time be supposed to be reckoned from the commencement of the motion, and let

$$u = f(x, y, z), \quad \frac{du}{dt} = F(x, y, z)$$

be the values of the unknown $u$, and of its differential coefficient with respect to $t$, which correspond to $t = 0$; so that $f(x, y, z)$ and $F(x, y, z)$ may be functions arbitrarily given for all values of the coordinates $x, y, z$, which belong to the points of a system whose vibrations are considered. After that all the arbitrary quantities which series (i) contains shall have been determined by means of the initial state of the system, and when also the particular equations which may have place at its extremities are taken into account, it is necessary that this series and its differential coefficient, when $t = 0$, should coincide with the functions $f(x, y, z)$ and $F(x, y, z)$ at the limits of the system. Hence it is necessary that (r)

$$\left.\begin{aligned} f(x, y, z) &= p + q + r + \&c., \\ F(x, y, z) &= \lambda p' + \mu q' + \nu r' + \&c.; \end{aligned}\right\} \quad (k)$$

which will furnish a development or transformation of a particular kind, for each of the functions $f(x, y, z)$ and $F(x, y, z)$; a transformation which will not be identical, and will only obtain for values of the variables $x, y, z$, that are confined within certain limits.

Although it is not possible in most cases to demonstrate directly the accuracy of these equations($k$), still there can exist no doubt on this head. In fact, it is evident from the

preceding considerations, that series (i) certainly represents the complete integral of the equation L = 0, that is to say, the most general value which can satisfy this equation. By hypothesis, the arbitrary quantities that this series contains, can be determined by means of the other conditions of the problem which has led us to this equation L = 0; therefore if these conditions are not incompatible, and if the problem is susceptible of a solution, it is necessary, after this determination, that series (i) should express the value of $u$ at any instant, and at any point whatever of the system; consequently, if $t = 0$ in this series, and in that which is deduced from it by the differentiation relative to $t$, they ought to represent the initial values of $u$ and $\frac{du}{dt}$, that is to say, the functions $f(x, y, z)$ and $F(x, y, z)$ whatever they may be, but solely for values of $x$, $y$, $z$ comprised within the limits of the system that is considered.

In the example of the longitudinal motion of an elastic rod, series (k) will represent for the entire length of this rod, and for the different hypotheses that are made respecting its extremities, the two arbitrary functions that have been designated $\phi x$ and $\phi' x$ in No. 495; and these series will coincide with the expressions for $\phi x$ and $\phi' x$ that have been made use of in that number, and which have been already demonstrated.

517. It follows from what precedes, that in order to express in a problem relative to the small vibrations of bodies, and also in other physical questions, each unknown quantity, by means of series (g) or (i), it is necessary previously to demonstrate that this series represents the most general value of the unknown that can satisfy the equation of partial differences on which it depends, and then to determine all the arbitrary quantities that this series contains, by means of the particular data of the problem, which belong to the extremities of the system and to its initial state; or what is the same thing, it is necessary to know, *à priori*, as in No. 495, the expressions in

series of the arbitrary initial value of each unknown, all whose terms multiplied by the sines or cosines of arcs proportional to the time, or by exponentials, satisfy separately the given equations of partial differences, and the other equations which respect the extreme points of the system. Any solution in which the generality of series (g) or series (i) that is employed, is not demonstrated *à priori*, or in which it is not verified *à posteriori*, that this series may represent the initial value of each unknown, whatever this value may be, in the entire extent of the system, including the extreme points, ought to be deemed incomplete. It appears from what precedes, that the first method will be always applicable; the second will be only so in a small number of particular cases.

## V. *Transversal Vibrations of an Elastic Rod.*

518. Elastic rods are susceptible of four kinds of vibrations, having corresponding tones, which may coexist in the same rod, whose natural state may be either straight or curved. These vibrations are longitudinal, transversal, normal, and those produced by torsion. The normal vibrations consist in alternate dilatations and condensations of sections of the rod perpendicular to its length; they have not been hitherto determined by theory. But the three other species have, and the relations between the tones which correspond to them, that are indicated by analysis, have been confirmed long since by experiment, and already pointed out by natural philosophers. Such is, for example, the curious observation for which we are indebted to Chladni, namely, that a rod firmly fixed at one end, and free at the other, gives out a tone that is graver by a *fifth*, when it is made to vibrate by torsion, than when it vibrates longitudinally; which implies that the tone which is given out in the first case is the same as would be heard in the second, if its length was increased in the ratio of three to two. Now M. Poisson has found that this ratio ought to be that of $\sqrt{10}$ to 2, which differs by a little less(*s*) than a twentieth from the

result stated by Chladni in round numbers. The reader is referred, for more extended developments of this important part of mathematical physics, to the author's memoir on the Equilibrium and Motion of Elastic Bodies, that has been already cited in No. 306, where the vibrations of flexible membranes and elastic plates are likewise discussed. In the present treatise it will be sufficient to consider the less complicated cases of this description of questions, which are those of vibrating cords and of the longitudinal vibrations of elastic rods, to which we now proceed to add the case of transversal vibrations.

519. It is supposed, as in the case of longitudinal vibrations (No. 493), that the rod is homogeneous, and taken in its natural state, of either a prismatic or cylindrical form; in like manner, it is assumed that it does not experience any torsion on itself, so that all the points of each longitudinal filament exist in the same plane during the entire continuance of the motion.

Let AMB (fig. 27) be the rectilinear direction of the mean filament in the natural state of the rod, $l$ its length, and $x$ the distance AM of any point whatever such as M from the extremity A. At the end of the time $t$, let M be supposed to be transferred to M′; from M′ let the perpendicular M′P be let fall on AB, and let us make

$$\text{MP} = u, \quad \text{M}'\text{P} = y.$$

If these two variables be supposed to be constantly very small, and if, in consequence, their squares and products be neglected, their values in functions of $x$ and $t$ will depend, as in the case of vibrating cords (No. 483), on linear equations, in which these unknown quantities are separated; hence the very small motions, in the longitudinal and also in the transversal direction, will coexist, without mutually influencing one another; and as the longitudinal motion has been completely determined, we need not now take it into account. Therefore we may make $u = 0$, in which case all the points of the mean

filament will oscillate on lines perpendicular to its natural direction, and $x$ and $y$ will be, at any instant, the coordinates of the plane curve A′M′B′ formed by this filament. We shall also abstract from the consideration of the small motions of dilatation or contraction which may have place in each section of the rod perpendicular to its length, in which case the motion which it is proposed to determine will be the same for all the points of the same section; so that it will be sufficient to consider that of the point which belongs to the mean filament.

The equation of this transversal motion may be deduced from equation (f) of No. 320, by substituting $Y - \frac{d^2y}{dt^2}$, or simply $-\frac{d^2y}{dt^2}$, in place of Y, if no given force is supposed to be applied to the different points of the rod. Therefore this equation will be

$$\frac{d^2y}{dt^2} + b^2\frac{d^4y}{dt^4} = 0; \qquad (1)$$

$b^2$ being a positive constant, which will depend on the material of the rod, and on the area and figure of the normal section. Besides this equation (1), which is common to all the points of the mean filament, there are also equations relative to its extremities, which will be the same as in the problem of equilibrium. In this respect, six different cases may occur, according as each of the two extremities of the rod will be firmly fixed, or merely pressed against, or entirely free. But as these cases may be treated in the same manner, we shall restrict ourselves to the consideration of one in detail, and we shall suppose that the rod is entirely free at its two extremities A and B, to which, moreover, no particular force will be applied. This being so, for all values of $t$, we shall have (No. 320) ($n$)

$$x = 0, \quad \frac{d^2y}{dx^2} = 0, \quad \frac{d^3y}{dx^3} = 0, \qquad (2)$$

at the extremity A′, and

$$x = l, \quad \frac{d^2y}{dx^2} = 0, \quad \frac{d^3y}{dx^3} = 0, \tag{3}$$

at the extremity B′.

At the commencement of the motion, the curve that the mean filament assumes, and the velocities of all its points will be known; if therefore the time $t$ be reckoned from this commencement, and if $\phi x$ and $\phi' x$ denote given functions from $x = 0$ to $x = l$, we shall have at once

$$t = 0, \quad y = \phi x, \quad \frac{dy}{dt} = \phi' x. \tag{4}$$

Nevertheless, these arbitrary functions should satisfy the conditions relative to $x = 0$ and $x = l$, which will be expressed by equations (2) and (3) for the function $\phi x$, and by their differentials relative to $t$ for the function $\phi' x$.

Thus the question to be resolved will consist in finding the value of $y$, as a function of $t$ and $x$, which satisfies equations (1), (2), (3), (4), the first of which is the only one which has place for all values of these two variables. But, it is useful, previously, to compare, for the same elastic rod, the coefficient $b$, which occurs in the equation of its transversal motion, and the coefficient $a$ that is contained in the equation of its longitudinal motion.

520. If $g$ denotes the gravity, $p$ the weight of the rod, and $q$ the tension which should be employed to double its length $l$, we have (No. 494)

$$a^2 = \frac{glq}{p}.$$

The product of the density of the rod and of the area of its normal section is equal to $\frac{p}{gl}(v)$; and by equation (f) of No. 320, from which equation (1) of the transversal motion is obtained, we shall have

$$b^2 = \frac{gla}{p} \int_{-k}^{h} vu^2 du; \tag{5}$$

in which $u$, $v$, $k$, $k'$ have the same signification as in No. 314, and the constant $a$ of the same number is the value of $q$, referred to the unit of surface, so that

$$q = a\omega,$$

in which $\omega$ denotes the area of the normal section of the rod. Likewise, if we make

$$\int_{-k}^{k} vu^2du = \omega h^2,$$

($h$ will be a line the value of which will depend on the area and form of its contour), we shall have

$$b^2 = \frac{glqh^2}{p};$$

hence there results

$$b = ah.$$

If the normal section of the rod is a rectangle whose base is perpendicular to the plane of the curve A′M′B′, and height equal to $2\varepsilon$, we shall have

$$\omega = 2v\varepsilon, \quad k' = k = \varepsilon, \quad \omega h^2 = v\int_{-\varepsilon}^{\varepsilon} u^2du = \tfrac{2}{3}v\varepsilon^3,$$

and we shall obtain ($x$)

$$b = \frac{a\varepsilon}{\sqrt{3}}.$$

In the case of a cylindrical rod, whose radius is represented by $\varepsilon$, we shall have

$$\omega = \pi\varepsilon^2, \quad k' = k = \varepsilon, \quad v = 2\sqrt{\varepsilon^2 - u^2};$$

from which we can deduce at once

$$\omega h^2 = \tfrac{1}{4}\pi\varepsilon^4,$$

and hence($y$)

$$b = \tfrac{1}{2}a\varepsilon.$$

If now the normal section of the rod is an isosceles triangle whose base is perpendicular to the plane of the curve A′M′B′,

and if, for greater clearness, we suppose that this plane is vertical, and that the base of the triangle corresponds to the upper face of the rod, then if we denote this base by $\lambda$, and the height by $2\varepsilon$, we shall have always($z$)

$$\omega = \lambda\varepsilon:$$

but the value of $h$ will be different, according as the convexity of upper face of the rod is directed upwards or downwards (No. 315). In the first case, we shall have ($a'$)

$$h = \frac{2\varepsilon}{3}, \quad h' = \frac{4\varepsilon}{3}, \quad v = \frac{\lambda}{2\varepsilon}\left(\frac{4\varepsilon}{3} + u\right);$$

from which there results

$$\omega h^2 = \frac{2\lambda\varepsilon^3}{9},$$

and, consequently,

$$b = \frac{a\varepsilon\sqrt{2}}{3}.$$

In the second case, we shall have($b'$),

$$h = \frac{4\varepsilon}{3}, \quad h' = \frac{2\varepsilon}{3}, \quad v = \frac{\lambda}{2\varepsilon}\left(\frac{2\varepsilon}{3} + u\right);$$

we may deduce from it

$$\omega h^2 = \frac{2\lambda\varepsilon^3}{3},$$

and, then,

$$b = a\varepsilon\sqrt{\tfrac{2}{3}}.$$

These results will be of use, farther on, in enabling us to compare together the tones of the same elastic rod, when it vibrates longitudinally, and when it vibrates transversally.

521. Let now $p$ and $q$ be functions of $x$, and $m$ a constant relatively to $t$ and $x$. In order to satisfy equation (1), let us assume($c'$)

$$y = p\sin m^2bt + q\cos m^2bt;$$

as this equation should obtain for all values of $t$, we must have

$$\frac{d^4p}{dx^4} = m^4p, \quad \frac{d^4q}{dx^4} = m^4q;$$

and, by integrating these two differential equations of the fourth order, there arises($d'$)

$$p = \text{A} \sin mx + \text{A}' \cos mx$$
$$+ \tfrac{1}{2} \text{B} (e^{mx} - e^{-mx}) + \tfrac{1}{2} \text{B}' (e^{mx} + e^{-mx})$$
$$q = \text{C} \sin mx + \text{C}' \cos mx$$
$$+ \tfrac{1}{2} \text{D} (e^{mx} - e^{-mx}) + \tfrac{1}{2} \text{D}' (e^{mx} + e^{-mx});$$

in which A, A′, B, B′, C, C′, D, D′ are eight arbitrary constants, and $e$ as usual denotes the base of the Naperian system of logarithms.

In consequence of the linear form of equation (1), it may be also satisfied by taking

$$y = \Sigma p \sin m^2bt + \Sigma q \cos m^2bt,$$

in which the sums $\Sigma$ are supposed to extend to all possible values, both real and imaginary, of $m$ and of the eight other constants, A, A′, &c. Moreover, it is evident from what has been stated in the preceding paragraph, that this value of $y$ will be the complete integral of equation (1).

If it be substituted in equations (2) and (3), which have place for all values of $t$, and, consequently, for all the terms of the sums $\Sigma$ taken separately, we shall have($e'$),

$$\frac{d^2p}{dx^2} = 0, \quad \frac{d^3p}{dx^3} = 0, \quad \frac{d^2q}{dx^2} = 0, \quad \frac{d^3q}{dx^3} = 0,$$

for $x = 0$, and $x = l$. If there be substituted for $p$ and $q$ their preceding values, there results at once

$$\text{B}' = \text{A}', \quad \text{B} = \text{A}, \quad \text{D}' = \text{C}', \quad \text{D} = \text{C},$$

and, besides,

$$\text{A} (2 \sin ml - e^{ml} + e^{-ml}) = \text{A}' (e^{ml} + e^{-ml} - 2 \cos ml),$$
$$\text{A}' (2 \sin ml + e^{ml} - e^{-ml}) = \text{A} (2 \cos ml - e^{ml} - e^{-ml}),$$
$$\text{C} (2 \sin ml - e^{ml} + e^{-ml}) = \text{C}' (e^{ml} + e^{-ml} - 2 \cos ml),$$
$$\text{C}' (2 \sin ml + e^{ml} - e^{-ml}) = \text{C} (2 \cos ml - e^{ml} - e^{-ml}).$$

Now, if the corresponding members of the two first, or the two last of these four equations, be respectively multiplied together, and if, in these products, the common factor AA′, or CC′, be suppressed, we obtain

$$4\sin^2 ml - (e^{ml} - e^{-ml})^2 + (2\cos ml - e^{ml} - e^{-ml})^2 = 0;$$

or, by reducing(f′)

$$(e^{ml} + e^{-ml})\cos ml - 2 = 0; \qquad \text{(a)}$$

this equation will enable us to determine the values of $m$. Moreover, the values of A, A′, &c., which may be obtained from the preceding equations, will be

$$\begin{aligned}
\text{B} = \text{A} &= \text{E}\ (e^{ml} + e^{-ml} - 2\cos ml),\\
\text{B}' = \text{A}' &= \text{E}\ (2\sin 2\,ml - e^{ml} + e^{-ml}),\\
\text{D} = \text{C} &= \text{E}'(e^{ml} + e^{-ml} - 2\cos ml),\\
\text{D}' = \text{C}' &= \text{E}'(2\sin 2\,ml - e^{ml} + e^{-ml});
\end{aligned}$$

E and E′ being two new constants which continue undetermined.

If for conciseness, we make

$$\begin{aligned}
\text{X} = {} & (e^{ml} + e^{-ml} - 2\cos ml)\,(\sin mx + \tfrac{1}{2}e^{mx} - \tfrac{1}{2}e^{-mx})\\
& + (2\sin ml - e^{ml} + e^{-ml})\,(\cos mx + \tfrac{1}{2}e^{mx} + \tfrac{1}{2}e^{-mx}),
\end{aligned}$$

the preceding value of $y$ will become(g′)

$$y = \Sigma\text{X}\,(\text{E}\sin m^2bt + \text{E}'\cos m^2bt); \qquad (b)$$

in which the sum $\Sigma$ extends as before to all possible values of E and E′, but with respect to $m$, only to those values of it that are furnished by equation (a). For all these values, we shall have

$$\frac{d^2\text{X}}{dx^2} = 0, \quad \frac{d^3\text{X}}{dx^3} = 0, \qquad \text{(c)}$$

when $x = 0$, and when $x = l$; and, whatever be the quantities $m$ and $x$, we shall have identically(h′),

$$\frac{d^4\text{X}}{dx^4} = m^4\text{X}. \qquad \text{(d)}$$

**522.** It only remains to determine the values of the coefficients E and E′ relative to each value of $m$, from knowing the initial state of the rod; this we proceed now to do, following the general process that has been adverted to in No. 515.

In the first place, we may remark that if $m$ is a root of equation (a), so also will $-m$, $m\sqrt{-1}$, $-m\sqrt{-1}(i')$; moreover, the corresponding values of X will only differ either in the sign, or in the factor $\sqrt{-1}$; hence it follows, that in formula (b), we can unite in one sole term, the terms which belong to these four roots, and then only extend the sum $\Sigma$ to real and positive values of $m$, or to values consisting only of a real and imaginary part, if there be such, and in which case the real part is positive. In this manner, if $m$ and $m'$ are two roots of equation (a) that are employed, $m'$ and $m'^2$ will differ from $\pm m$ and $\pm m^2$.

This being agreed on, if equation (1) be multiplied by X$dx$, and if it be then integrated from $x=0$ to $x=l$, there results($k'$)

$$\int_0^l \frac{d^2.\mathrm{X}y}{dt^2}\,dx+b^2\int_0^l \mathrm{X}\frac{d^4y}{d^4x}\,dx=0;$$

and we shall obtain by partial integration($l'$)

$$\int_0^l \mathrm{X}\frac{d^4y}{dx^4}dx=\left(\mathrm{X}\frac{d^3y}{dx^3}-\frac{d\mathrm{X}}{dx}\frac{d^2y}{dx^2}+\frac{d^2\mathrm{X}}{dx^2}\frac{dy}{dx}-\frac{d^3\mathrm{X}}{dx^3}y\right)$$
$$-\left[\mathrm{X}\frac{d^3y}{dx^3}-\frac{d\mathrm{X}}{dx}\frac{d^2y}{dx^2}+\frac{d^2\mathrm{X}}{dx^2}\frac{dy}{dx}-\frac{d^3\mathrm{X}}{dx^3}y\right]+\int_0^l \frac{d^4\mathrm{X}}{dx^4}y\,dx.$$

The terms comprised between the parentheses belong to $x=l$, and those contained between the brackets, to $x=0$; they mutually destroy one another, in virtue of equations (2), (3), (4), relative to these limits; and, by equation (d), which obtains for all values of $x$, if $m^4$X be substituted in place of $\frac{d^4\mathrm{X}}{dx^4}$ under the sign $\int$, we shall have

$$\int_0^l \mathrm{X}\frac{d^4y}{dx^4}\,dx=m^4\int_0^l \mathrm{X}y\,dx.$$

Therefore, because

$$\int_0^l \frac{d^2.\text{X}y}{dt^2}\,dx = \frac{d^2.\int_0^l y\text{X}\,dx}{dt^2},$$

we shall have

$$d^2.\frac{\int \text{X}y\,dx}{dt^2} + b^2m^4\int \text{X}y\,dx = 0.$$

The complete integral of this differential equation of the second order is

$$\int_0^l \text{X}y\,dx = \text{H}\cos m^2bt + \text{H}'\sin m^2bt;$$

in which H and H′ denote two arbitrary constants. In order to determine them, if $t = 0$, in this formula and in its differential with respect to $t$, there will result

$$\text{H} = \int_0^l \text{X}\phi x\,dx, \quad \text{H}' = \frac{1}{bm^2}\int_0^l \text{X}\phi' x\,dx.$$

Whatever may be the magnitude of $t$, we shall have

$$\left.\begin{aligned} \int_0^l \text{X}y\,dx = \int_0^l \text{X}\phi x\,dx.\cos m^2bt, \\ + \frac{1}{bm^2}\int_0^l \text{X}\phi' x\,dx.\sin m^2bt. \end{aligned}\right\} \quad (e)$$

If formula (b) be substituted in place of $y$ in the first member of this equation (e); as its second member only contains $\cos m^2bt$ and $\sin m^2bt$, if $m'$ be a root of equation (a), such that $m'$ and $m'^2$ differ from $\pm m$ and $\pm m^2$, as we have supposed above, the term corresponding to $m'$ must disappear from the first member; in order to this, it is necessary that

$$\int_0^l \text{X}\text{X}'dx = 0; \quad (f)$$

in which X′ denotes what X becomes when $m$ is changed into $m'(m')$. But in the case of $m' = m$, it follows from this same equation (e), that

$$\text{E}\int_0^l \text{X}^2dx = \frac{1}{bm^2}\int_0^l \text{X}\phi' x\,dx,$$

$$\text{E}'\int_0^l \text{X}^2 dx = \int_0^l \text{X}\phi x\,dx\,;$$

by means of this the values of E and E′ will be known in functions of $m$, and formula (b) will become

$$y = \Sigma\,\text{X}\left[\frac{\int_0^l \text{X}\phi x\,dx}{\int_0^l \text{X}^2 dx}\cos m^2bt + \frac{\int_0^l \text{X}\phi' x\,dx}{bm^2\int_0^l \text{X}^2 dx}\sin m^2bt.\right] \qquad \text{(g)}$$

As this expression for $y$, and the value of $\frac{dy}{dt}$ which follows from it, no longer contains any unknown quantity, they will make known at each instant, the ordinate and velocity of M′ any point whatever of the curve A′M′B′; which is the complete solution of the problem. The integral $\int_0^l \text{X}^2 dx$ may be obtained, under a finite form, by the ordinary rules, but the values of the integrals $\int_0^l \text{X}\phi x\,dx$ and $\int_0^l \text{X}\phi' x\,dx$ cannot, in general, be computed but by the method of quadratures.

523. If $t$ be made equal to cipher in the expressions for $y$ and $\frac{dy}{dt}$, we shall have, by equations (4) and (g),

$$\left.\begin{aligned} \phi x &= \Sigma\left(\frac{\int_0^l \text{X}\phi x\,dx}{\int_0^l \text{X}^2\phi\,x dx}\right)\text{X}, \\ \phi' x &= \Sigma\left(\frac{\int_0^l \text{X}\phi' x\,dx}{\int_0^l \text{X}^2 dx}\right)\text{X}. \end{aligned}\right\} \qquad \text{(h)}$$

These two similar formulæ obtain for any functions whatever of $x$, such as $\phi x$ and $\phi' x$, whether continuous or discontinuous, but solely from $x = 0$ to $x = l$; and it should be observed, that they have not place for the extreme values of $x$, unless those of these functions satisfy the condition stated above (No. 519). Although these formulæ cannot be directly

demonstrated, they are, however, not less certain on that account, as has been explained in No. 516.

In the particular case in which all the points of the rod have received at the commencement a common velocity, and also a velocity proportional to their distances from its middle point, it is evident that it ought to be actuated by a motion of translation and a rotatory motion, without experiencing any curvature, or undergoing any vibrations. This might also have been inferred from these equations (h), and from formula (g).

In this case if $c$ and $\gamma$ denote two constant quantities, we have

$$\phi x = 0, \quad \phi' x = c + \gamma\,(x - \tfrac{1}{2} l);$$

and if the second equation (h) be differentiated, there results, by taking into account equation (d),

$$\Sigma \left( \frac{\int_0^l \mathrm{x}\phi' x\, dx}{\int_0^l \mathrm{x}^2 dx} \right) \mathrm{x} m^{4i} = 0;$$

in which $i$ may be any positive integer whatever($n'$). Consequently, the development according to the powers of $t$, of the part

$$\Sigma \left( \frac{\int_0^l \mathrm{x}\phi' x\, dx}{\int_0^l \mathrm{x}^2 dx} \right) \frac{\mathrm{x} \sin m^2 bt}{m^2 b},$$

of the formula (g), will be reduced to its first term($o'$)

$$t\Sigma \left( \frac{\int_0^l \mathrm{x}\phi' x\, dx}{\int_0^l \mathrm{x}^2 dx} \right) \mathrm{x},$$

the value of which will be, in virtue of the second equation (h), $t\phi' x$. Hence, in consequence of the equation $\phi x = 0$, and of the value of $\phi' x$, formula (g) will be simply

$$y = [c + \gamma (x - \tfrac{1}{2}l)]\, t,$$

as it ought to be.

524. It may be demonstrated by means of equation (f), that equation (a) does not admit of a root consisting of a real and imaginary part. In fact, if there exists such a root as $f + g\sqrt{-1}$; there will be also another which only differs from this in the sign of $\sqrt{-1}$, which will be $f - g\sqrt{-1}$; therefore, we may assume in equation ($f$),

$$m = f + g\sqrt{-1}, \quad m' = f - g\sqrt{-1};$$

in which $f$ and $g$ denote real quantities, none of which can be cipher. We shall also, at the same time, have

$$\text{X} = \text{F} + \text{G}\sqrt{-1}, \quad \text{X}' = \text{F} - \text{G}\sqrt{-1};$$

in which F and G are likewise real quantities. Consequently, equation $f$ will become

$$\int_0^l (\text{F}^2 + \text{G}^2)\, dx = 0;$$

which is impossible, since, in order that it should obtain, it would be necessary that an integral of which all the elements are positive, and which expresses their sum (No. 13), should be equal to cipher. Hence the supposition of a root $f + g\sqrt{-1}$ is inadmissible. This last equation would be also inadmissible, if $f$ or $g$ were cipher; but then we could no longer assume $f + g\sqrt{-1}$, and $f - g\sqrt{-1}$ for $m$ and $m'$; for equation (f) implies that $m'$ and $m'^2$ differ from $\pm m$ and $\pm m^2$; which would not be the case, if one of the two quantities $f$ and $g$ was cipher.

525. When in equation (a) the root $m$ is equal to cipher, the corresponding term of formula (g) assumes the form $\frac{0}{0}$; its true value is obtained by supposing that $m$ is only an infinitely small quantity; in this case, we shall have ($p'$)

$$\text{X} = 4m^3l^3(x - \tfrac{1}{2}l), \quad \cos m^2bt = 1, \quad \sin m^2bt = m^2bt,$$

and the term in question will be

$$\left[\int_0^l (3x-l)\,\phi x\,dx + t\int_0^l (3x-l)\,\phi' x\,dx\right]\frac{3x-l}{l^3}.$$

It refers to the motions of translation and rotation that are common to all the points of the rod; as we shall not take them into account, nor consider the root $m = 0$, all the terms of the series (g) will be periodic.

But, if we advert to the circumstance that the different values of $m$ are incommensurable, it will appear that all the points of the rod will never in general revert, at the same time, to their primitive state, or, in other words, an elastic rod will not, in all cases, perform, like a stretched cord, isochronal transversal vibrations. In order that the isochronism may have place, and that the rod may produce an unique appreciable tone, it is necessary, that in consequence of its curvature, and of the velocities of its several points at the commencement of the motion, all the terms of formula (g) should disappear, except one only, by which means it will be reduced to the form

$$y = \mathrm{X}\,(\mathrm{E}\sin m^2bt + \mathrm{E}'\cos m^2bt), \tag{i}$$

in which the constants $\mathrm{E}$ and $\mathrm{E}'$ are substituted, for the sake of abridging, in place of their values found above. When formula (g) is reduced to a small number of terms, the rod will cause to be heard at the same time several distinct sounds, the tones of which cannot be accurately compared together.

526. If $\lambda$ denotes a numerical value of $ml$ deduced from equation (a), so that $m = \frac{\lambda}{l}$, and if $\mathrm{T}$ be the duration of an entire oscillation of the rod, corresponding to this root $m$, and $n$ the number of vibrations in the unit of time, we shall have, by means of equation (i),

$$\mathrm{T} = \frac{2\pi l^2}{\lambda^2 b}, \quad n = \frac{\lambda^2 b}{2\pi l^2};$$

so that the different tones, which a rod bent in the same direction, and vibrating transversally, produces, when free at its two extremities, will depend on the values of $\lambda$, and the gravest or fundamental tone corresponds to the least of these values. It is evident that the quantity $b^2$, furnished by formula (5), depends only on $l$ the length of the rod; if the rod be cylindrical or circular, it likewise appears that this quantity is proportional to the square of the diameter. Consequently in the case of two cylindrical rods, consisting of the same materials, and of which the order of vibrations is the same, i. e. for which the value of $\lambda$ is the same, the number $n$ will be in the direct ratio of the thickness, and inversely as the square of the length ($q'$).

In the case of a rod of a prismatic form, there will, in general, be two different sounds produced, according as it will vibrate transversely in one direction or another. Thus, if for example, the normal section be supposed to be a rectangle, and if the rod be made to vibrate successively($r'$), so that the base or altitude of the rectangle may be perpendicular to the plane of the curve A′M′B′, the successive values of $n$ will be to each other as this altitude and this base, for the same order of vibrations. When the normal section is triangular, as in the third example of No. 520, the value of $b^2$ will not be the same for two successive semi-vibrations; their durations will consequently be unequal; this, however, will not prevent their entire vibrations from being isochronous, if formula (g) is always reducible to one sole term($s'$).

By putting the factor X of formula (i) equal to cipher, we can determine the values of $x$ which correspond to the *nodes* of vibrations, that is to say, to the immoveable points on the line AB, for each value of $m$, or for each tone that the rod can cause to be heard.

527. When the elastic rod that is considered is firmly secured at its extremity A, and free at its other end, the mean

filament will be a tangent at A to the line AB, during the entire continuance of the motion; so that equations (2) of the preceding problem ought to be replaced by

$$x = 0, \quad y = 0, \quad \frac{dy}{dx} = 0.$$

The preceding analysis must be modified in consequence of this, which it can be, without any difficulty; and the general value of $y$ will be still expressed by formula (g); but the values of $m$, that ought to be made use of, should be deduced from the equation ($t'$)

$$(e^{ml} + e^{-ml}) \cos ml + 2 = 0, \qquad (a')$$

which only differs from equation (a) in the sign of the last term; at the same time, the value of X should be determined by means of the equation

$$\begin{aligned} \mathrm{X} = {} & (e^{ml} + e^{-ml} + 2\cos ml)\,(\sin mx - \tfrac{1}{2} e^{mx} + \tfrac{1}{2} e^{-mx}) \\ & + (2\sin ml + e^{ml} - e^{-ml})\,(\cos mx - \tfrac{1}{2} e^{mx} - \tfrac{1}{2} e^{-mx}). \end{aligned}$$

In order that the rod, when vibrating thus transversally, may produce only one sound, it is necessary that formula (g) should be reduced, by means of its initial state, to one sole term or to formula (i), as in the preceding case. If $\lambda'$ be a positive value of $ml$ deduced from equation (a'), T$'$ the duration of a vibration corresponding to $m' = \frac{\lambda'}{l}$, and $n'$ the number of vibrations in the unit of time, we shall have

$$\mathrm{T}' = \frac{2\pi l^2}{\lambda'^2 b}, \quad n' = \frac{\lambda'^2 b}{2\pi l^2},$$

and every thing that has been stated in the preceding numbers respecting the comparison of the tones of rods free at their two extremities, is equally applicable to the case of rods firmly secured at one of their extremities; in like manner the nodes of vibrations which accompany each tone given out by the

same rod, will be determined by putting the preceding value of x equal to cipher.

528. In order to resolve equations (a) and (a′) by approximation, let

$$ml = \lambda = \tfrac{1}{2}(2i+1)\pi \mp \delta,$$

in equation (a), and

$$ml = \lambda' = \tfrac{1}{2}(2i+1)\pi \pm \delta',$$

in equation (a′); in which $i$ denotes any positive integer number or cipher, and $\delta$, $\delta'$ two positive quantities, which cannot surpass $\frac{1}{2}\pi$ ($u'$). These new unknown quantities should be affected with the superior or inferior signs, according as $i$ is even or odd; and, by this means, equations (a) and (a′) become

$$\left.\begin{aligned} \sin\delta &= \frac{2}{e^{\frac{1}{2}(2i+1)\pi}\, e^{\mp\delta} + e^{-\frac{1}{2}(2i+1)\pi}\, e^{\pm\delta}}, \\ \sin\delta' &= \frac{2}{e^{\frac{1}{2}(2i+1)\pi}\, e^{\pm\delta'} + e^{-\frac{1}{2}(2i+1)\pi}\, e^{\mp\delta'}}. \end{aligned}\right\} \quad (k)$$

It is easy to perceive that in consequence of the limits of $\delta$ and $\delta'$, neither of these unknown quantities can have more than one value for each value of $i$; that of $\delta$, for $i = 0$, will be $\delta = \frac{1}{2}\pi$ ($v'$); and as it corresponds to $ml = 0$, we should not take it into account. For $i = 0$, we have $\delta = 0,\ 01797$, by neglecting $\delta$ in the second member of the preceding equation($k$); and if in it this first approximate value of $\delta$ be substituted, we shall have, more accurately,

$$\delta = 0,\ 01765.$$

The values of $\delta$ relative to $i = 2$, $i = 3$, will be still($x'$) less than this; therefore the values of $\lambda$ will differ very little from the odd multiples of $\frac{1}{2}\pi$; and, it appears from the expression of the number $n$, that the tones of a rod that is free at its two extremities will constitute, very nearly, a series increasing as the squares of the numbers 3, 5, 7, &c. The least value of $\lambda$, which corresponds to the gravest sound, is

$$\lambda = \tfrac{3}{2}\pi + \delta = 4{,}74503.$$

In the case of $i = 0$, we find after some trials, to a sufficient degree of approximation ($y'$),

$$\delta' = 0{,}30431.$$

Therefore the least value of $\lambda'$ will be

$$\lambda' = \tfrac{1}{2}\pi + \delta' = 1{,}87011.$$

Hence there results, by comparing its square with that of the preceding value of $\lambda$, and observing that these squares are to each other as $n'$ and $n$ their respective numbers of vibrations,

$$\frac{n'}{n} = 0{,}15715,$$

for the ratio of the gravest tone given by the rod firmly fixed at one of its extremities, to that of the same rod, when free at its two extremities. The other values of $\delta'$ are very small; therefore the corresponding values of $\lambda'$ will be, very nearly, the 3, 5, 7, &c., multiples of $\frac{1}{2}\pi$; and the tones of the rod, one of whose extremities is firmly fixed, will, with the exception of the gravest, constitute a series increasing as the squares of these odd numbers.

Experiments made long since have confirmed every thing indicated here by theory, relative to the series of tones produced by elastic rods, which are either free or fixed firmly at their extremities, to the position of the nodes which accompany these different modes of vibration, and to the relations of the tones, in the directions of their lengths and thicknesses.

We proceed now to compare together the tones or number of transversal and longitudinal vibrations of the same elastic rod. Observation has likewise confirmed on this point the results of the calculus.

529. We shall suppose for greater clearness, that in these two descriptions of vibrations, the rod is free at its two extremities, and we shall restrict ourselves to the consideration of the gravest tone that is given out by each of them.

There results, by making use of the least value of $\lambda$ found in the preceding number,

$$n = \frac{(4,74503)^2 b}{2\pi l^2} = (3,56082)\frac{b}{l^2},$$

for the number of transversal vibrations in the unit of time. If $n_1$ denotes the number of longitudinal vibrations in the same time, we shall have, by the third case of No. 495,

$$n_1 = \frac{a}{2l};$$

and as $b = ah$ (No. 520), there will result

$$n = (7,12164)\frac{hn_1}{l},$$

a formula that is independent of the material of which the rod consists, and by means of which the transversal tone may be obtained when the longitudinal tone is given, and *vice versa.*

The magnitude of $h$ which occurs in the expression, will depend on the figure of the normal section, and will be proportional, every thing else being the same, to the thickness. As this dimension is very small relatively to the length $l$, it follows that the tone of the transversal vibrations will be very grave relatively to the other; which agrees with the observations that have been most usually made. By No. 520, if the rod is a cylinder of which the radius is $\varepsilon$, we have $h = \frac{1}{2}\varepsilon$; and if it is a parallellopiped, and that $2\varepsilon$ is also the thickness, we have $h = \frac{\varepsilon}{\sqrt{3}}$; therefore, in these two cases, we shall have

$$n = (7,12164)\frac{\varepsilon n_1}{2l}, \quad n = (7,12164)\frac{\varepsilon n_1}{l\sqrt{3}}.$$

As the number $n_1$ is independent of the figure and dimensions of the normal section, it follows that for the same values of $\varepsilon$ and $l$, the number of transversal vibrations is less in the first case than in the second, in the ratio of $\sqrt{3}$ to $2\,(z')$.

# CHAPTER IX.

## GENERAL EQUATIONS AND PROPERTIES OF THE MOTION OF A SYSTEM OF BODIES.

### I. *General Equations of this Motion.*

530. SINCE the forces lost by all the points of a system during each instant, ought to constitute an equilibrium, (No. 530), if the principle of virtual velocities be applied to these forces, a general formula will be obtained, from which can be deduced, in each particular case, all the equations respecting the motion of bodies, just as all those relating to their equilibrium have been deduced from the general equation of virtual velocities. But, however natural and obvious this combination of the general principle of dynamics with that of equilibrium may now appear, still it was not made at the time when the first of these two principles was discovered, although the second had been previously given in all its generality.

We are indebted to Lagrange for thus connecting these two principles; by this means the solutions of all the problems of mechanics, or at least the formation of the differential equations on which they depend, are reduced to one uniform process. It is this general process which we now proceed to explain. It may not, however, be superfluous to apprise the reader here that the order which has been pursued in this treatise, in which problems relative to solid or flexible bodies have been directly resolved, is by proceeding from the simpler to the more complex and difficult cases, and the reason is, because it appears to be most suitable to a profound study of mechanics and knowledge of this science, which should not be solely con-

sidered in an abstract point of view, and independently of physical circumstances.

531. Let $m$, $m'$, $m''$, &c., denote the masses of the points of the system in question. At the end of $t$, the time reckoned from the commencement of the motion, let $x$, $y$, $z$ denote the three rectangular coordinates of $m$, and X, Y, Z the components of the accelerating force applied to this material point, acting in the direction of the positive productions of $x$, $y$, $z$. Let the same letters, with corresponding accents, represent the homologous quantities which refer to the other points $m'$, $m''$, &c. The components estimated in the directions of X, Y, Z, of the forces lost by any point $m$ during the instant $dt$, will be

$$m\left(\mathrm{X}-\frac{d^2x}{dt^2}\right), \quad m\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right), \quad m\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right);$$

consequently, the system will be in equilibrio if the point $m$ be supposed to be solicited by these forces, and each of the other points $m'$, $m''$, &c., by similar forces. Now, the general equation of this equilibrium will be formed by substituting in equation (e) of No. 341, the three preceding components in place of X, Y, Z; this gives

$$\Sigma m\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)\delta x+\Sigma m\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right)\delta y+\Sigma m\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right)\delta z=0; \quad (1)$$

in which, as the sums $\Sigma$ extend to all the points $m$, $m'$, $m''$, &c., of the system, they consequently consist of a number of parts equal to the number of these points.

We shall suppose, as in the number cited, that the mode in which these material points are connected together, is expressed by the equations

$$\mathrm{L}=0, \quad \mathrm{L}'=0, \quad \mathrm{L}''=0, \ \&c. \qquad (2)$$

in which L, L′, L″, &c., are given functions of the variables $x$, $y$, $z$, $x'$, &c., or of a part of them, which may also contain the time $t$ explicitly. If, for example, the point $m$ is constrained to remain on a surface which gradually changes its form, or

which is in motion in space, and if $L = 0$ represents the equation of this surface, then $L$ will be a given function of $x, y, z, t$.

Although the forces of which equation (1) expresses the equilibrium refer to the quantities of motion lost during the interval of time $dt$, and that during this instant the positions of the points $m, m', m''$, &c., change by infinitely small quantities, we can, nevertheless, suppose that this equilibrium obtains in the positions that these points occupy at the end of the time $t$, that is to say, we need not take into account their change of position during the instant $dt$, which can alter the quantities of motion lost while it is taking place, only by an infinitely small quantity of the second order, and the corresponding motive forces by an infinitely small quantity of the first order($a$). Consequently, the infinitely small displacements which the principle of virtual velocities implies, and which are expressed, in the directions of the coordinates, by $\delta x, \delta y, \delta z$, for the point $m$, by $\delta x', \delta y', \delta z'$, for the point $m'$, &c.; must satisfy the conditions of the system, such as they are at the end of the time $t$; hence equations (2) must also have place when $x + \delta x,\ y + \delta y,\ z + \delta z,\ x' + \delta x'$, &c. are substituted in place of $x, y, z, x'$, &c., the time $t$, which they may contain explicitly, being supposed not to vary; therefore we obtain, as in No. 341,

$$\left.\begin{array}{l}
\frac{dL}{dx}\delta x + \frac{dL}{dy}\delta y + \frac{dL}{dz}\delta z + \frac{dL}{dx'}\delta x' + \&c. = 0, \\
\frac{dL'}{dx}\delta x + \frac{dL'}{dy}\delta y + \frac{dL'}{dz}\delta z + \frac{dL'}{dx'}\delta z' + \&c. = 0, \\
\frac{dL''}{dx}\delta x + \frac{dL''}{dy}\delta y + \frac{dL''}{dz}\delta z + \frac{dL''}{dx'}\delta x' + \&c. = 0, \\
\&c.
\end{array}\right\} \quad (3)$$

By means of these equations, a part of the quantities $\delta x$, $\delta y$, &c., can be eliminated in the first member of equation (1), and then the coefficients of each of the remaining quan-

tities may be put equal to cipher. If, as in No. 342, the method of indeterminate factors be employed, we shall obtain the following equations:

$$\left.\begin{array}{l} m\dfrac{d^2x}{dt^2} = mX + \lambda\dfrac{dL}{dx} + \lambda'\dfrac{dL'}{dx} + \lambda''\dfrac{dL''}{dx} + \&c., \\ m\dfrac{d^2y}{dt^2} = mY + \lambda\dfrac{dL}{dy} + \lambda'\dfrac{dL'}{dy} + \lambda''\dfrac{dL''}{dy} + \&c., \\ m\dfrac{d^2z}{dt^2} = mZ + \lambda\dfrac{dL}{dz} + \lambda'\dfrac{dL'}{dz} + \lambda''\dfrac{dL''}{dz} + \&c., \\ m'\dfrac{d^2x'}{dt^2} = m'X' + \lambda\dfrac{dL}{dx'} + \lambda'\dfrac{dL'}{dx'} + \lambda''\dfrac{dL''}{dx'} + \&c., \\ \&c. \end{array}\right\} \quad (4)$$

in which $\lambda$, $\lambda'$ $\lambda''$, &c., are factors whose values make known the forces arising from the connexion of the points of the system, and from the resistance of the surfaces or curves on which they are constrained to move (No. 343).

The number of equations (2) and (4) taken together, will be always the same as that of all the unknown quantities of the problem, that is, the number of quantities $\lambda, \lambda', \lambda''$, &c., is equal to that of equations (2), and the number of coordinates of the points $m, m', m''$, &c., is triple that of these moveables, and equal to the number of equations (4); consequently, they are sufficient in all cases to determine the values of all these unknown quantities in functions of the time.

532. If the given forces which act on all the points of the system be supposed to be distributed into two groups, so that we may have

$$\begin{array}{lll} X = P + U, & Y = Q + V, & Z = R + W, \\ X' = P' + U', & Y' = Q' + V', & Z' = R' + W'; \\ \&c.; & & \end{array}$$

and if we assume also, that the differential equations of the problem can be integrated by considering solely the forces $P, Q, R, P', Q', R'$, &c.; $a, b, c$, &c., being the arbitrary constants introduced by these integrations; we can extend this

solution to the complete forces $X, Y, Z, X', Y', Z'$, &c., by means of the method founded on the variation of arbitrary constants, the principle of which was explained in No. 229. The differentials of the quantities $a$, $b$, $c$, &c., supposed to become variable, will be linear relative to $U, V, W, U', V', W'$, &c., and of the form

$$da = AU + BV + CW + A'U' + \&c.,$$

$$db = A_1U + B_1V + C_1W + A_1'U' + \&c.,$$

$$dc = A_2U + B_2V + C_2W + A_2'U' + \&c.,$$

&c.

$A$, $B$, &c., being functions of the same unknown quantities, $a, b, c$, &c.(*b*) By this means, the second differential equations of the problem will be changed into twice the number of differential equations of the first order; but this transformation will be principally useful, when the secondary forces $U, V, W, U'$, &c., are very small with respect to the primitive forces $P, Q, R, P'$, &c.; for this circumstance will enable us to consider, in the first approximation, the quantities $a$, $b$, $c$, &c., which the coefficients $A$, $B$, &c., contain, as constant, and consequently, to deduce from the preceding formulæ by immediate integration, or by the method of quadratures, the variable parts of these unknown quantities.

It was Lagrange who thus extended to all problems of mechanics the method of the variation of arbitrary constants, to which he had before reduced the theory of the particular solutions of differential equations, and of which he also made other applications less general. But he restricted himself to assigning the general expressions of the quantities $U, V, W, U', V', W'$, &c., in linear functions of the differentials $da$, $db$, $dc$, &c.; and it still remained to find the inverse formulæ which give, directly, in the general case, the differentials of the unknown quantities $a, b, c$, &c., in linear functions of the forces $U, V, W$, &c., and to demonstrate, in a

direct and general manner, the important properties which their coefficients A, B, C, &c., possess. This has been done by M. Poisson in the memoirs relating to this subject, which have been inserted in the 15th No. *of the Journal of the Polytechnic School*, and *in the first volume of the Memoires of the Academy of Sciences*, to which the reader, who wishes to know this theory in all its details, and the consequences which follow from it, is referred. If the general expressions of $da$, $db$, $dc$, &c., be successively applied to the problem of the motion of a material point attracted to a fixed centre by a force varying according to any function whatever of the distance, and also to the problem of the motion of a solid body about a fixed point, the same expressions will be obtained for the differentials of the homologous constants in these two problems, which in other respects are so different from each other; and it thus appears, that the two principal questions in astronomy, namely, the determination of the motion of the heavenly bodies, considered as isolated material points, and the determination of the motion of these bodies about their respective centres of gravity, are reducible to the same formulæ, and depend on the same analysis.

533. It is evident, that if one of equations (2) is a consequence, or may be deduced from the others, then one of the quantities $\lambda$, $\lambda'$, $\lambda''$, &c., must remain undetermined, since then this superfluous equation may be suppressed or retained, just as we please. If, for example, $a$ and $b$ are given constants, and if

$$L'' = aL + bL',$$

each of the three first equations (2) will add nothing to the conditions expressed by the two others, and, consequently, one of the three unknown $\lambda$, $\lambda'$, $\lambda''$, must remain indeterminate. This is, in fact, what will be the case; for if we make

$$\lambda + a\lambda'' = \mu, \quad \lambda' + b\lambda'' = \mu',$$

these three unknown will be reduced, in equations (4), to the

two quantities $\mu$ and $\mu'$, which can alone be determined by means of these equations, and from which the values of only two of the three quantities $\lambda, \lambda', \lambda''$, can be obtained.

If the material point $m$ is constrained to remain at constant and given distances from the three fixed points A, A′, A″, (fig. 28), its position will be completely determined; the values of its coordinates will consequently be constant; and the three first equations (4) will be reduced to the equations of equilibrium, that will determine the tensions of the threads A$m$, A′$m$, A″$m$, by which the moveable $m$ is attached to the three fixed points. If this material point is constrained to remain at a constant distance from a fourth fixed point A‴, one of the four distances A$m$, A′$m$, A″$m$, A‴$m$, will be determined by means of the three others; and as one of the four given conditions is thus a consequence of of the three others, it appears from what has been already stated, and agreeably to what has been established in No. 292, that the tension of one of the four threads A$m$, A$m'$, A$m''$, A$m'''$, will remain undetermined. In fact, if the four given distances be denoted by $l, l', l'', l'''$, the three coordinates of A, by $a, b, c$, those of A′, by $a', b', c'$, &c.; we shall have for equations (2),

$$\text{L} = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} - l = 0,$$

$$\text{L}' = \sqrt{(x-a')^2+(y-b')^2+(z-c')^2} - l' = 0,$$

$$\text{L}'' = \sqrt{(x-a'')^2+(y-b'')^2+(z-c'')^2} - l'' = 0,$$

$$\text{L}''' = \sqrt{(x-a''')^2+(y-b''')^2+(z-c''')^2} - l''' = 0;$$

and if $\alpha, \beta, \gamma$, denote the constant values of $x, y, z$, which satisfy these four equations, we shall have for equations (4)($c$),

$$m\text{X} + \frac{\lambda(\alpha-a)}{l} + \frac{\lambda'(\alpha-a')}{l'} + \frac{\lambda''(\alpha-a'')}{l''} + \frac{\lambda'''(\alpha-a''')}{l'''} = 0,$$

$$m\text{Y} + \frac{\lambda(\beta-b)}{l} + \frac{\lambda'(\beta-b')}{l'} + \frac{\lambda''(\beta-b'')}{l''} + \frac{\lambda'''(\beta-b''')}{l'''} = 0,$$

$$m\text{Z} + \frac{\lambda(\gamma-c)}{l} + \frac{\lambda'(\gamma-c')}{l'} + \frac{\lambda''(\gamma-c'')}{l''} + \frac{\lambda'''(\gamma-c''')}{l'''} = 0;$$

from which it appears, that one of the four quantities $\lambda$, $\lambda'$, $\lambda''$, $\lambda'''$, which denote, as has been stated in No. 345, the tensions of the threads $\text{A}m$, $\text{A}'m$, $\text{A}''m$, $\text{A}'''m$, will be undetermined. But, however little extensible these threads may be, if this physical circumstance be taken into account, the material point $m$ will make small vibrations, which may be completely determined, and also the tensions of the four threads, at each instant.

534. In order to demonstrate this, let us assume, for greater clearness, that the force which acts on the point $m$ is that of gravity, which we shall denote by $g$. If we suppose that the axis of the coordinates $z$ is vertical, and that it is drawn in the direction of this force, its three components will be $\text{X} = 0$, $\text{Y} = 0$, $\text{Z} = g$. Let $\varepsilon$, $\varepsilon'$, $\varepsilon''$, $\varepsilon'''$, be the extensions which the four threads $l, l', l'', l'''$, would experience if the weight $mg$ was suspended vertically at their inferior extremity, and $\zeta, \zeta', \zeta'', \zeta'''$, the extensions of these threads at the end of the time $t$, during the motion; the values of their tensions at the same instant will be (No. 288),

$$\frac{gm\zeta}{\varepsilon}, \quad \frac{gm\zeta'}{\varepsilon'}, \quad \frac{gm\zeta''}{\varepsilon''}, \quad \frac{gm\zeta'''}{\varepsilon'''}.$$

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As the moveable $m$ is no longer constrained to remain at constant distances from $\text{A}$, $\text{A}'$, $\text{A}''$, $\text{A}'''$, the terms of equations (4), of which the factors are $\lambda, \lambda', \lambda'', \lambda'''$, and which arise from these conditions, should be suppressed; but, on the other hand, the four preceding forces directed from $m$ towards $\text{A}$, from $m$ towards $\text{A}'$, from $m$ towards $\text{A}''$, from $m$ towards $\text{A}'''$, must be joined to the weight of this material point; this is the same thing, as if the preceding values of $\text{L}, \text{L}', \text{L}'', \text{L}'''$, were substituted in equations (4), and if, at the same time, we made

$$\lambda = \frac{-gm\zeta}{\varepsilon}, \quad \lambda' = \frac{-gm\zeta'}{\varepsilon'}, \quad \lambda'' = \frac{-gm\zeta''}{\varepsilon''}, \quad \lambda''' = \frac{-gm\zeta'''}{\varepsilon'''}.$$

Likewise, let

e end of the time $t$; $\alpha, \beta, \gamma$ being the same constants as e, and $u, v, w$ being very small variables, the squares products of which may be neglected; there will re-d)

$$\zeta = \frac{1}{l}[(\alpha - a)u + (\beta - b)v + (\gamma - c)w],$$

$$\zeta' = \frac{1}{l'}[(\alpha - a')u + (\beta - b')v + (\gamma - c')w],$$

$$\zeta'' = \frac{1}{l''}[(\alpha - a'')u + (\beta - b'')v + (\gamma - c'')w],$$

$$\zeta''' = \frac{1}{l'''}[(\alpha - a''')u + (\beta - b''')v + (\gamma - c''')w];$$

relatively to the unknown quantities $u, v, w$, equations will be linear, and will be reduced to($e$)

$$+g\left[\frac{(\alpha-a)\zeta}{l\varepsilon} + \frac{(\alpha-a')\zeta'}{l'\varepsilon'} + \frac{(\alpha-a'')\zeta''}{l''\varepsilon''} + \frac{(\alpha-a''')\zeta'''}{l'''\varepsilon'''}\right] = 0,$$

$$+g\left[\frac{(\beta-b)\zeta}{l\varepsilon} + \frac{(\beta-b')\zeta'}{l'\varepsilon'} + \frac{(\beta-b'')\zeta''}{l''\varepsilon''} + \frac{(\beta-b''')\zeta'''}{l'''\varepsilon'''}\right] = 0,$$

$$+g\left[\frac{(\gamma-c)\zeta}{l\varepsilon} + \frac{(\gamma-c')\zeta'}{l'\varepsilon'} + \frac{(\gamma-c'')\zeta''}{l''\varepsilon''} + \frac{(\gamma-c''')\zeta'''}{l'''\varepsilon'''}\right] = 0.$$

eir integrals will be obtained by the ordinary methods; the six arbitrary constants which are introduced by these egrations, are determined by making the lengths of the eads A$m$, A$'m$, A$''m$, A$'''m$, at the commencement of the motion, al to their natural lengths $l, l', l'', l'''$, and by supposing t the initial velocity is cipher, that is to say, that the six ntities $u, v, w, \frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}$, are nothing when $t = 0$. This ng done, these integrals will make known at any instant atever, the values of $u, v, w$, or the position of the point and $\zeta, \zeta', \zeta'', \zeta'''$, the extensions of the four threads, and o their tensions at the same instant, will be likewise deter-ned. The same analysis may be easily extended to the case

in which the point $m$ is retained by five or a greater number of threads attached to fixed points.

If the quantities $u$, $v$, $w$ be supposed to be cipher, and if, in consequence, the first terms of the three last of the seven preceding equations be suppressed, the values of $u$, $v$, $w$, $\zeta$, $\zeta'$, $\zeta''$, $\zeta'''$, which result from these seven equations, refer to the state of equilibrium of the point $m$ and of the four threads of suspension($f$).

535. In No. 353, it was shown how the principle of D'Alembert obtains, in the case of a sudden change of velocity produced by forces termed *impulsions* or *percussions*, which act on the moveables with great intensity, during extremely short intervals of time, and impress on them velocities which may be very considerable, although the points of these bodies are not sensibly displaced. Therefore, the equation furnished by the combination of this principle with that of virtual velocities, is equally applicable to this description of cases. Thus, if forces of this kind are simultaneously applied to $m$, $m'$, $m''$, &c., the material points of the system which we are considering, and if A, B, C denote the given velocities parallel to the axes of the coordinates, which these forces would impress on the point $m$, if it were free, and $a$, $b$, $c$, the unknown velocities which it will actually assume in these respective directions, and if A$'$, B$'$, C$'$, $a'$ $b'$ $c'$, A$''$, B$''$, C$''$, $a''$, $b''$, $c''$, &c., be the corresponding quantities relatively to the points $m'$, $m''$, &c., then the quantities of motion lost in the directions of these coordinates will be

$$m(\text{A}-a), \quad m(\text{B}-b), \quad m(\text{C}-c),$$

for the point $m$, and similar expressions may be obtained for the quantities of motion lost by each of the other points; consequently, if to this system of forces equation (e) of No. 341 be applied, we shall have

$$\Sigma m\left[(\text{A}-a)\,\delta x+(\text{B}-b)\,\delta y+(\text{C}-c)\,\delta z\right]=0\,; \qquad (5)$$

in which the sum $\Sigma$ is supposed to extend to all the points of the system, and $\delta x$, $\delta y$, $\delta z$, are increments of the coordinates of $m$, any point whatever.

If the manner in which the points of the system are connected is always expressed by equations (2), $\delta x$, $\delta y$, $\delta z$, and also $\delta x'$, $\delta y'$, $\delta z'$, $\delta x''$, &c., the increments of the coordinates of the other points $m'$, $m''$, &c., must satisfy equations (3); consequently, by means of these equations, a certain number of the quantities $\delta x$, $\delta y$, &c., may be eliminated from formula (5), and then the coefficients of the remaining quantities can be put equal to zero. If, as has been done above, the method of indeterminate factors be employed, their values will make known the percussions the connecting strings of the points of the system undergo, by the effect of a sudden change of the velocity, and also the percussions which are normal to the surfaces or curves on which these points are constrained to move.

536. When it is proposed to apply equation (5) to a sudden change of velocity produced by the impact of the bodies of the system among each other, or against fixed obstacles, there are several important observations which should be premised.

Let M and M′ (fig. 29) be two of these solid bodies, K their point of contact, HKH′ the normal common to their surfaces at this point. As the displacements of the different points of these moveables during the entire continuance of the impact, may be considered as insensible, the equilibrium of the quantities of motion lost may be supposed to refer to whatever instant we please of this continuance (No. 353); so that, if A, B, C be supposed to represent the components of the velocity of any point whatever at the commencement of the impact, we may, at the same time, assume for $a$, $b$, $c$, the components of its velocity at any instant whatever of this phenomenon, and the velocities of all the points of the system, which vary very rapidly during this continuance, ought always satisfy the conditions of this equilibrium. But in order that these

conditions may be expressed by the equation of virtual velocities, the infinitely small displacements which are attributed to the points of the system, should be compatible with its nature and with the relative disposition of its parts at the instant in question; and, it is besides, necessary that the same thing should obtain with respect to the displacements which are directly opposed to them (No. 331).

Thus, for example, if a material point is laid on the surface of a solid body against which it presses, this point may move in *every* direction on the plane which touches this surface, and only in one direction on the normal. This being so, the equation of virtual velocities is true for all *tangential* displacements, since the opposite displacements are equally possible; but it does not subsist for the normal displacements, on account of the impossibility of the contrary displacement.

It follows from this observation, that if it be proposed to apply equation (5) to a determinate epoch of the continuance of the impact, and if $\mu$ and $\mu'$ are, at this instant, the material points of M and M′ which refer to the point of contact K, we may ascribe to $\mu$ and $\mu'$ infinitely small displacements, altogether arbitrary and independent of each other, in the plane which is a tangent to the surface at K; but the displacements of $\mu$ and $\mu'$ along the normal must be *equal*, and directed along the *same* part KH or KH′ of this line; for if they were *unequal*, or if they did not take place in the *same* direction, these displacements, or the contrary displacements of the points $\mu$ and $\mu'$, would be impossible, and equation (5) would be no longer applicable to them. In consequence of not paying attention to this *essential* condition, some authors have fallen into error, in the explanation which they gave of the equation in question.

If a third solid body M″ touches M′ at the point K′, where the common normal to their surfaces is the line LK′L′, and of which $m'$ and $m''$ are the material points of M′ and M″ that correspond to this point K′, at the instant which is considered

while the impact is taking place; it is also necessary that the infinitely small displacements attributed to $m'$ and $m''$ along this normal, should be equal and estimated in the same direction in equation (5); and the same should be the case for all the points of contact of the bodies of the system, when several of them impinge simultaneously on each other. In the case of the impact of one of these bodies against a fixed obstacle, the normal displacement of the point of contact must be supposed to be cipher, since it will not be possible in the opposite direction.

537. When the two moveables M and M′ slide the one on the other during the continuance of the shock, it is necessary to take into account, in equation (5), the friction which results from it, and which may be very considerable, as has been already stated in No. 353.

The infinitely small quantity of motion which this force will abstract, during each instant, from the two moveables M, M′, in a direction contrary to that in which the velocities of the material points $\mu$ and $\mu'$, which refer to the point of contact K, are estimated, will be proportional to that which M will have communicated to M′, in the direction KH′, or M′ to M along KH, during the same instant. Hence, if the friction retains the same direction for each moveable, during the entire continuance of the shock, and if the finite quantity of motion communicated by one moveable to the other, along the parts KH or KH′ of the normal, during this same interval, be denoted by U, the entire friction can be represented by $f$U; $f$ being a coefficient which depends solely on the nature of the two bodies near to their point of contact. This force should be applied to M in a direction opposite to that in which the sliding of $\mu$ takes place, and to M′ in a direction opposite to that of $\mu'$. Therefore if these two motions take place along KF and KF′ parts of FKF′ a tangent to the two moveables, and if the projections of the infinitely small displacements, which in equation (5) are ascribed to the points $\mu$ and $\mu'$, be denoted

by $p$ and $p'$, the term which should be added to the first member of this equation, in consequence of the friction in question, will be equal to

$$-f\text{U}(p+p');$$

the quantity $p$ is positive or negative, according as this projection falls on KF, the direction of the motion of $\mu$, or on its production KF′; and in like manner the projection $p'$ will be positive or negative, according as it falls on KF′ or on KF.

Similar terms should be introduced into equation (5), for all the points in which two bodies of the system impinge on each other. It is necessary in practice to take these terms into consideration; the example of No. 477 is sufficient to show the influence that these terms, or the frictions from which they arise, may have on the percussions; but when the question is relative to general theorems on the impact of bodies, they are not taken into account, and accordingly, in the sequel, they will be supposed to be cipher or insensible.

In like manner, the quantities of motion produced by the weight of the moveables during the continuance of the impact, are neglected, for since these quantities are proportional to this duration, they must be insensible. With respect to the quantities of motion produced by the molecular attractions which are developed during the impact, either from one body to another, when the distances of their surfaces become insensible, or in the interior of each body, in consequence of the compressions or dilatations which it experiences, they have been already taken into account in equation (5), and their total components are precisely the quantities which have been represented by $m$A, $m$B, $m$C, for $m$ any point whatever.

538. When a system of material points is entirely free in space, so that equations (2), which express their connexion, contain only the mutual distances of these points, none of which is supposed to be fixed, or constrained to remain on a given curve or surface, the motion of such a system in space may be decomposed, as has been already done in the case of a

solid body entirely free (No. 433), into two simpler motions; namely, a motion of translation common to all the points of the system, and which will be that of its centre of gravity, and one of rotation about this centre. We now proceed to deduce successively from formula (1), the differential equations of these two motions.

It is evident from the nature of the system, that all its points may be displaced by the same quantity, in any given direction. If $\alpha$, $\beta$, $\gamma$ denote the projections of this common displacement on the three axes of the coordinates, we shall then have

$$\alpha = \delta x = \delta x' = \delta x'', \text{ \&c.},$$
$$\beta = \delta y = \delta y' = \delta y'', \text{ \&c.},$$
$$\gamma = \delta z = \delta z' = \delta z'', \text{ \&c.};$$

equation (1) will then become

$$\alpha \Sigma m \left(\mathrm{X} - \frac{d^2x}{dt^2}\right) + \beta \Sigma m \left(\mathrm{Y} - \frac{d^2y}{dt^2}\right) + \gamma \Sigma m \left(\mathrm{Z} - \frac{d^2z}{dt^2}\right) = 0;$$

and as the three quantities $\alpha$, $\beta$, $\gamma$ are independent of each other, this equation can be decomposed into the following,

$$\Sigma m \frac{d^2x}{dt^2} = \Sigma m\mathrm{X}, \quad \Sigma m \frac{d^2y}{dt^2} = \Sigma m\mathrm{Y}, \quad \Sigma m \frac{d^2z}{dt^2} = \Sigma m\mathrm{Z}. \qquad (6)$$

Now, if $x_1$, $y_1$, $z_1$ be the three coordinates of the centre of gravity of the system, we shall have

$$x_1 \Sigma m = \Sigma m x, \quad y_1 \Sigma m = \Sigma m y, \quad z_1 \Sigma m = \Sigma m z;$$

hence we obtain

$$\frac{d^2x_1}{dt^2} \Sigma m = \Sigma m \frac{d^2x}{dt^2},$$
$$\frac{d^2y_1}{dt^2} \Sigma m = \Sigma m \frac{d^2y}{dt^2},$$
$$\frac{d^2z_1}{dt^2} \Sigma m = \Sigma m \frac{d^2z}{dt^2};$$

consequently, we shall have

$$\frac{d^2x_1}{dt^2}\Sigma m = \Sigma m\text{X}, \quad \frac{d^2y_1}{dt^2}\Sigma m = \Sigma m\text{Y}, \quad \frac{d^2z_1}{dt^2}\Sigma m = \Sigma m\text{Z}, \qquad (7)$$

for the differential equations of the motion of the centre of gravity, which will be the motion of translation of the system. It appears from inspection of them that the motion of the centre of gravity of every system entirely free, is the same as if the masses of all the moveables were condensed into it, and their motive forces were transferred to it parallel to their directions, as in the case of a solid body (No. 438).

If, among the points $m$, $m'$, $m''$, &c., there are any which are constrained to move on given surfaces, equations (4) and (7) may still subsist, by joining to the given forces, other forces of an unknown magnitude, whose directions may be perpendicular to these surfaces, and which will express their resistances; we are thus enabled to abstract from the consideration of the given surfaces, and consider the points $m$, $m'$, $m''$, &c., as belonging to a system entirely free.

539. The nature of such a system enables us also to make all its points to turn at the same time about the same axis, with the same angular velocity, so that their mutual distances will not vary. If this line be supposed to pass through the origin of the coordinates, and if $\lambda$, $\mu$, $\nu$ are the angles which its arbitrary direction makes with the axes of $x$, $y$, $z$, the cosines of the angles which the direction of the displacement of $m$ makes with parallels to the three axes drawn through this point, will be $\frac{\delta x}{h}$ $\frac{\delta y}{h}$, $\frac{\delta z}{h}$, in which $h^2$ is supposed to be equal to $\delta x^2 + \delta y^2 + \delta z^2$; and as this direction exists in a plane perpendicular to the axis of rotation, we must have

$$\frac{\delta x}{h}\cos\lambda + \frac{\delta y}{h}\cos\mu + \frac{\delta z}{h}\cos\nu = 0.$$

Moreover, as the axis of rotation passes through the origin of the coordinates, the quantity $x^2 + y^2 + z^2$ will not vary

during the displacement of $m$; consequently we shall have also

$$x\delta x + y\delta y + z\delta z = 0;$$

and from these two last equations there may be obtained without any difficulty ($g$),

$$\left.\begin{aligned} \delta z &= (y\cos\lambda - x\cos\mu)\varepsilon, \\ \delta y &= (x\cos\nu - z\cos\lambda)\varepsilon, \\ \delta x &= (z\cos\mu - y\cos\nu)\varepsilon; \end{aligned}\right\} \qquad (8)$$

$\varepsilon$ being an indeterminate factor. In like manner we shall have

$$\delta z' = (y'\cos\lambda - x'\cos\mu)\,\varepsilon',$$

$$\delta y' = (x'\cos\nu - z'\cos\lambda)\,\varepsilon',$$

$$\delta x' = (z'\cos\mu - y'\cos\nu)\,\varepsilon';$$

$\varepsilon'$ being also an indeterminate factor which must be equal to $\varepsilon$, in order that the motion of rotation may be the same for $m$ and for $m'$, and that the distance of these two points may not vary. In fact, as the square of this distance is $(x-x')^2 + (y-y')^2 + (z-z')^2$, and as it appears from the preceding formulæ that the two parts $x^2 + y^2 + z^2$ and $x'^2 + y'^2 + z'^2$ of this quantity are constant, the variation of $xx' + yy' + zz'$ must be likewise cipher, hence there results

$$x'\delta x + y'\delta y + z'\delta z + x\delta x' + y\delta y' + z\delta z' = 0;$$

or which comes to the same thing, by substituting for $\delta x$, $\delta x'$, &c., their values($h$),

$$[(x'y - y'x)\cos\nu + (z'x - x'z)\cos\mu$$
$$+ (y'z - z'y)\cos\lambda]\,(\varepsilon - \varepsilon') = 0;$$

an equation which evidently cannot obtain for all the points of the system, unless $\varepsilon' = \varepsilon$.

If in equation (1), formulæ (8) be substituted for $\delta x$, $\delta y$, $\delta z$, there results, by observing that $\varepsilon$, $\cos\lambda$, $\cos\mu$, $\cos\nu$, are quantities common to all the points of the system,

$$\varepsilon \cos \nu \Sigma m \left[x\left(\frac{d^2y}{dt^2} - \mathrm{Y}\right) - y\left(\frac{d^2x}{dt^2} - \mathrm{X}\right)\right]$$

$$+ \varepsilon . \cos \mu \Sigma m \left[z\left(\frac{d^2x}{dt^2} - \mathrm{X}\right) - x\left(\frac{d^2z}{dt^2} - \mathrm{Z}\right)\right]$$

$$+ \varepsilon . \cos \lambda \Sigma m \left[y\left(\frac{d^2z}{dt^2} - \mathrm{Z}\right) - z\left(\frac{d^2y}{dt^2} - \mathrm{Y}\right)\right] = 0;$$

and as the coefficients of the sums $\Sigma$ are three quantities independent of each other, this equation may be resolved into three others, namely,

$$\left.\begin{aligned} \Sigma m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) &= \Sigma m\,(x\mathrm{Y} - y\mathrm{X}), \\ \Sigma m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right) &= \Sigma m\,(z\mathrm{X} - x\mathrm{Z}), \\ \Sigma m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) &= \Sigma m\,(y\mathrm{Z} - z\mathrm{Y}), \end{aligned}\right\} \quad (9)$$

which equations will be those of the motion of rotation of a system entirely free, about a fixed point which may be arbitrarily assumed, and where the origin of coordinates is placed. These equations will still subsist, when all the points $m$, $m'$, $m''$, &c., or a part of them, are constrained to exist at given distances from this origin, since in that case the values of $\delta x$, $\delta x'$, &c., which we have made use of, satisfy this condition.

If the system is reduced to one sole point, the motion of rotation will not then be distinguished from that of translation; in fact, equations (9) will only be a combination of equations (6); and moreover, it is evident, that each of them, is a consequence of the two others; for if the system be reduced to a point $m$, and if equations (9) be multiplied by $z$, $y$, $x$, respectively, and then added together, there will result an identical equation.

In place of causing the system to turn about any axis whatever, in order to obtain at once the three equations (9), each of them can be obtained more simply, by making this

line to coincide, as in No. 340, with one of the axes of the coordinates; but the advantage of the preceding calculus is to show, that the consideration of the motion about any axis whatever, can only furnish the three equations (9), in the same manner as a consideration of the motion parallel to any axis whatever, can only give the three equations (6).

If there is a fixed axis in the system, and if it be assumed to be that of $z$, for example, the first of the three equations (9) will be the only one that will subsist, and it will be that of the motion of rotation about a fixed axis, as in the case of a solid body (No. 391).

540. If in the case of a system entirely free, in which equations (6) and (9) obtain simultaneously, the origin of the coordinates be transferred to the point of the system of which the variable coordinates are denoted by $x_1$, $y_1$, $z_1$, relatively to the first origin; and if, for this purpose, we make

$$\begin{aligned} &x = x_1 + x_{,}, \quad y = y_1 + y_{,}, \quad z = z_1 + z_{,}, \\ &x' = x_1 + x_{,}', \quad y' = y_1 + y_{,}', \quad z' = z_1 + z_{,}', \\ &\&c. \end{aligned}$$

so that $x_{,}, y_{,}, z_{,}$, $x_{,}', y_{,}', z_{,}'$, &c., may be the coordinates of $m$, $m'$, &c., referred to the new origin. The first equation (9) may be at once written as follows:

$$\begin{aligned} x_1\Sigma m\left(\frac{d^2y}{dt^2} - \text{Y}\right) &- y_1\Sigma m\left(\frac{d^2x}{dt^2} - \text{X}\right) \\ + \Sigma m\left(x_{,}\frac{d^2y}{dt^2} - y_{,}\frac{d^2x}{dt^2}\right) &= \Sigma m(x_{,}\text{Y} - y_{,}\text{X}); \end{aligned}$$

the terms multiplied by $x_1$ and $y_1$ are respectively equal to cipher, in virtue of the two first equations (6); and by actually substituting the preceding values of $x$, $x'$, &c., in the remaining part of the first member, we shall have

$$\frac{d^2y_1}{dt^2}\Sigma m x_{,} - \frac{d^2x_1}{dt^2}\Sigma m y_{,} + \Sigma m\left(x_{,}\frac{d^2y_{,}}{dt^2} - y_{,}\frac{d^2x_{,}}{dt^2}\right)$$
$$= \Sigma m(x_{,}\text{Y} - y_{,}\text{X}),$$

whatever may be the moveable origin of the coordinates. But if this point is the centre of gravity of the system, the sums $\Sigma mx_{,}$ and $\Sigma my_{,}$ will be cipher; consequently, the terms multiplied by $\frac{d^2x_1}{dt^2}$ and $\frac{d^2y_1}{dt^2}$ will disappear also, as well as those of which the factors are $x_1$ and $y_1$; by which means the preceding equation will be simplified. By making similar reductions on the two other equations (9), we shall have

$$\left.\begin{aligned}\Sigma m\left(x_{,}\frac{d^2y_{,}}{dt^2}-y_{,}\frac{d^2x_{,}}{dt^2}\right)&=\Sigma m\,(x_{,}\mathrm{Y}-y_{,}\mathrm{X}),\\ \Sigma m\left(z_{,}\frac{d^2x_{,}}{dt^2}-x_{,}\frac{d^2z_{,}}{dt^2}\right)&=\Sigma m\,(z_{,}\mathrm{X}-x_{,}\mathrm{Z}),\\ \Sigma m\left(y_{,}\frac{d^2z_{,}}{dt^2}-z_{,}\frac{d^2y_{,}}{dt^2}\right)&=\Sigma m\,(y_{,}\mathrm{Z}-z_{,}\mathrm{Y});\end{aligned}\right\}\quad(10)$$

for the three equations of the motion of rotation of the system about its centre of gravity. It is evident from a comparison of them with equations (9), that this motion will be the same as if the centre of gravity was a fixed point and the given forces which act on all the points of the system were not changed; a property which is peculiar to the centre of gravity, and which has been already demonstrated (No. 438) in the case of a solid body entirely free.

541. If the same values as in No. 538, be assigned to the quantities $\delta x$, $\delta y$, &c., which equation (5) contains; there will result

$$\Sigma ma=\Sigma m\mathrm{A},\quad \Sigma mb=\Sigma m\mathrm{B},\quad \Sigma mc=\Sigma m\mathrm{C};\qquad(11)$$

from this it appears, that in the sudden changes of velocity, the sum of the quantities of motion of all the points of a system entirely free, parallel to each of the axes of the coordinates, remains unchanged, and, consequently, this must be the case in any direction whatever. It likewise follows, that the magnitude and direction of the velocity of the centre of gravity, does not undergo any change; for the components of this velocity, before and after the sudden change, are the

respective members of each of equations (11), divided by the total mass $\Sigma m$. Hence, in the impact of two, or a greater number of bodies, of any nature or form whatever, the velocity of their centre of gravity, and the entire quantity of motion estimated in any direction, never experience any change, as has been already remarked in a particular case (No. 364.)

If $a, b, c, a', b', c'$, &c., are the components of the initial velocities of $m, m'$, &c., and $\text{A}, \text{B}, \text{C}, \text{A}', \text{B}', \text{C}'$, &c., the components of the velocities which would be impressed on them in any manner whatever, at the commencement of the motion, if these material points were detached and isolated from each other, we shall have

$$\frac{dx_1}{dt}\Sigma m = \Sigma ma, \quad \frac{dy_1}{dt}\Sigma m = \Sigma mb, \quad \frac{dz_1}{dt}\Sigma m = \Sigma mc;$$

and, consequently,

$$\frac{dx_1}{dt}\Sigma m = \Sigma m\text{A}, \quad \frac{dy_1}{dt}\Sigma m = \Sigma m\text{B}, \quad \frac{dz_1}{dt}\Sigma m = \Sigma m\text{C},$$

for $t = 0$; by means of these equations, the given velocities $\text{A}, \text{A}'$, &c., or even the entire sum of the quantities of motion communicated to the system, parallel to the three axes of coordinates, will make known the components of the initial velocity of the centre of gravity.

If the values of $\delta x, \delta y, \delta z$, given by formulæ (8), are again substituted in equation (5), there results($i$)

$$\left.\begin{aligned} \Sigma m\,(xb - ya) &= \Sigma m\,(x\text{B} - y\text{A}), \\ \Sigma m\,(za - xc) &= \Sigma m\,(z\text{A} - x\text{C}), \\ \Sigma m\,(yc - zb) &= \Sigma m\,(y\text{C} - z\text{B}); \end{aligned}\right\} \quad (12)$$

from which it appears, that in all sudden changes of velocity, the moments of the quantities of motion of all the points of a system entirely free remain unchanged, with respect to any axis whatever; this theorem still obtains, though there should be

one or more fixed points in the system, provided that these points exist on the axis of the moments.

If these equations (12) be supposed to refer to the commencement of the motion, so that we may have

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b, \quad \frac{dz}{dt} = c,$$

when $t = 0$, and if, as in the preceding number, the origin of the coordinates be transferred to the centre of gravity of the system, which is assumed to be entirely free, these equations (12) will be changed into the following:

$$\Sigma m \left(x_{,} \frac{dy_{,}}{dt} - y_{,} \frac{dx_{,}}{dt}\right) = \Sigma m \left(x_{,}\text{B} - y_{,}\text{A}\right),$$

$$\Sigma m \left(z_{,} \frac{dx_{,}}{dt} - x_{,} \frac{dz_{,}}{dt}\right) = \Sigma m \left(z_{,}\text{A} - x_{,}\text{C}\right),$$

$$\Sigma m \left(y_{,} \frac{dz_{,}}{dt} - z_{,} \frac{dy_{,}}{dt}\right) = \Sigma m \left(y_{,}\text{C} - z_{,}\text{B}\right),$$

in which $x_{,}, y_{,}, z_{,}$, are the coordinates of $m$ any point whatever, with respect to the new origin. These equations will be those of the initial motion of the system about its centre of gravity; and as they do not contain the components of the velocity of this point, it follows that this motion of rotation will be the same as if the centre of gravity was a fixed point, and the given velocities A, A′, &c., which occur in their second members were not changed; a result which agrees with what has been already established, in another manner, in the case of a solid body (No. 436).

542. It may be observed here, that equations (11) and (12) can be deduced from equations (6) and (9), by supposing in these, that $m$X, $m$Y, $m$Z, $m'$X′, $m'$Y′, $m'$Z′, &c. are the components of motive forces, which acting on the points $m$, $m'$, &c., with great intensity, are capable of producing in a very short interval of time, which we shall denote by $\theta$, the given quantities of motion, $m$A, $m$B, $m$C, $m'$A′, $m'$B′, $m'$C′, &c.

In fact, according to this, we shall have,

$$\int_0^\theta \text{X}dt = \text{A}, \quad \int_0^\theta \text{Y}dt = \text{B}, \quad \int_0^\theta \text{Z}dt = \text{C};$$

and if the points of the system are supposed to be in repose at the commencement of the time $\theta$, and if $a, b, c, a', b', c'$, &c. are the components of their velocities at the end of this interval of time, we shall have also

$$\int_0^\theta \frac{d^2x}{dt^2}\,dt = a, \quad \int_0^\theta \frac{d^2y}{dt^2}\,dt = b, \quad \int_0^\theta \frac{d^2z}{dt^2}\,dt = c,$$

for $m$ any point whatever.

Now, if the first equation (6) be multiplied by $dt$, and if its two members be then integrated from $t = 0$ to $t = \theta$, we obtain

$$\Sigma m \int_0^\theta \frac{d^2x}{dt^2}\,dt = \Sigma m \int_0^\theta \text{X}dt;$$

which coincides, by what precedes, with the first equation (11), and the same is the case for the two other equations (6) and (11).

Moreover, if the displacements of the points $m, m', m''$, &c., during the time $\theta$, are not taken into account, and if, consequently, their coordinates are considered as constant during the action of the given forces, we can deduce from the first equation (9)

$$\Sigma m \left(x.\int_0^\theta \frac{d^2y}{dt^2}\,dt - y.\int_0^\theta \frac{d^2x}{dt^2}dt\right) = \Sigma m \left(x.\int_0^\theta \text{Y}dt - y.\int_0^\theta \text{X}dt\right);$$

which, in virtue of the preceding suppositions, is the same expression as the first equation (12); and in the same manner, the two other equations (12) may be obtained from the two last equations (9).

543. As in the expressions for the increments of the coordinates given in No. 539, it is assumed that the distances of the respective points of the system from each other, and from the origin of the coordinates, are invariable; their expressions divided by $dt$, should coincide with the components of the ve-

locity, relative to the elements of a solid body, turning about a fixed point; it will not be useless to verify this.

For this purpose, let o$x$, o$y$, o$z$ (fig. 30), be the axes of the coordinates, and oI the axis to which the angles $\lambda$, $\mu$, $\nu$ of No. 539, about which the system is made to turn by an infinitely small quantity, refer. In this motion, the points of the system describe similar arcs of a circle, and they have all the same angular velocity; in order to determine this velocity, it will be sufficient to determine that of a point K, which may, for example, exist on the axis o$z$ at the commencement of the instant $dt$. Now, in the case of this point, we have $x = 0$, and $y = 0$; this reduces formulæ (8) to

$$\delta x = z\varepsilon \cos\mu, \quad \delta y = -z\varepsilon\cos\lambda, \quad \delta z = 0;$$

consequently, the absolute velocity of the point K will be

$$\sqrt{\frac{\delta x^2 + \delta y^2 + \delta z^2}{dt^2}} = \frac{z\varepsilon \sin\nu}{dt},$$

since $\cos^2\lambda + \cos^2\mu + \cos^2\nu = 1$. And as its distance from the axis oI is $z \sin\nu$, its angular velocity will be equal to $\frac{\varepsilon}{dt}$, which will be that of the entire system($k$).

Hence if it be denoted by $\omega$, we shall have $\varepsilon = \omega dt$, and by making

$$\omega\cos\lambda = p, \quad \omega\cos\mu = q, \quad \omega\cos\nu = r,$$

formulæ (8) will become

$$\frac{\delta z}{dt} = (yp - xq), \quad \frac{\delta y}{dt} = (xr - zp), \quad \frac{\delta x}{dt} = (zq - yr);$$

results which evidently coincide with those of No. 408, when in these last, the directions of the moveable axes o$x_{,}$, o$y_{,}$, o$z_{,}$, at the instant that they are considered, are assumed to be those of the fixed and arbitrary axes o$x$, o$y$, o$z$.

It may be observed, that if the plane $moz$ describes first an angle $rdt$ about the axis o$z$, and if the motion has place from o$x$ towards o$y$, or in the direction indicated by the sa-

gitta $s$, the increments of $x$, $y$, $z$, the coordinates of the point $m$, will be obtained, by making $p = 0$ and $q = 0$, in the values of $\delta x$, $\delta y$, $\delta z$; consequently, its three coordinates will become

$$x - yrdt, \quad y + xrdt, \; z.$$

If, after this first motion, the plane $moy$ describes an angle $qdt$ about the axis $oy$, by revolving from the axis $oz$ towards the axis $ox$, the increments of the coordinates of $m$, will be obtained by making $p = 0$, and $r = 0$, in the values of $\delta x$, $\delta y$, $\delta z$, and then by substituting the three preceding coordinates in the place of $x$, $y$, $z$; from which it follows that after this second motion the coordinates of the point $m$ will be

$$x - yrdt + zqdt, \quad y + xrdt, \quad z - (x - yrdt)qdt;$$

and if infinitely small quantities of the second order be neglected, the third coordinate will be reduced $z - xqdt$. Finally, if after the second motion, the plane $mox$ describes an angle $pdt$ about the axis $ox$, by turning from the axis $oy$ towards the axis $oz$, we shall find, by neglecting infinitely small quantities of the second order,

$$x + (zq - yr)dt, \quad y + (xr - zp)dt, \quad z + (yp - xq)\,dt,$$

for the values of the three coordinates of the point $m$, at the end of the third motion, which were originally equal to $x$, $y$, $z$.

Hence it follows that if a point $m$ turns successively in equal intervals of time, about three rectangular axes, with angular velocities denoted respectively by $p$, $q$, $r$, its final displacement will be the same, as if it turned during one of these instants, with an angular velocity denoted by $\omega$, about one sole axis, which makes with the three first, angles whose cosines are $\frac{p}{\omega}$, $\frac{q}{\omega}$, $\frac{r}{\omega}$. This remark relative to the three velocities of rotation $p$, $q$, $r$, which are termed the *components* of the velocity $\omega$ (No. 407), may be also applied to the components of the velocity of translation.

The composition of velocities of rotation follows the same laws, and is comprised in the same formulæ as that of velocities of translation; by means of this analogy between these two descriptions of motion, we are enabled to deduce the identity of the composition of forces and of the composition of moments, which was already inferred (No. 281) from a similar analogy existing between the projections of right lines and the projections of surfaces.

## II. *General Laws of Small Oscillations.*

544. Besides the motions of translation and rotation, in which, as all the points of any system participate, their mutual distances do not vary, there are likewise other motions, in which the moveables recede from or approach to one another. Now, if their displacements are always very small, we can reduce the problem to linear equations, and determine, by approximation, the coordinates of the moveables in functions of the time. A great variety of interesting phenomena depend on these small oscillatory motions, of which we now propose to explain the general laws.

Let $i$ denote the number of the moveables $m$, $m'$, $m''$, &c., and $\nu$ the number of equations (2) of No. 531, which express the conditions of the system. The number of coordinates of these material points will be denoted by $3i$, and, if we make $3i - \nu = n$, equations (2) will determine a number $\nu$ of the coordinates in functions of the $n$ others, or, more generally, all the coordinates may be determined by means of these equations, in functions of $n$ independent variables. Let $\alpha$, $\beta$, $\gamma$, &c. be the initial values of these $n$ variables, and $\alpha + u$, $\beta + v$, $\gamma + w$, &c., their values at the end of the time $t$, in which $u$, $v$, $w$, &c. are supposed to continue very small quantities during the motion. Each of the coordinates of the moveables will be a given function of $\alpha + u$, $\beta + v$, $\gamma + w$, &c., which may, besides, contain the time $t$, if this variable should occur

icitly in equations (2). These functions may be developed very convergent series, ranged according to the powers products of $u, v, w$, &c. Let these developments be re-ented as follows:

$$= p + au + bv + cw + \&c.$$
$$+ \tfrac{1}{2}eu^2 + \tfrac{1}{2}fv^2 + \tfrac{1}{2}gw^2 + huv + kuw + lvw + \&c.,$$

$$= p_1 + a_1u + b_1v + c_1w + \&c.$$
$$+ \tfrac{1}{2}e_1u^2 + \tfrac{1}{2}f_1v^2 + \tfrac{1}{2}g_1w^2 + h_1uv + k_1uw + l_1vw + \&c.,$$

$$= p_2 + a_2u + b_2v + c_2w + \&c.$$
$$+ \tfrac{1}{2}e_2u^2 + \tfrac{1}{2}f_2v^2 + \tfrac{1}{2}g_2w^2 + h_2uv + k_2uw + l_2vw + \&c.,$$

$$' = p' + a'u + b'v + c'w + \&c.$$
$$+ \tfrac{1}{2}e'u^2 + \tfrac{1}{2}f'v^2 + \tfrac{1}{2}g'w^2 + h'uv + k'uw + l'vw + \&c.,$$

$$' = p_1' + a_1'u + b_1'v + c_1'w + \&c.$$
$$+ \tfrac{1}{2}e_1'u^2 + \tfrac{1}{2}f_1'v^2 + \tfrac{1}{2}g_1'u^2 + h_1'uv + k_1'uw + l_1'vw + \&c.,$$

$$' = p_2' + a_2'u + b_2'v + c_2'w + \&c.$$
$$+ \tfrac{1}{2}e_2'u^2 + \tfrac{1}{2}f_2'v^2 + \tfrac{1}{2}g_2'w^2 + h_2'uv + k_2'uw + l_2'vw + \&c.,$$

&c.;

as by hypothesis, equations (2) are not supposed to con- the time $t$ explicitly, all the coefficients of the powers products of $u, v, w$, &c., in these series, will be given stants.

If the system is actuated by a motion of translation or ro-on common to all its points, the variable parts of their co-inates, which result from this motion, must be comprised he first terms $p, p_1$, &c.; but, for greater simplicity, we ll assume that this circumstance does not obtain, conse-ntly, these first terms will be also given constants.

As the components of the forces which act on the points $m', m''$, &c., are given functions of their coordinates, if the ues of $x, y$, &c. be substituted in their expressions, they can

be then developed according to the powers and products of $u$, $v$, $w$, &c. By this means, we shall have consequently

$$
\begin{aligned}
\mathrm{X} &= \mathrm{P} + \mathrm{A}u + \mathrm{B}v + \mathrm{C}w + \&c.,\\
\mathrm{Y} &= \mathrm{P}_1 + \mathrm{A}_1u + \mathrm{B}_1v + \mathrm{C}_1w + \&c.,\\
\mathrm{Z} &= \mathrm{P}_2 + \mathrm{A}_2u + \mathrm{B}_2v + \mathrm{C}_2w + \&c.,\\
\mathrm{X}' &= \mathrm{P}' + \mathrm{A}'u + \mathrm{B}'v + \mathrm{C}'w + \&c.,\\
\mathrm{Y}' &= \mathrm{P}_1' + \mathrm{A}_1'u + \mathrm{B}_1'v + \mathrm{C}_1'w + \&c.,\\
\mathrm{Z}' &= \mathrm{P}_2' + \mathrm{A}_2'u + \mathrm{B}_2'v + \mathrm{C}_2'w + \&c.,\\
&\&c.;
\end{aligned}
$$

the first terms $\mathrm{P}$, $\mathrm{P}_1$, &c., and all the coefficients $\mathrm{A}$, $\mathrm{A}_1$, &c., are given functions of $p$, $p_1$, &c., $a$, $b$, &c., which may, besides, contain the time $t$, if this variable occurs explicitly in the expressions of the given forces. However we shall suppose that this is not the case, and shall therefore assume that the quantities $\mathrm{P}$, $\mathrm{P}_1$, &c., $\mathrm{A}$, $\mathrm{A}_1$, &c., are given constants.

545. This being established, in order to apply equation (1) of No. 531 to the motion in question, we should attribute to the independent variables $u$, $v$, $w$, &c., infinitely small increments, which we shall denote by $\delta u$, $\delta v$, $\delta w$, &c.; and then substitute in this equation (1), the corresponding values of $\delta x$, $\delta y$, $\delta z$, which, if infinitely small quantities of the second order be neglected, will be(*l*)

$$
\begin{aligned}
\delta x = {} & (a + eu + hv + kw + \&c.)\,\delta u\\
& + (b + fv + hu + lw + \&c.)\,\delta v\\
& + (c + gw + ku + lv + \&c.)\,\delta w\\
& + \&c.,\\
\delta y = {} & (a_1 + e_1u + h_1v + k_1w + \&c.)\,\delta u\\
& + (b_1 + f_1v + h_1u + l_1w + \&c.)\,\delta v\\
& + (c_1 + g_1w + k_1u + l_1v + \&c.)\,\delta w\\
& + \&c.,
\end{aligned}
$$

$$\begin{aligned}\delta z = {} & (a_2 + e_2 u + h_2 v + k_2 w + \&c.)\,\delta u \\ & + (b_2 + f_2 v + h_2 u + l_2 w + \&c.)\,dv \\ & + (c_2 + g_2 w + k_2 u + l_2 v + \&c.)\,\delta w \\ & + \&c.\end{aligned}$$

The values of $\delta x'$, $\delta y'$, $\delta z'$ may be obtained from these formulæ by marking all the constants with a superior stroke; and those of $\delta x''$, $\delta y''$, $\delta z''$, by marking them with two, &c. After the substitution of these values of $\delta x$, $\delta y$, &c., is effected, as the quantities $\delta u$, $\delta v$, $\delta w$, &c., are arbitrary and independent, the coefficient of each of them should be put equal to cipher, in the first member of equation (1). By this means, we shall have

$$\begin{aligned}\Sigma m\Big[ & \left(\frac{d^2x}{dt^2} - \text{X}\right)(a + eu + hv + kw + \&c.) \\ & + \left(\frac{d^2y}{dt^2} - \text{Y}\right)(a_1 + e_1 u + h_1 v + k_1 w + \&c.) \\ & + \left(\frac{d^2z}{dt^2} - \text{Z}\right)(a_2 + e_2 u + h_2 v + k_2 w + \&c.)\Big] = 0,\end{aligned}$$

$$\begin{aligned}\Sigma m\Big[ & \left(\frac{d^2x}{dt^2} - \text{X}\right)(b + fv + hu + lw + \&c.) \\ & + \left(\frac{d^2y}{dt^2} - \text{Y}\right)(b_1 + f_1 v + h_1 u + l_1 w + \&c.) \\ & + \left(\frac{d^2z}{dt^2} - \text{Z}\right)b_2 + f_2 v + h_2 u + l_2 w + \&c.)\Big] = 0,\end{aligned}$$

$$\begin{aligned}\Sigma m\Big[ & \left(\frac{d^2x}{dt^2} - \text{X}\right)(c + gw + ku + lv + \&c.) \\ & + \left(\frac{d^2y}{dt^2} - \text{Y}\right)(c_1 + g_1 w + k_1 u + l_1 v + \&c.) \\ & + \left(\frac{d^2z}{dt^2} - \text{Z}\right)(c_2 + g_2 w + k_2 u + l_2 v + \&c.)\Big] = 0,\end{aligned}$$

&c.

in which the sums $\Sigma$ are always supposed to extend to all the points $m$, $m'$, $m''$, &c., of the system.

It still remains to substitute in these equations, in place of $x$, $y$, &c., X, Y, &c., their preceding values. If when this substitution is made, the squares and products of $u$, $v$, $w$, &c., and also the products of these unknown quantities, and of their differential coefficients $\frac{d^2u}{dt^2}$, $\frac{d^2v}{dt^2}$, $\frac{d^2w}{dt^2}$, &c., which likewise are always very small, be neglected in a first approximation, there will then result a number $n$ of linear equations with constant coefficients, which we shall denote by (a), each of which will be of the form

$$\left.\begin{aligned} &\mathrm{D}\frac{d^2u}{dt^2} + \mathrm{E}\frac{d^2v}{dt^2} + \mathrm{F}\frac{d^2w}{dt^2} + \&c. \\ &+ \mathrm{G}u + \mathrm{H}v + \mathrm{K}w + \&c. = \mathrm{Q}\,; \end{aligned}\right\} \quad \text{(a)}$$

in which the coefficients D, E, F, &c., G, H, K, &c., and also the quantity Q, denote given functions of the constants that occur in the preceding values of $x$, $y$, &c., X, Y, &c. After having determined the approximate values of $u$, $v$, $w$, &c., by means of these $n$ equations, we should substitute them in the terms of the rigorous equations, which have been neglected in this first approximation; the new equations which result will differ from the first in this, that their second members, instead of being constant, will be known functions of $t$; we can obtain from them other values of $u$, $v$, $w$, &c., more accurate than the first, and so on, by the method of successive approximations. According to the usual mode of proceeding in questions of this kind, we shall restrict ourselves to the first approximation. When the number of material points $m$, $m'$, $m''$, &c. is infinite, equations (a) will be changed into equations of partial differences, common to all the points of the system, the number of which will be always equal to that of the unknown quantities $u$, $v$, $w$, &c. This has been already observed, for example, in the problem of vibrating cords (No. 483), in which the number of these unknown quantities, that express the displacements of any point whatever of the cord, in the direction of three rectangular coordinates, and whose

values depend on three equations of partial differences of the second order with respect to $t$, is three.

546. As the second members of equations (a), and the coefficients which occur in the first, are constant quantities, we can always make these second members to disappear, by increasing each of the unknown quantities, $u$, $v$, $w$, &c., by a constant, which it is easy to determine. Consequently we can suppose, without limiting at all the generality of the question, that $Q = 0$, in each of equations (a); this implies that $\alpha$, $\beta$, $\gamma$, &c., the initial values of the $n$ independent variables, refer to a state of equilibrium($m$) of the system, from which it has been made to deviate, by displacing by ever so little the points $m$, $m'$, $m''$, &c., and impressing on them very small velocities. As these displacements and velocities must be compatible with the connexions of the points of the system, they are not the *initial* values of the coordinates $x$, $y$, &c., and of their first differential coefficients $\frac{dx}{dt}$, $\frac{dy}{dt}$, &c., which are given *arbitrarily* in each case, but solely the initial values of the independent unknown quantities $u$, $v$, $w$, &c., and of their first differential coefficients $\frac{du}{dt}$, $\frac{dv}{dt}$, $\frac{dw}{dt}$, &c.

Equations (a), in which the second members are supposed to be cipher, may be satisfied by assuming

$$\left.\begin{aligned} u &= \text{RN} \sin(t\sqrt{\rho} - r), \\ v &= \text{RN}' \sin(t\sqrt{\rho} - r), \\ w &= \text{RN}'' \sin(t\sqrt{\rho} - r), \\ &\text{\&c.} \end{aligned}\right\} \qquad \text{(b)}$$

R and $r$ being arbitrary constants, the second of which may be supposed to be positive and less than $\pi$, and $\rho$, N, N$'$, N$''$, &c., denote constants which we shall have to determine. The substitution of these values of $u$, $v$, $w$, &c., in equations (a), will evidently give a number of equations equal to $n$, and of the following form($n$),

$$(\text{D}\text{N} + \text{E}\text{N}' + \text{F}\text{N}'' + \&c.)\rho = \text{G}\text{N} + \text{H}\text{N}' + \text{K}\text{N}'' + \&c.$$

If between these equations $n-1$ of the quantities $\text{N}, \text{N}', \text{N}''$, &c., be eliminated, the $n^{th}$ quantity will disappear at the same time, and we shall have, in order to determine $\rho$, an equation of the $n^{th}$ degree, which we shall denote by

$$\Delta = 0.$$

Moreover, the values of the $n-1$ quantities $\text{N}$, $\text{N}'$, $\text{N}''$, &c., for example, of $\text{N}'$, $\text{N}''$, &c., which are obtained from these same equations, will be rational functions of the degree $n$, $n-1$ with respect to $\rho$, having a common denominator, and all multiplied by the quantity $\text{N}$, which remains undetermined. If this be made equal to the common denominator, the $n$ quantities $\text{N}$, $\text{N}'$, $\text{N}''$, &c., will be symmetrically expressed by polynomials of the degree $n$, relatively to $\rho$. Now, in consequence of the linear form of equations (a), they may be satisfied not only by the preceding values of $u, v, w$, &c., relative to such a root as we please of equation $\Delta = 0$, but also by taking for $u, v, w$, &c., the sums of all these particular values, in which the constants $\text{R}$ and $r$ will be changed at the same time as the root of $\Delta = 0$. Therefore, if $\rho, \rho_1, \rho_2$, &c., be the roots of this equation, and if $\text{N}, \text{N}_1, \text{N}_2$, &c., denote the corresponding values of $\text{N}$; $\text{N}', \text{N}_1', \text{N}_2'$, &c., those of $\text{N}'$, &c.; equations (a) may be satisfied by means of

$$\left.\begin{aligned}
u &= \text{R}\text{N}\sin(t\sqrt{\rho}-r) + \text{R}_1\text{N}_1\sin(t\sqrt{\rho_1}-r_1) + \&c.,\\
v &= \text{R}\text{N}'\sin(t\sqrt{\rho}-r) + \text{R}_1\text{N}_1'\sin(t\sqrt{\rho_1}-r_1) + \&c.,\\
w &= \text{R}\text{N}''\sin(t\sqrt{\rho}-r) + \text{R}_1\text{N}_1''\sin(t\sqrt{\rho_1}-r_1) + \&c.,\\
&\&c.,
\end{aligned}\right\} \quad (c)$$

in which $\text{R}, \text{R}_1, \text{R}_2$, &c., $r, r_1, r_2$, &c., are arbitrary constants, and as the number of them is $2n$, it follows that these formulæ (c) will be the $n$ complete integrals of equations (a), each of which is of the second order.

In each case, the values of $\text{R}\cos r$, $\text{R}_1\cos r_1$, &c., $\text{R}\sin r$, $\text{R}_1\sin r_1$, &c. can be determined by means of the given values

of the $2n$ quantities $u, v, w$, &c., $\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}$, &c., when $t = 0$. As all these values are supposed to be very small, those of $R, R_1, R_2$, &c., will be so likewise; consequently, if $\rho, \rho_1, \rho_2$, &c., the roots of the equation $\Delta = 0$, are all real and positive, the values of $u, v, w$, &c., in functions of $t$, will be periodic, and will continue very small, as has been supposed, during the entire continuance of the motion. But, if one or more of these roots are imaginary, or negative, the terms which correspond to them in equations (8), will be changed, by known formulæ, into exponentials, and will increase indefinitely; consequently, however small the values of $u, v, w$, &c., may have been at the commencement of the motion, they will eventually cease to be so, so that formulæ (c) will no longer represent approximate values of these unknown quantities, except for very inconsiderable intervals of time. In the first case, which we propose to examine particularly, the state of equilibrium, from which the system has been made to deviate a little, is one of stability; in the second case, this equilibrium is instable, at least relatively to the primitive derangements, for which the coefficients $R, R_1, R_2$, &c., of the terms that are not periodic, are not equal to cipher.

547. When all the coefficients $R, R_1, R_2$, &c., are cipher, except, for example, the first, formulæ (c) become reduced to formulæ (b). Therefore, if the squares and products of $v, u, w$, &c., be always neglected, we shall have simply (No. 544),

$$\left.\begin{aligned}
x &= p + (aN + bN' + cN'' + \&c.)R.\sin(t\sqrt{\rho} - r),\\
y &= p_1 + (a_1N + b_1N' + c_1N'' + \&c.)R.\sin(t\sqrt{\rho} - r),\\
z &= p_2 + (a_2N + b_2N' + c_2N'' + \&c.)R.\sin(t\sqrt{\rho} - r),\\
x' &= p' + (a'N + b'N' + c'N'' + \&c.)R.\sin(t\sqrt{\rho} - r),\\
y' &= p_1' + (a_1'N + b_1'N' + c_1'N'' + \&c.)R.\sin(t\sqrt{\rho} - r),\\
z' &= p_2' + (a_2'N + b_2'N' + c_2'N'' + \&c.)R.\sin(t\sqrt{\rho} - r),\\
&\&c.
\end{aligned}\right\} \quad (d)$$

As the first terms $p, p_1$, &c., and also the coefficients of the second terms are constant, it follows, that in this case, all the points of the system, will perform, in the direction of each of their coordinates, oscillations which will be isochronous and of a constant amplitude, the common duration for all these moveables, and in all possible directions, will be equal to $\frac{2\pi}{\sqrt{\rho}}(o)$. The relations which will exist between the amplitudes for two different points, or two different directions, will be determinable, and will depend on the nature of the system, and on $\rho$ the root of the equation $\Delta = 0$. As their absolute magnitude depends on the factor R, it will be arbitrary, and can not influence the *duration* of the oscillations. All the moveables will return at the same time to their position of equilibrium, which answers, by hypothesis (No. 546) to $u = 0$, $v = 0$, $w = 0$, &c., or to $x = p$, $y = p_1$, &c.; the first return will take place after the lapse of an interval $t = \frac{r}{\sqrt{\rho}}$, which will depend, as also R, on their primitive derangement.

If a system of material points, in which the connexion of these points allows a number $n$ of independent variables, be deranged very little from the state of equilibrium, it may assume a number $n$ of motions similar to the preceding, which correspond to the $n$ roots of the equation $\Delta = 0$. Moreover, in virtue of formulæ (c), and of the corresponding values of the coordinates $x$, $y$, &c., all these small motions, or only a certain number of them, may have place at the same time in this system; and conversely, whatever the initial derangement may be, the motion of each of these points can be always resolved in directions parallel to each axis of the coordinates, into a number $n$, or less than $n$, of simple oscillations (like those which respect equations (d)), the independent durations of the initial derangement of which will be $\frac{2\pi}{\sqrt{\rho}}$, $\frac{2\pi}{\sqrt{\rho_1}}$, $\frac{2\pi}{\sqrt{\rho_2}}$, &c. When all these durations are commen-

surable, the entire system will revert to the same state, after the lapse of each interval equal to the longest, this is what, for example, obtains in the motion of vibrating chords, and has not place in the transversal motion of elastic rods (Nos. 490 and 525).

It is in this general theorem, that *the principle of the co-existence of small oscillations* consists. It likewise obtains when the number of points $m$, $m'$, $m''$, &c., of the system is infinite; and the number of simple oscillations which are then possible may be also infinite; but notwithstanding this, both their durations and the relations which exist between their amplitudes are still determinable quantities. Thus, in the transversal motion of a stretched chord, the length, the weight, and the tension of which are denoted by $l, p, \omega$, respectively, and the gravity by $g$, the durations of the simple oscillations can be no other than the quantity $2\sqrt{\frac{pl}{g\omega}}$ and its submultiples; and by formula (d) of No. 489, the amplitudes of the oscillation which corresponds to any submultiple $i$, are to each other in the ratio of $\sin.\frac{i\pi x}{l}$ to $\sin.\frac{i\pi x'}{l}$, for those points of the chord, of which $x$ and $x'$ denote the distances from one of its extremities($p$).

548. When the points $m$, $m'$, $m''$, &c., oscillate in a resisting medium, X, Y, &c., the components of the forces which solicit them, will contain in their expressions, $\frac{dx}{dt}$, $\frac{dy}{dt}$, &c., the components of the velocities of these moveables. If the resistance of the medium be proportional to the square, or to a higher power of the velocity, it will not influence the motion of the system, at the degree of approximation to which we have restricted ourselves, since the terms $\frac{du^2}{dt^2}$, $\frac{dv^2}{dt^2}$, $\frac{dw^2}{dt^2}$, &c., which result from it, in the expressions of $x$, $y$, &c., are of

the same order of smallness as the quantities which have been neglected. But if the motions of the points $m$, $m'$, $m''$, &c., are not very rapid, then we should suppose, as in the case of very small oscillations of a simple pendulum, that the resistance is proportional to the first power of the velocity; this supposition will introduce into the expressions X, Y, &c., and by consequence, into equations (a), terms multiplied by $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, &c., which ought not to be neglected.

These equations which we shall denote by (e), will then be of the form

$$\left.\begin{aligned} &\mathrm{D}\frac{d^2u}{dt^2} + \mathrm{E}\frac{d^2v}{dt^2} + \mathrm{F}\frac{d^2w}{dt^2} + \&c., \\ &+ \mathrm{G}u + \mathrm{H}v + \mathrm{K}w + \&c., \\ &= \mathrm{D}'\frac{du}{dt} + \mathrm{E}'\frac{dv}{dt} + \mathrm{F}'\frac{dw}{dt} + \&c.; \end{aligned}\right\} \quad (e)$$

D′, E′, F′, &c., being likewise constant coefficients, which will be proportional to the density of the medium, and, for the most part, extremely small relatively to those which occur in the first member. Now this system of equations may be satisfied by means of formulæ (b) multiplied by exponentials, that is to say, by assuming

$$\left.\begin{aligned} &u = \mathrm{RN}\sin(t\sqrt{\rho} - r)\,e^{-\omega t}, \\ &v = \mathrm{RN}'\sin(t\sqrt{\rho} - r)\,e^{-\omega' t}, \\ &w = \mathrm{RN}''\sin(t\sqrt{\rho} - r)\,e^{-\omega'' t}, \\ &\&c.; \end{aligned}\right\} \quad (f)$$

in which $\omega$, $\omega'$, $\omega''$, &c. are very small constant quantities, and $e$ denotes, as usual, the base of the Naperian system of logarithms. The squares and products of these unknown quantities, and of the coefficients D′, E′, F′, &c., are supposed to be neglected, and as the values of $u$, $v$, $w$, &c., already satisfy equations (e) if their second number be cipher, when the exponentials are

not taken into account, their substitution in equations (e), will furnish a number $n$ of equations of the form

$$2\,\mathrm{DN}\omega+2\,\mathrm{EN}'\omega'+2\,\mathrm{FN}''\omega''=-(\mathrm{D}'\mathrm{N}+\mathrm{E}'\mathrm{N}'+\&c.),$$

by means of which the values of the $n$ unknown quantities $\omega$, $\omega'$, $\omega''$, &c., can be obtained($q$).

Since the effect of the resistance of a medium is to diminish gradually the amplitudes of the oscillations, these values will be positive. This diminution will be more or less rapid for the different independent variables $u$, $v$, $w$, &c.; it will also depend on $\rho$ the root of the equation $\Delta=0$, which occurs in the values of $\mathrm{N}$, $\mathrm{N}'$, $\mathrm{N}''$, &c.; so that the amplitudes of the simple oscillations of which the system is susceptible, will not all decrease with the same rapidity. However, the resistance of the medium will not otherwise have any influence on the duration of each of these oscillations, which will be always $\frac{2\pi}{\sqrt{\rho}}$ for that which corresponds to the root $\rho$. By taking the sums of formulæ (f), relative to all the roots of the equation $\Delta=0$, we shall have, as before, the most general values of $u$, $v$, $w$, &c.

549. It results from the principle of No. 547, that if the points of the system are so connected together, that there remains only one independent variable, they can perform only one species of oscillations, for which the duration and relations between the amplitudes in the case of the several moveables, will depend on the forces that solicit them, and on the nature of the system. This case will obtain, for example, in the motion of two material points $m$ and $m'$, attached the one to the other by a thread of a constant length, and constrained to move on given curves.

If, on the contrary, the points $m$, $m'$, $m''$, &c., are not connected together, nor constrained to remain on given surfaces or curves, a circumstance that does not prevent them from being subjected to their mutual attractions or repulsions, and to other similar forces directed towards fixed points, all their coordi-

nates will be independent variables; and in this case, which is in some measure the inverse of the preceding, the number of simple oscillations which may have place, will be triple of that of these material points($r$). This is what in fact obtains in the problem of No. 534, relative to the very slight motion of a point $m$, considered as entirely free, and subjected to the action of forces directed towards four fixed points.

For another example of the application of the preceding principle, let us consider the small oscillations of a heavy material point such as $m$, on the surface of an ellipsoid, one of whose axes is vertical. Let $2c$ denote the length of this axis, $2a$ and $2b$ those of the two horizontal axes, and, consequently,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

is the equation of the surface, the origin of the coordinates being at its centre. If this origin be transferred to the lowest point, and if the postive $z$s be supposed to be directed upwards, $c-z$ should be substituted in place of $z$ in this equation. As the oscillations of the moveable on each side of this inferior point are supposed to be very small, the horizontal abscissæ $x$ and $y$ will be so likewise, and its vertical ordinate will be very small relatively to $x$ and $y$. Therefore, if after the substitution of $c-z$ in place of $z$, the square of $z$ be neglected, we shall have($s$)

$$z=\frac{cx^2}{2a^2}+\frac{cy^2}{2b^2};$$

and if $h$ and $k$ denote the radii of curvature of the two principal sections of the surface, with respect to the lowest point, where $x=0$, $y=0$, we shall have,

$$h=\frac{a^2}{c},\quad k=\frac{b^2}{c}.$$

This being established, as in this question there are only two independent variables, namely $x$ and $y$, the moveable can

only perform two sorts of simple oscillations; and its most general motion will result from the coexistence of these two particular motions. Now, if the moveable be made to move from the lowest point of the ellipsoid, by impressing on it, in the section of which the horizontal axis is $2a$, a velocity directed in the plane of this section, it is evident, that it will not deviate from it during its entire motion. Therefore, if the force of gravity be as usual denoted by $g$, the duration of these small oscillations will be then $2\pi\sqrt{\frac{h}{g}}$, like to that of the simple pendulum, the length of which is $h$ (No. 183); and, at any instant whatever, we shall have

$$x = \text{R} \sin\left(t\sqrt{\frac{g}{h}} - r\right), \quad y = 0;$$

R and $r$ being as before, two arbitrary constants. In the case, in which the small oscillations are performed in the plane of the section whose horizontal axis is $2b$, their duration would be $2\pi\sqrt{\frac{h}{g}}$, and we should have, at any instant whatever,

$$x = 0, \quad y = \text{R}'\sin\left(t\sqrt{\frac{g}{h}} - r'\right);$$

R$'$ and $r'$ being also arbitrary constants. Consequently, the most general values of $x$ and of $y$ will be the sums of these particular values, that is to say,

$$x = \text{R} \sin\left(t\sqrt{\frac{g}{h}} - r\right), \quad y = \text{R}' \sin\left(t\sqrt{\frac{g}{h}} - r'\right).$$

In order to determine the four arbitrary constants, R, R$'$, $r$, $r'$, let us suppose that at the commencement of the motion,

$$t = 0, \quad x = p, \quad y = q, \quad \frac{dx}{dt} = p', \quad \frac{dy}{dt} = q';$$

then there results from this,

$$\text{R} \sin r = -p, \quad \text{R}' \sin r' = -q,$$

$$\text{R} \cos r = p' \sqrt{\frac{h}{g}}, \quad \text{R}' \cos r' = q' \sqrt{\frac{k}{g}};$$

hence, by substituting for the values of $h$ and $k$, we shall have, at any instant whatever($t$),

$$x = p \cos t \frac{\sqrt{gc}}{a} + \frac{p'a}{\sqrt{gc}} \sin t \frac{\sqrt{gc}}{a},$$

$$y = q \cos t \frac{\sqrt{gc}}{b} + \frac{q'b}{\sqrt{gc}} \sin t \frac{\sqrt{gc}}{b}.$$

In the case of $a = b = c$, these formulæ ought to coincide with those of No. 207, by making, as in them,

$$x = a\theta \cos \psi, \quad y = a\theta \sin \psi.$$

In fact, they then become,

$$a\theta \cos \psi = p \cos t \sqrt{\frac{g}{a}} + p' \sqrt{\frac{a}{g}} \sin t \sqrt{\frac{g}{a}},$$

$$a\theta \sin \psi = q \cos t \sqrt{\frac{g}{a}} + q' \sqrt{\frac{a}{g}} \sin t \sqrt{\frac{g}{a}};$$

but in this number we have supposed

$$\theta = \alpha, \quad \psi = 0, \quad \frac{d\theta}{dt} = 0, \quad a\theta \frac{d\psi}{dt} = \beta \sqrt{ga},$$

when $t = 0$; this requires that we should assume

$$p = a\alpha, \quad p' = 0, \quad q = 0, \quad q' = \beta \sqrt{ga};$$

there will consequently result($u$)

$$\theta \cos \psi = \alpha \cos t \sqrt{\frac{g}{a}}, \quad \theta \sin \psi = \beta \sin t \sqrt{\frac{g}{a}};$$

hence we obtain

$$\theta^2 = \tfrac{1}{2}(\alpha^2 + \beta^2) + \tfrac{1}{2}(\alpha^2 - \beta^2)\cos 2t\sqrt{\frac{g}{a}},$$

$$\alpha \operatorname{tang} \psi = \beta \operatorname{tang} t\sqrt{\frac{g}{a}},$$

as in the number already cited($v$).

(550.) Let us suppose generally, that

$$u = \text{U}, \quad v = \text{V}, \quad w = \text{W}, \ \&c.,$$

are the values of the independent variables at *any instant whatever*, when

$$u = u_1, \quad v = v_1, \quad w = w_1, \ \&c.,$$

$$\frac{du}{dt} = u_{,}, \quad \frac{dv}{dt} = v_{,}, \quad \frac{dw}{dt} = w_{,}, \ \&c.,$$

at the commencement of the motion; likewise let us suppose that we have at any instant whatever,

$$u = \text{U}', \quad v = \text{V}', \quad w = \text{W}', \ \&c.,$$

when

$$u = u_1', \quad v = v_1', \quad w = w_1', \ \&c.,$$

$$\frac{du}{dt} = u_{,}', \quad \frac{dv}{dt} = v_{,}', \quad \frac{dw}{dt} = w_{,}', \ \&c.,$$

for $t = 0$; and so on. Then, at the end of any time whatever, we shall have

$$\left.\begin{aligned} u &= \text{U} + \text{U}' + \text{U}'' + \&c., \\ v &= \text{V} + \text{V}' + \text{V}'' + \&c., \\ w &= \text{W} + \text{W}' + \text{W}'' + \&c., \\ &\&c., \end{aligned}\right\} \quad (g)$$

when we suppose that at the commencent of the motion,

$$u = u_1 + u_1' + u_1'' + \&c.,$$

$$v = v_1 + v_1' + v_1'' + \&c.,$$

$$w = w_1 + w_1' + w_1'' + \&c.,$$

$$\&c.,$$

$$\frac{du}{dt} = u_{,} + u_{,}' + u_{,}'' + \&c.,$$

$$\frac{dv}{dt} = v_{,} + v_{,}' + v_{,}'' + \&c.,$$

$$\frac{dw}{dt} = w_{,} + w_{,}' + w_{,}'' + \&c.,$$

&c.

In fact, it is evident, from the initial values which have been supposed in the first instance, to have been given to $u, v, w$, &c., $\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}$, &c., that formulæ (g) will satisfy these last equations, when $t = 0$; moreover, as by hypothesis, the particular values of $u, v, w$, &c., satisfy the differential equations of the motion, their sums, or formulæ (g), will also satisfy them, since these equations are linear, and contain no terms independent of the unknown $u, v, w$, &c. (No. 546); formulæ (g) will therefore satisfy all the conditions of the question, and will, consequently, be the values of the unknown at any instant whatever.

551. This general theorem may be denominated the *principle of the superimposition of small motions.* We should take care not to confound it with that of the coexistence of small oscillations; it is independent of the particular laws of the small motions that have been considered, and results solely from this, that the displacements and velocities of the moveables are treated as infinitely small quantities, since their products and all powers superior to the first are neglected.

It is in virtue of this principle, that sonorous waves which issue from different points are propagated and superimposed in the air, without producing any modification in each other; so that at each instant the displacement and velocity of a molecule of the air in any direction whatever, are the sums of the displacements and velocities which would belong to all these waves separately considered; which circumstance enables us to

hear distinctly, and without confusion, several sounds produced by different sonorous bodies. Simultaneous sounds may also result from the coexistence of small oscillations in the same sonorous body. Thus, for example, when a stretched cord performs, at the same time, isochronous oscillations which correspond to its entire length, and also oscillations which correspond to the third of this length, the motion of the air is precisely the same, as if two cords, whose lengths were as one to three, performed *simultaneously* the slowest vibrations of which they are susceptible; and, the fundamental tone of the given cord, and another more elevated tone, which is the *fifth* of the upper *octave*, is heard at the same time. This is also the reason why the sounds produced by the longitudinal vibrations and by the transversal vibrations which have place at the same time, in the same stretched cord, or in the same elastic rod, are heard distinctly.

In consequence of the same principle, the waves produced in several points of the surface of water, are simultaneously propagated round about from these different centres, and may cross in all directions on this surface, without modifying one another, so that at any instant whatever, the elevation of the water at each point will be the sum of the positive or negative elevations which would have place in virtue of all these waves separately considered.

The explanation which is given of the phenomenon of *interferences* in the theory of luminous undulations, is also founded on the principle of the superimposition of small motions, which, it may be observed, is susceptible of numerous applications.

In order to complete it, we may add, that if forces emanating from moveable centres, act on the points of the system, the second members of equations (a) of their small motions (No. 545), will be linear functions of the components of these given forces. The same will be the case with respect to the complete integrals of these same differential equations; hence

it follows, that the parts of $u$, $v$, $w$, &c., independent of the initial state of the system, and consequently, the similar parts of the coordinates of the moveables, will be the sums of the values which they would have, if the given forces acted separately. Thus, for example, in the phenomenon of the tides, the total elevation of the sea at each point, and at each instant, is the sum of the elevations which would be produced by the separate actions of the sun and moon; and this is the reason why, every thing else being considered as equal, the height of the tides is greatest in the syzygies, and least in the quadratures.

## II. *Principles of the Conservation of the Motion of the Centre of Gravity, and of the Conservation of Areas.*

552. Since the motion of the centre of gravity of a system entirely free is the same as if the masses of all the moveables being united in it, their motive forces were transferred to it parallel to their respective directions, it follows that the given forces, whose components parallel to each ordinate are equal and contrary, will not occur in the differential equations of this motion. Now, this case is that of the motive forces arising from the mutual actions of the points of the system, in virtue of the general law of *action equal to reaction*, which is always *observed* in nature, as we now proceed to explain.

If a material point situated at M acts on another point situated at M′, and impresses on it, or tends to impress on it, in an instant, an infinitely small quantity of motion which we shall denote by $\mu$, it is invariably observed,

1st. That this action is directed along the line drawn from the point M′ towards the point M, or along its production beyond M′.

2ndly. That at the same time the point situated at M′, reacts on the point situated at M, along the line drawn from M towards M′, or along its production beyond M.

3rdly. That this reaction communicates, or tends to communicate to the point situated at M, a quantity of motion precisely equal to $\mu(x)$.

The mutual action between these two material points is termed attraction or repulsion, according as it tends to increase or diminish the distance MM′; if they are entirely free, and if their masses are represented by $m$ and $m'$, the velocities by which they will be respectively actuated will be $\frac{\mu}{m}, \frac{\mu}{m'}$, that is to say, in the inverse ratio of their masses, and the quantities by which they approach to or recede from each other, in an infinitely small portion of time, will be equal to these velocities multiplied by the half of this time (No. 114)($x$). Moreover, the quantity $\mu$ will either depend on the nature of the bodies to which $m$ and $m'$ belong, or be independent of them, and proportional to $mm'$ the product of these masses (No. 241), as in the case of universal attraction ($y$).

The first of the three propositions which have been now stated, may be considered as self-evident; for when the quantities of matter $m$ and $m'$ are reduced to infinitely small dimensions, and placed at a finite distance from each other, there is no reason why the action of one of these points on the other should be exercised on one side of the line which joins them, and about which every thing is similar, rather than on the other; but with respect to the two other propositions, they can only be considered by us as the results of experiment, generalized indeed by induction, and confirmed by all the consequences which have been obtained from them. In fact, it cannot be considered, *à priori*, as impossible for a material point $m$ to act on another $m'$, without the latter reacting on the first, in an opposite direction with equal intensity. Thus, the principle of reaction equal and contrary to action may be admitted as a general law of nature, which is established by observation, in like manner as the law of

universal attraction, in the inverse ratio of the square of the distance.

553. This being established, if all the material points of a system entirely free are only subject to their mutual actions, these motive forces, transferred to the centre of gravity of the system, will destroy each other, two by two, consequently, the motion of this system will be rectilinear and uniform, and will constantly preserve its initial velocity and direction; hence this theorem has been denominated *the principle of the conservation of the motion of the centre of gravity.*

This motion is not altered by the impact of bodies, whatever may be their degree of elasticity (No. 541); and, in fact, the phenomenon of the impact is produced, as has been already stated (No. 499), by the mutual actions of the molecules of the striking and struck body, which actions are exerted at distances, which, although insensible, are of a finite magnitude, and for which the law of reaction, equal and contrary to action, must have place. For the same reason, if the parts of a solid body in motion are separated by any internal explosion, the direction and velocity of the centre of gravity of all its parts after the explosion, will be the same as the direction and velocity of the centre of gravity, which have place previously to this event. In general, the sudden changes of velocity which accompany these impacts or explosions are the effects of the mutual actions of the molecules; when the molecules approach to or recede from each other, these forces vary in very high ratios, and they, consequently, cause the velocities of the bodies to vary also considerably, during very short intervals of time.

The principle in question is independent of the connexion of the points of the system, provided that none of them is either attached to other points foreign from the system that is considered, nor constrained to move on a fixed or moveable curve or surface. Conditions of this kind, when they exist,

give rise to forces which should be transferred to the centre of gravity, and which may cause its velocity to vary. This will be also the case, when there are forces applied to the points of the system, which do not arise from their mutual actions; and, in this case, the mutual actions may influence indirectly the motion of the centre of gravity, by diminishing or increasing the distances of the points of the system from the fixed or moveable points from which the foreign forces emanate, and, consequently, changing their intensities.

The inertia of *a material point* consists in this, that it cannot excite any motion in itself, nor in any way modify the motion which it has received, without the aid of forces emanating from other points; in like manner, the inertia of a *system of bodies* consists in this, that the mutual action of its parts can neither produce nor modify the motion of its centre of gravity, without the intervention of points against which the moveables are pressed, or of foreign forces. Hence, the motion of the centre of gravity of the sun, the planets, the satellites, and comets, must be uniform and rectilinear in space, when the action that the fixed stars exert on all these bodies, and the resistance of the medium in which they move, if any such exists, is not taken into account.

The manner in which the different parts of a muscle act on each other, in order to produce its motions, is unknown to us; and perhaps we shall be for ever ignorant of the means by which the will puts these parts, that are of a different nature, in the respective dispositions which are required, in order that they may actually produce their mutual attractions or repulsions; whatever it may be, it cannot be doubted but that these actions are subjected to the law of reciprocity, like to all other natural forces; hence it follows, that an animal, in whatever manner it exerts itself, can never, by the mere act of volition, displace its centre of gravity, without the intervention of an exterior point against which it may press. A man or any other animal may cause their centre of gravity to rise or fall vertically, by

pressing against the earth; they may also cause it to advance horizontally by the aid of friction against its surface; but their locomotion would be impossible on a perfectly polished plane, on which this resistance would be entirely insensible. In the flight of birds, it is the centre of gravity of the animal and of the *entire mass* of the air that it puts in motion, that remains constantly at rest, and in a vacuo, it would be unable to displace its proper centre of gravity, whatever might be the rapidity with which it moved its wings.

Neither can it be doubted, but that imponderable fluids are subject to the law of reaction equal and contrary to action, and that consequently, the principle of the conservation of the motion of the centre of gravity which follows immediately from it, must likewise be observed in their motions as in that of all other substances, from which they differ only in their extreme tenuity. Thus, when electricity, heat, and light emanate from one side of a moveable, this body should recede in a contrary direction, in order that the centre of gravity of the entire system may remain at rest; but the quantity of motion by which it will be actuated will not be appreciable by the senses, unless that of the imponderable fluid is so likewise, notwithstanding the extreme smallness of its mass, and it will be proportional to the magnitude of its velocity. This can only be decided by means of very delicate experiments.

## III. *Conservation of the Motion of Rotation.*

554. It has been demonstrated that the forces arising from the mutual action of the points $m$, $m'$, $m''$, &c., of a system entirely free, do not occur in equation (7) of their motion of translation; it may be likewise shown that they disappear from the equations of its motion of rotation about the origin of the coordinates (Nos. 538 and 539).

In fact, let $F$ be the force arising from the mutual action of $m$ and $m'$, which is the same for these two material points,

and directed, if the force be attractive, from $m$ towards $m'$ for the point $m$, and from $m'$ towards $m$ for the point $m'$. If the distance of these two points be denoted by $\rho$, the cosines of the angles which the line $mm'$ makes with lines, drawn parallel to the axes of $x$, $y$, $z$, through the point $m$, will be

$$\frac{x'-x}{\rho}, \quad \frac{y'-y}{\rho}, \quad \frac{z'-z}{\rho};$$

hence then, relatively to the force F,

$$m\text{X} = \frac{(x'-x)\text{F}}{\rho}, \quad m\text{Y} = \frac{(y'-y)\text{F}}{\rho}, \quad m\text{Z} = \frac{(z'-z)\text{F}}{\rho},$$

will be the components of the motive force of $m$; in the same manner we shall find

$$m'\text{X}' = \frac{(x-x')\text{F}}{\rho}, \quad m'\text{Y}' = \frac{(y-y')\text{F}}{\rho}, \quad m'\text{Z}' = \frac{(z-z')\text{F}}{\rho},$$

for the components of the motive force of $m'$, arising from this force F. Now, it is evident from these values, that

$$m(x\text{Y}-y\text{X}) + m'(x'\text{Y}'-y'\text{X}') = 0,$$
$$m(z\text{X}-x\text{Z}) + m'(z'\text{X}'-x'\text{Z}') = 0,$$
$$m(y\text{Z}-z\text{Y}) + m'(y'\text{Z}'-z'\text{Y}') = 0;$$

consequently, the terms arising from the mutual action of the points of the system mutually destroy each other, two by two, in the second members of equation (9) of No. 539.

If therefore no other motive forces act on the points $m$, $m'$, $m''$, &c., the motion of rotation of the system about the origin of the coordinates, will arise solely from the initial velocities impressed on these points; so that unless there be some extraneous forces acting on the system, or some points against which the moveables press, taken without it, the sole mutual action of its parts can neither produce any motion of translation or of rotation common to all its points, or in any degree modify that which it has primitively received.

555. The second members of equations (9) will be likewise zero, when the points of the system, besides their mutual actions, are also solicited by forces directed towards the origin of the coordinates; for $m\text{X}$, $m\text{Y}$, $m\text{Z}$, the components of such a force applied to the point $m$, are to each other as $x$, $y$, $z$, the coordinates of this point, consequently we have

$$x\text{Y} = y\text{X}, \quad z\text{X} = x\text{Z}, \quad y\text{Z} = z\text{Y};$$

and the term which arises from it disappears from each of equations (9).

Thus, in every system entirely free, or which contains only one fixed point, and whose points $m$, $m'$, $m''$, &c., are only subject to their mutual actions, and to forces directed towards this fixed point, when this point is taken for the origin of the coordinates, we shall have

$$\left.\begin{aligned} \Sigma\, m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) &= 0, \\ \Sigma\, m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right) &= 0, \\ \Sigma\, m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) &= 0. \end{aligned}\right\} \quad (a)$$

If there is no fixed point in the system, and if the moveables are only subject to their mutual actions, we can assume any point we please, for the origin of the coordinates; and as in this same case, equations (7) of No. 538, become

$$\Sigma m\frac{d^2x}{dt^2} = 0, \quad \Sigma m\frac{d^2y}{d^2t} = 0, \quad \Sigma m\frac{d^2z}{dt^2} = 0, \qquad (b)$$

it follows that we can assume for this origin, a point which has a uniform and rectilinear motion in space.

In fact, if the coordinates of this moveable point be denoted by $\alpha$, $\beta$, $\gamma$, we shall have

$$\frac{d^2\alpha}{dt^2} = 0, \quad \frac{d^2\beta}{dt^2} = 0, \quad \frac{d^2\gamma}{dt^2} = 0;$$

and if the origin of the coordinates be transferred to it, we should make

$$x = \alpha + x_{,}, \quad y = \beta + y_{,}, \quad z = \gamma + z_{,},$$

relatively to $m$ any point whatever; now by substituting these values in the first equation (a), it can be made to assume the form ($z$)

$$\Sigma m \left(x_{,}\frac{d^2y_{,}}{dt^2} - y_{,}\frac{d^2x_{,}}{dt^2}\right) + \frac{d^2\beta}{dt^2}\Sigma m x_{,} - \frac{d^2\alpha}{dt^2}\Sigma m y_{,}$$

$$+ \alpha\Sigma m\frac{d^2y}{dt^2} - \beta\Sigma m\frac{d^2x}{dt^2} = 0;$$

which, in consequence of the preceding equations, will be reduced to

$$\Sigma m \left(x_{,}\frac{d^2y_{,}}{dt^2} - y_{,}\frac{d^2x_{,}}{dt^2}\right) = 0;$$

and in like manner the two other equations (a) will become

$$\Sigma m \left(z_{,}\frac{d^2x_{,}}{dt^2} - x_{,}\frac{d^2z_{,}}{dt^2}\right) = 0,$$

$$\Sigma m \left(y_{,}\frac{d^2z_{,}}{dt^2} - z_{,}\frac{d^2y_{,}}{dt^2}\right) = 0,$$

when the origin of the coordinates is transferred to a point, the motion of which is uniform and rectilinear.

As in the case in question, the motion of the centre of gravity of the system is uniform and rectilinear (No. 553), it follows, that when this centre is taken for the origin of the coordinates, equations (a) should obtain.

556. Since

$$x\frac{d^2y}{dt} - y\frac{d^2x}{dt} = d\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right),$$

$$z\frac{d^2x}{dt} - x\frac{d^2z}{dt} = d\left(z\frac{dz}{dt} - x\frac{dz}{dt}\right),$$

$$y\frac{d^2z}{dt} - z\frac{d^2y}{dt} = d\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right);$$

if equations (a) be multiplied by $dt$, and then integrated, there results

$$\left.\begin{aligned}\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)&=c,\\ \Sigma m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right)&=c',\\ \Sigma m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)&=c'',\end{aligned}\right\}\qquad(c)$$

$c$, $c'$, $c''$, denoting the arbitrary constants introduced by the integration; it appears from these equations, that in the motion of a system entirely free, in which the moveables are only subject to their mutual actions, or to the actions of forces directed towards a fixed centre; the moments of the quantities of motion of all the points of the system, with respect to three rectangular axes, that intersect in this point, and consequently, with respect to any other right line drawn through this point, are constant quantities. The value of these moments will not be changed in the impact of the bodies of a system, or when an explosion takes place in one or more of them, for the forces which produce these phenomena, are the mutual actions of their molecules; this agrees with the result of No. 541.

It results from the preceding number, that if there are no forces directed to a fixed point, this theorem also obtains, with respect to any axis whatever, which moves parallel to itself, and passes constantly through the centre of gravity of the system, or more generally, through any point the motion of which is uniform and rectilinear. Likewise it follows from equations (b), that in this same case, the sums of the quantities of motion of all the points of the system estimated in the direction of three rectangular coordinates, and consequently, in any direction whatever, are also constant quantities; this theorem may be regarded as contained in that which refers to the moments of these forces, the centre of the moments, and the origin of the coordinates being supposed to be infinitely distant.

557. The values of the constants $c, c', c''$, will depend on the direction of the rectangular axes which are taken for those of the coordinates; but if we make

$$c^2 + c'^2 + c''^2 = \gamma^2,$$

the quantity $\gamma$ will be not only independent of $t$, but also of this direction; for it expresses the *principal* moment of a system of forces (No. 281), the value of which does not depend on the arbitrary direction of the lines along which those forces are decomposed. Hence, when there is no fixed point in the system, the value of $\gamma$ will be the same when it is computed at two different epochs of the motion, the centre of gravity being taken for the origin of the coordinates, whatever may be otherwise their direction, whether the same or different, at these two epochs. In these calculations, it is the relative positions and velocities of the moveables at the given epochs that are employed, namely, $x, y, z$, the perpendiculars let fall from each point $m$ on three rectangular planes drawn arbitrarily through the centre of gravity, and $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, the differences between the components of the velocity of $m$, parallel to their intersections, and the components of the velocity of the centre of gravity in the same directions. Even if one or more impact or explosion of the bodies of the system should take place in the interval between the two epochs for which the value of $\gamma$ has been calculated, this value will not be changed, provided that in the case of an explosion, all the parts of the broken body are taken into account, in the calculation made at the second epoch. It follows, therefore, if this value of $\gamma$ is not the same at the second, as it was at the first epoch, that in the interval, foreign forces have acted on the moveables, or other bodies which do not constitute a part of the system, have impinged on them.

If $a, a', a''$, be the angles which the axis of the principal

moment makes with the axes of $x$, $y$, $z$, whose origin is at the centre of gravity, we shall have (No. 281)

$$\cos \alpha = \frac{c}{\gamma}, \quad \cos \alpha' = \frac{c'}{\gamma}, \quad \cos \alpha'' = \frac{c''}{\gamma};$$

hence, if these axes are constantly parallel to themselves, the quantities $c$, $c'$, $c''$, will not undergo any change, and the direction of the axis of the principal moment will be also invariable, as well as the magnitude of its principal moment. The same thing has place with respect to a fixed point, when there is one in the system, and the origin of the coordinates is placed in it; this has been already observed in No. 416, relatively to a solid body.

558. It is important to observe, that the terms arising from the mutual action of the bodies composing the system disappear in the second members of equations (9) of No. 539, even when the intensity of this action varies with the time, independently of the distance, that is to say, when the components of this force contain the time $t$ explicitly. Equations (c), and, consequently, the invariability of the principal moment, and of the direction of its axis, therefore, has still place in this case, which occurs, for example, when the points of the system lose, under a radiating form, a part of their proper heat, a circumstance that diminishes, at equal distances, the intensity of their mutual action. Thus, if the action of the sun and moon on the mass of the earth is not taken into account, and if we suppose that our planet was formerly in a gaseous state, and then that it became solid by the effect of cooling, without losing any part of its ponderable matter, we may be assured, that in this transformation, neither the magnitude nor axis of direction of the principal moment of the quantities of motion of all its points, undergo any change. This axis becomes that of the figure of the earth about which it now turns; and it is easy to perceive (No. 386) that in this motion

$$\gamma = \omega M k^2,$$

is the value of the principal moment; $\omega$ being the angular velocity of rotation, M the mass, and M$k^2$ the moment of inertia with respect to the axis of figure. If the cooling of the earth is still going on, and if, in consequence, its radius diminishes, the value of $k$ will diminish in the same ratio; and since the quantity $\gamma$ is constant, the value of $\omega$ will increase in the inverse ratio of the square of $k$, and the duration of the day will decrease proportionably to the square of the radius. A diminution, arising from this cause, of a ten millioneth part in the duration of the day, would imply a decrease of a twenty millioneth part in the length of the radius; and as we are certain, that for the last 2500 years, the day has not experienced this diminution (No. 443), it follows that the mean radius of the earth has not varied a twenty millioneth, or three metres very nearly in this long interval of time, by the effect of cooling, if the mean temperature of the earth has not yet attained to a permanent state.

No change can arise in the quantity $\gamma$ from earthquakes, volcanic explosions, the blowing of the winds against its surface, or the friction and pressure of the sea on the solid parts of the terrestrial spheriod, for all these phenomena are cases of the mutual actions of the parts of the system; and as the displacements of these parts, which take place under all these circumstances, are not considerable enough to produce any sensible change in the value of $k$, these different causes will not produce any appreciable alteration in the rapidity of $\omega$ the earth's velocity, or in the duration of the day.

559. The theorems which have been deduced from equations (c) may be also stated in another manner.

For this purpose, it may be observed, that the formula $\frac{1}{2}(xdy - ydx)$ expresses the area described during the instant $dt$, or the differential of the area described during the time $t$, by the radius vector of the projection of the point $m$, on the plane of the axes of $x$ and $y$, reckoning from the axis of the positive $x$s towards the axis of the positive $y$s (No. 154). In the same

manner, $\frac{1}{2}(zdx - xdz)$ is the differential of the area described by the radius vector of the projection of the same point $m$, on the plane of the axes of $z$ and $x$, reckoning from the axis of $z$ towards the axis of $x$; and $\frac{1}{2}(ydz - zdy)$ expresses the differential of the area described by the radius vector of the projection of this point on the plane of the axes of $y$ and $z$, reckoning from the axis of $y$ towards the axis of $z$.

This being established, the areas should be considered as positive or negative, according as they are described in each plane, in the direction indicated above, or in the opposite direction. Let $\frac{1}{2}\lambda$ be the sum of the areas described during the time $t$ by the radii vectores of the projections of all the points of the system on the plane of the axes of $x$ and $y$, and multiplied respectively by their masses $m, m', m''$, &c. Let $\frac{1}{2}\lambda'$ denote the sum of the areas described on the plane of the axes of $z$ and $x$ during the same time, by the radii vectores of the projections of these material points, and likewise multiplied by their respective masses. Finally, let $\frac{1}{2}\lambda''$ be the sum of the areas described on the plane of the axes of $y$ and $z$ during this time $t$, by the radii vectores of the projections of these same points, multiplied also by their masses. These three sums will be functions of $t$, the values of which will vanish with this variable, and their differentials will be

$$\frac{1}{2}d\lambda = \frac{1}{2}\Sigma m(xdy - ydx),$$

$$\frac{1}{2}d\lambda' = \frac{1}{2}\Sigma m(zdx - xdz),$$

$$\frac{1}{2}d\lambda'' = \frac{1}{2}\Sigma m(ydz - zdy).$$

Consequently we shall have, in virtue of equations (c)

$$d\lambda = cdt, \quad d\lambda' = c'dt, \quad d\lambda'' = c''dt;$$

and, by integrating, we obtain

$$\lambda = ct, \quad \lambda' = c't, \quad \lambda'' = c''t.$$

Therefore, in the motion of a system entirely free, whose points are only subject to their mutual actions, the sum of the

areas represented by $\frac{1}{2}\lambda$, $\frac{1}{2}\lambda'$, $\frac{1}{2}\lambda''$, are proportional to the time in which they are described, and the sums of the areas described in the unit of time, retain their initial values during the continuance of the motion; the centre of the areas being a fixed point, either the centre of gravity of the system, or indeed any other point, the motion of which is uniform and rectilinear.

It is in this that the *principle of the conservation of areas* consists. It obtains also, when there is a fixed point in the system towards which the forces acting on one or more of the moveables are directed, provided that this fixed point is taken for the centre of the areas; this comprehends the theorem of No. 154 relative to a single material point.

It may be remarked here, that when the points $m$, $m'$, $m''$, &c., turn in the same direction about the centre of areas, as the centres of the planets do about the sun, this will be also the case with respect to their projections on the planes of the coordinates; so that the signs of all the terms of the sums $\frac{1}{2}\lambda$, $\frac{1}{2}\lambda'$, $\frac{1}{2}\lambda''$ will be the same: on the contrary, they will have different signs, and these sums may be either positive or negative, when a part of the moveables turn in one direction, and the other part in the contrary direction, as is the case in the motion of the comets about the sun.

560. Now, let O (fig. 31) be the fixed or moveable centre of the areas; O$x$, O$y$, O$z$, the directions of the rectangular axes of the coordinates; M and $\text{M}_1$ the positions of any point $m$ at the end of the times $t$ and $t+dt$; P and $\text{P}_1$ the projections of M and $\text{M}_1$ on the plane of the axes of $x$ and $y$. The triangle $\text{MOM}_1$ will be the plane area described during the instant $dt$ by the radius vector of $m$, and the triangle $\text{POP}_1$ will be its projection on the plane of the axes of $x$ and $y$, or the area described during this instant, by the radius vector of the projection of $m$ on this plane. The projections of $\text{MOM}_1$ on the two other planes of the coordinates, will be likewise the areas

described by the radii vectores of the projections of $m$ on these planes.

This will be the case likewise for the areas described in space during all the infinitely small portions of $t$, by the radii vectores of all the points of the system, and multiplied by their masses, or in other words, for all these areas increased in the ratio of the masses $m, m', m''$, &c., to unity. Consequently, the quantities $\frac{1}{2}\lambda$, $\frac{1}{2}\lambda'$, $\frac{1}{2}\lambda''$ considered above, will be the sums of the projections of these infinitely small areas on the three planes of the coordinates, and the theorems of No. 276 and the following numbers, may be applied to this system of plane areas and to their projections.

Thus, among all the planes which can be made to pass through O, there is one on which the sum of the projections of the plane areas, respectively affected with the sign which results from the direction of the motion relative to each of them, is a *maximum*. If the value of this greatest area be denoted by $\mu$, we shall have

$$\mu^2 = \lambda^2 + \lambda'^2 + \lambda''^2;$$

and, if OH be the perpendicular from the centre O to this plane, by making

$$z\text{OH} = \beta, \quad y\text{OH} = \beta', \quad x\text{OH} = \beta'',$$

we shall have also

$$\cos\beta = \frac{\lambda}{\mu}, \quad \cos\beta' = \frac{\lambda'}{\mu}, \quad \cos\beta'' = \frac{\lambda''}{\mu}.$$

Now, from the values of $\lambda, \lambda', \lambda''$, given above, it is evident that these formulæ are the same thing as

$$\cos\beta = \frac{c}{\gamma}, \quad \cos\beta' = \frac{c'}{\gamma}, \quad \cos\beta'' = \frac{c''}{\gamma}; \tag{d}$$

$c, c', c'', \gamma$ being the same constants as before. Hence it appears that the direction of the plane of the greatest area will

remain constant during the continuance of the motion, and that the normal to this plane drawn through the centre of the areas will always coincide with the axis of the principal moment of the quantities of motion of all the points of the system.

It follows from this, that in the motion of every system entirely free, the material points composing which are only subject to their mutual actions, there exists a plane passing through the centre of gravity, which remains constantly parallel to itself, and whose position can be determined at each instant, by means of the masses of all these points, of their coordinates referred to the centre of gravity as their origin, an of the excess of the components of their velocities over those of the velocity of this centre.

We are indebted to Laplace for this theorem, who has denominated the plane in question *the invariable plane*, and he proposed to make use of it in astronomy, in order to refer to its constant direction the variable directions (No. 244) of the planes of the planetary orbits.

561. It is to the plane of the orbit of the earth, and to a right line drawn in this plane through the centre of the sun, and in a direction parallel to the line of the equinoxes, that astronomers refer the positions of the stars, and the directions of the planes in which they move. As the true ecliptic and the line of the equinoxes are in motion in space, their positions, at a given instant, are determined by comparing them with those of the stars; but as the proper motions of the stars, which are for the most part unknown, may, after the lapse of several ages, lead us into error as to the absolute displacements of the orbit of the earth, it is useful, in order to prevent mistakes, to be able to assign its true direction at any instant whatever.

Let us therefore suppose that the plane of the axes of $x$ and $y$ is the plane of the ecliptic at a given instant, or more accurately, a plane parallel to that of this ecliptic, and drawn through o the centre of gravity of the solar system. Through

the point O (fig. 32), let the axes O$x$ and O$y$ be drawn arbitrarily in this plane; and also let the values of the quantities $c$, $c'$, $c''$ be supposed to be computed by means of the coordinates and the actual velocities of all the points of the solar system, referred to the rectangular axes O$x$, O$y$, O$z$, and the masses of these points. If OH is the perpendicular to the invariable plane of this system, and EOE′ the intersection of this plane with that of the axes of $x$ and $y$, then by equations (d) we shall have

$$\cos \mathrm{HO}z = \frac{c}{\gamma}, \quad \operatorname{tang} \mathrm{EO}x = \frac{c'}{c''}; \tag{e}$$

by means of which the direction of the invariable plane relatively to that of the axes of $x$ and $y$, can be determined. But in order to be able to infer reciprocally, the absolute direction of the plane of the ecliptic parallel to that of the axes of $x$ and $y$, it is moreover requisite that there should exist, on the invariable plane, a line which remains constantly parallel to itself, and whose direction may be known at each instant. OK being this line, the angle KO$x$ will be known at the epoch in question. The angle EOK can be deduced from this angle, and the angles HO$z$ and EO$x$; and the two angles HO$z$ and EOK will completely determine the absolute direction of the plane of the ecliptic.

Now, the existence of the invariable plane in the solar system, supposes, implicitly, that the action of the stars on the sun and on the planets, is not taken into account, and that all the parts of the system are subjected solely to their mutual actions. But, in this case, the motion of O, the centre of gravity of the system is uniform and rectilinear; consequently, unless the direction of this motion is exactly perpendicular to the invariable plane, the line which the point O describes in space, when projected on this plane, will continue parallel to itself. There does not appear to be any other line that can be taken for the line OK; but to be able to make that use of it which we have indicated, the direction of the motion of the

centre of gravity of our universe should be previously determined by observation, which is at present very imperfectly known.

If the values of the angles HO$x$ and EOK be determined at two different epochs, the real displacements of the ecliptic which take place in this interval will be known, and in this determination it will appear that the angle HO$z$, or the inclination of the moveable plane to the invariable plane, is not sufficient of itself to determine them completely. Nevertheless, if the quantities $c'$ and $c''$ are very small relatively to $c$, the angle HO$z$ will be also very small, and very small differences in the values of $c'$ and $c''$ will produce considerable ones in the values of EO$x$, and consequently of EOK; so that in this case, it would appear that OE, the intersection of the ecliptic and of the invariable plane, will have experienced a considerable displacement on this plane. But, in general, this displacement will be only apparent, for the small differences of the values of $c'$ and $c''$ will arise, in a great measure, from the inevitable errors in the observations, by means of which these values are determined, and from the quantities which we are obliged to neglect in calculating them.

In fine, when the angle HO$z$ is very small, that is to say, when the inclination of the ecliptic to the invariable plane is inconsiderable, the angle EOK, which it is then extremely difficult to determine, has very little influence on the true direction of the plane of the orbit of the earth.

562. As the number and masses of the comets are unknown, the terms which correspond to them in the values of $c$, $c'$, $c''$, relative to the solar system, must be neglected; however, the values of $c$, $c'$, $c''$, furnished by formulæ (c), will be very little affected by this omission, in consequence of the smallness of these masses, and also, because the terms of these formulæ, which respect the comets, are in a great measure destroyed by the opposition of their signs (No. 559.)

The parts of $c$, $c'$, $c''$, which belong to the sun, the planets,

and the satellites, may be determined in the following manner.

Let M be the mass of one of these bodies, and $dm$ the element of this mass, whose coordinates referred to the axes O$x$, O$y$, O$z$, are $x, y, z$. Let $x_1, y_1, z_1$, denote the coordinates of the centre of gravity of M, with respect to the same axes, and $x_{\prime}, y_{\prime}, z_{\prime}$, the coordinates of $dm$, referred to parallels to these axes, drawn through this centre of gravity; so that, at any instant whatever, we may have

$$x = x_1 + x_{\prime}, \quad y = y_1 + y_{\prime}, \quad z = z_1 + z_{\prime},$$

$$\frac{dx}{dt} = \frac{dx_1}{dt} + \frac{dx_{\prime}}{dt}, \quad \frac{dy}{dt} = \frac{dy_1}{dt} + \frac{dy_{\prime}}{dt}, \quad \frac{dz}{dt} = \frac{dz_1}{dt} + \frac{dz_{\prime}}{dt}.$$

The origin of the coordinates $x_{\prime}, y_{\prime}, z_{\prime}$, being at the centre of gravity of M, we have

$$\int x_{\prime} dm = 0, \quad \int y_{\prime} dm = 0, \quad \int z_{\prime} dm = 0,$$

and, consequently,

$$\int \frac{dx_{\prime}}{dt} dm = 0, \quad \int \frac{dy_{\prime}}{dt} dm = 0, \quad \int \frac{dz_{\prime}}{dt} dm = 0;$$

in which the integrals are supposed to extend to the entire mass. Now, if these values of $x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ be substituted in the first equation (c), and if $m$ and $\Sigma$ be changed into $dm$ and $\int$, there results ($a$)

$$c = \text{M}\left(x_1 \frac{dy_1}{dt} - y_1 \frac{dx_1}{dt}\right) + \int \left(x_{\prime} \frac{dy_{\prime}}{dt} - y_{\prime} \frac{dx_{\prime}}{dt}\right) dm;$$

from which it appears, that the moment of the quantities of motion of M with respect to the axis O$z$, consists of two parts; the first depends only on the motion of the centre of gravity of M, and is the same as if this mass was concentrated in this point; the second is independent of this motion, and the same as if the centre of gravity of M was at rest, and the axis O$z$ was transferred to this point, parallel to itself. The same result is

applicable to the quantities $c'$ and $c''$, and also obtains with respect to any axis whatever. Now, if A, B, C be the moments of inertia of M, with respect to the three principal axes which intersect at its centre of gravity; $p$, $q$, $r$, the components of its angular velocity of rotation relative to the same axes; $\lambda$, $\mu$, $\nu$ the angles which their directions make with a line drawn parallel to the axis $oz$, through their point of intersection; then by what has been established in No. 409, we shall have

$$\int\left(x_{\prime}\frac{dy_{\prime}}{dt}-y_{\prime}\frac{dx_{\prime}}{dt}\right)dm = \text{A}p\cos\lambda + \text{B}q\cos\mu + \text{C}r\cos\nu,$$

for the moment, with respect to this parallel, of the quantities of motion of all the points of M, arising from its rotation about its centre of gravity. Hence it follows that the complete value of $c$ will be

$$c = \Sigma\text{M}\left(x_1\frac{dy_1}{dt}-y_1\frac{dx_1}{dt}\right)+\Sigma(\text{A}p\cos\lambda+\text{B}q\cos\mu+\text{C}r\cos\nu); \quad \text{(f)}$$

in which the two sums $\Sigma$ comprehend the sun, all the planets and their satellites, and therefore consist of thirty terms. Now, as the relations of A, B, C to M, depend on the internal constitution of this body M, they will be, without doubt, always unknown; all that we know respecting them is, that these three relations differ very little from each other, in consequence of the nearly spherical form of the heavenly bodies, and also that they are less than if these bodies were homogeneous, because the densities of the concentrical strata decrease from the centre to the surface of M. It would therefore be impossible to calculate the values of $c$, $c'$, $c''$, if it was necessary to take into account that part of each of these quantities, which arises from the rotation of the heavenly bodies. But whatever may be the form and internal constitution of M, the part of $\text{A}p\cos\lambda + \text{B}q\cos\mu + \text{C}r\cos\nu$, which is due to the initial state of this solid body, remains constant during the entire continuance of its motion (No. 416); so that this quan-

tity cannot vary, for each heavenly body, except on account of the attractions of the other bodies, inasmuch as their resultant does not pass exactly through the centre of gravity of M, that is to say, as far as they are exerted on the non-spherical part of M, i. e. the part by which it deviates from a sphere. It follows from this, that for each heavenly body the variable part of the second term of the formula (f) is very small, and may be neglected with respect to the part of the first term, which is relative to the same body. Thus for example, if the radius of the terrestrial globe be denoted by $h$, the angular velocity of its motion of rotation about its axis of figure by $\omega$, and the angle which this axis makes with the parallel to the axis $Oz$ by $\delta$, the second term of formula (f), which respects the earth, will be less than $\frac{2\text{M}h^2}{5}\omega\cos\delta$, which would be its value if the earth was homogeneous; likewise, if $\rho$ and $\theta$ be the mean radius of the orbit of the earth, and its mean velocity in its annual motion; the value of the first term of formula (f), relatively to the earth, will be consequently $\text{M}\rho^2\theta$. Now, if the axis $Oz$ is perpendicular to the plane of this orbit, in which case $\delta$ will denote the obliquity of the ecliptic, then a variation of five degrees in the magnitude of this angle will not produce a variation in the value of $\frac{2\text{M}h^2}{5}\omega\cos\delta$, which is a hundredth millioneth part of the quantity $\text{M}\rho^2\theta$. It is easy to be assured of this, by observing that the ratio of $\omega$ to $\theta$ hardly exceeds 366, that that of $\rho$ to $h$ is about 2400, and the angle $\delta$ very nearly 23° 28′. The same will be the case with respect to all the other planets. In the case of the sun, there is reason to think that the variable part of the second term of formula (f) which corresponds to it, is altogether insensible, because all observations indicate that its form is nearly spherical, and as the centrifugal force, compared with the weight, is extremely small in different points of this body (No. 260), we may also assume

that its interior strata are likewise nearly spherical; hence it follows that the resultant of the attractions of the planets must constantly pass through its centre of gravity, and, consequently, cannot cause any perturbation in its motion of rotation about this point($b$).

It follows from these considerations, that if the second term of the second member of equation (f), be made to pass into its first member, the invariable part of this second term may be comprised in the constant $c$, by affecting it with a contrary sign; and if its variable part be solely neglected, and if similar operations be performed on the equations which furnish the values of $c'$ and $c''$, we shall obtain, to a degree of accuracy much superior to that given by observation,

$$\left.\begin{aligned} c &= \Sigma \mathrm{M}\left(x_1\frac{dy_1}{dt} - y_1\frac{dx_1}{dt}\right), \\ c' &= \Sigma \mathrm{M}\left(z_1\frac{dx_1}{dt} - x_1\frac{dz_1}{dt}\right), \\ c'' &= \Sigma \mathrm{M}\left(y_1\frac{dz_1}{dt} - z_1\frac{dy_1}{dt}\right), \end{aligned}\right\} \qquad (\mathrm{g})$$

563. The origin of the coordinates $x_1, y_1, z_1$, which occur in these formulæ, is at the centre of gravity of the solar system; it will be more convenient to transfer it to the centre of gravity of the sun. For this purpose, let $g, h, k$, be the coordinates of this point referred to the same axes as $x_1, y_1, z_1$; and let $x, y, z$ denote the coordinates of the centre of gravity of M, referred to parallel axes passing through the centre of gravity of the sun; we shall then have

$$x_1 = x - g, \quad y_1 = y - h, \quad z_1 = z - k;$$

hence, if these values be substituted in the first equation (g), it will become($c$)

$$c = \Sigma \mathrm{M}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) + \left(g\frac{dh}{dt} - h\frac{dg}{dt}\right)\Sigma \mathrm{M}$$
$$- g\Sigma \mathrm{M}\frac{dy}{dt} + h\Sigma \mathrm{M}\frac{dx}{dt} + \frac{dg}{dt}\Sigma \mathrm{M}y - \frac{dh}{dt}\Sigma \mathrm{M}x;$$

and the expressions of $c'$ and $c''$ may be transformed in the same manner. Moreover, as the origin of the coordinates $x_1, y_1, z_1$, is at the centre of gravity of the system, we have

$$g\Sigma M = \Sigma Mx, \quad h\Sigma M = \Sigma My, \quad k\Sigma M = \Sigma Mz,$$

and, consequently,

$$\frac{dg}{dt}\Sigma M = \Sigma M\frac{dx}{dt}, \quad \frac{dh}{dt}\Sigma M = \Sigma M\frac{dy}{dt}, \quad \frac{dk}{dt}\Sigma M = \Sigma M\frac{dz}{dt};$$

if by means of these equations, $g, h, k, \frac{dg}{dt}, \frac{dh}{dt}, \frac{dk}{dt}$, be eliminated from the expressions of $c, c', c''$, they will finally become

$$\left.\begin{aligned}
c &= \Sigma M\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) \\
&\quad - \frac{1}{\Sigma M}\left(\Sigma Mx.\Sigma M\frac{dy}{dt} - \Sigma My.\Sigma M\frac{dx}{dt}\right), \\
c' &= \Sigma M\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) \\
&\quad - \frac{1}{\Sigma M}\left(\Sigma Mz.\Sigma M\frac{dx}{dt} - \Sigma Mx.\Sigma M\frac{dz}{dt}\right), \\
c'' &= \Sigma M\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) \\
&\quad - \frac{1}{\Sigma M}\left(\Sigma My.\Sigma M\frac{dz}{dt} - \Sigma Mz.\Sigma M\frac{dy}{dt}\right).
\end{aligned}\right\} \quad \text{(h)}$$

The coordinates $x, y, z$, of the centre of gravity of each planet or satellite, and $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, the components of its velocity, may be regarded as the data furnished by observations made at the different epochs, for which it is proposed to calculate the values of $c, c', c''$, and, consequently, the angles HO$z$ and EO$x$ relative to the direction of the invariable plane, by means of equations (e). As the origin of the coordinates is now supposed to be at the centre of the sun, the sums $\Sigma$ which relate to them will not contain the mass of the sun,

which as it will therefore solely occur in the denominator $\Sigma\text{M}$, the second term of each of the formulæ (h) will be very small relatively to the first($c$).

## IV. *Principles of living Forces and least Action.*

564. When equations (2) of No. 531 do not contain the time explicitly, equations (3) of the same number may be satisfied, by assuming for $\delta x$, $\delta y$, $\delta z$, $\delta x'$, &c., the differentials $dx$, $dy$, $dz$, $dx'$, &c., relative to this variable; for then these last equations will become the complete differentials of the first, namely,

$$d\text{L} = 0, \quad d\text{L}' = 0, \quad d\text{L}'' = 0, \text{ \&c.},$$

and since by hypothesis, the quantities L, L$'$, L$''$, &c., are cipher, during the entire continuance of the motion, their complete differentials taken by considering $x, y, z, x'$, &c., as functions of $t$, are likewise cipher. But if L, for example, contains the time explicitly, its complete differential will be

$$d\text{L} = \frac{d\text{L}}{dt}dt + \frac{d\text{L}}{dx}dx + \frac{d\text{L}}{dy}dy + \text{\&c.};$$

and by assuming

$$\delta x = dx, \quad \delta y = dy, \quad \delta z = dz, \quad \delta x' = dx', \text{ \&c.};$$

the first equation (3) will not agree with the equation $d\text{L} = 0$, except for those particular values of $t$, if any such exist, for which $\frac{d\text{L}}{dt} = 0$. We shall suppose in what follows, that the condition of the system of material points $m, m', m''$, &c., expressed by equations (2), are independent of the time $t$; moreover, the quantities L, L$'$, L$''$, &c., may be any functions whatever, of the coordinates of these moveables, which may not involve solely their mutual distances; for the system may contain fixed points, and also points constrained to remain on immoveable surfaces or curves.

This being agreed upon, if the preceding values of $\delta x$,

$\delta y$, &c., be substituted in equation (1) of the number cited, it will become

$$\Sigma m\left(\frac{d^2x}{dt^2}dx + \frac{d^2y}{dt^2}dy + \frac{d^2z}{dt^2}dz\right) = \Sigma m\,(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz).$$

But if $v, v', v''$, &c., denote the velocities of the points $m, m', m''$, &c., at the end of the time $t$, we shall have, relatively to $m$ any point whatever,

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = v^2;$$

and by differentiating with respect to $t$, there will result

$$\tfrac{1}{2}\,d.v^2 = \frac{d^2x}{dt^2}dx + \frac{d^2y}{dt^2}dy + \frac{d^2z}{dt^2}dz;$$

from which it appears, that the preceding equation may be changed into the following,

$$\tfrac{1}{2}d.\Sigma mv^2 = \Sigma m\,(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz). \qquad (a)$$

Now, if the points of the system are attracted or repelled by forces emanating from fixed centres, and if these forces be any functions whatever of the distance, the formulæ $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$ will be an exact differential (No. 158), for each of the moveables in particular. Moreover, if the points $m, m', m''$, &c., are subjected to their mutual actions, the intensities of which are likewise functions of their respective distances, and which satisfy the law of reaction equal and contrary to action, the sum of the quantities $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$ and $\mathrm{X}'dx' + \mathrm{Y}'dy' + \mathrm{Z}'dz'$ relative to the mutual action of $m$ and $m'$, will be also an exact differential (No. 346); and the same is the case for all the other parts of the sum $\Sigma$, taken two by two. It follows, therefore, that if there is no force directed to an extraneous moveable centre, which would introduce the time $t$ into the expressions $\mathrm{X}, \mathrm{Y}$, &c., nor any resistance of a medium, for which these expressions would contain the velocities of the moveables, so that the points $m, m', m''$, &c., may be only subject to their mutual actions,

and to attractions or repulsions emanating from fixed centres, we shall have

$$\Sigma m(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz) = d\phi(x, y, z, x', \&c.);$$

in which $\phi$ denotes a given function of the coordinates of $m, m', m''$, &c., depending on the laws of these forces in functions of the distances. Then, if equation (a) be integrated, we shall obtain, by denoting the arbitrary constant, introduced by the integration, by C,

$$\Sigma mv^2 = \mathrm{C} + 2\phi(x, y, z, x', \&c.).$$

In order to eliminate C, if the initial velocities of $m, m', m''$, &c., be denoted by $h, h', h''$, &c., the initial coordinates of $m$, by $a, b, c$, those of $m'$, by $a', b', c'$, &c., we shall have at the commencement of the motion,

$$\Sigma mk^2 = \mathrm{C} + 2\phi(a, b, c, a', \&c.);$$

and if this equation be taken from the preceding, there will result, at any instant whatever,

$$\Sigma mv^2 - \Sigma mk^2 = 2\phi(x, y, z, x', \&c.) - 2\phi(a, b, c, a', \&c.) \quad \text{(b)}$$

The quantities $\Sigma mv^2$ and $\Sigma mk^2$ are the sums of the living forces of all the points of the system, at this instant, and at the commencement of the motion; it therefore appears from this equation, that the difference of these two sums depends solely on the coordinates of the moveables, at those two epochs, and not at all on the manner in which they are connected together, or the routes which they describe in passing from their initial positions to those which they occupy at the end of the time $t$. It is in this, that the law of motion, which has been denominated the *principle of living forces,* consists.

565. It follows immediately from this principle thus expressed:

1st. That the sum of the living forces is constant when the points of the system are not subject to any motive force, and that their velocities do not vary in magnitude and direction,

except in consequence of their mutual connexions, or because they are constrained to move on fixed and given curves or surfaces.

2ndly. That if all the points of the system occupy the same positions at two different epochs, the sums of their living forces will be also the same at these two epochs.

As the forces which produce the impact of bodies of any nature whatever, arise from the reciprocal actions of their molecules (No. 553), it follows that equation (b) has place during the entire continuance of this phenomenon. Now, in the impact of bodies endowed with perfect elasticity, the moveables are supposed to resume, after the percussion is over, the exact form which they had previously, so that all their points revert to their primitive positions; if, therefore, this has actually place in the case of two or more bodies of any form whatever, then when they commence to separate from each other after the impact, the sum of the living forces of all their points will be the same at this instant, as it was the first moment of the percussion, or, in other words, there will be no loss of living force in the system, as has been already observed (No. 361), in the particular case of two homogeneous spheres, whose centres move in the same right line.

566. If the force of attraction or repulsion which emanates from a fixed centre, and acts on the point $m$, be denoted by R, and if $r$ be the mutual distance of these two points, we shall have (No. 158)

$$m(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz) = \pm \mathrm{R}dr;$$

in which the superior sign has place when the force is repulsive, and the inferior sign in the case of an attractive force, and R denotes a given function of $r$, which may be always regarded as positive. Consequently, if the distance $r$ is $a$ at the commencement of the motion, and $u$ at the end of the time $t$, $\pm 2\int_a^u \mathrm{R}dr$ will express the variation of the living force of the system, produced by the force R during the time $t$,

that is to say, it will be the part of the second member of equation (b), which belongs to this force. Consequently, when this force is repulsive, there will be an increase or diminution of living force, according as the distance $r$ shall have been increased or diminished, and the contrary will be the case when the force R is attractive. It is evident from what has been observed in No. 346, that this will also have place with respect to the mutual actions of the points of the system; and in fact, if, as in No. 554, we denote the mutual action of $m$ and $m'$, for example, by F, and the distance $mm'$ by $\rho$, we shall have

$$mX = \mp \frac{(x'-x)F}{\rho}, \quad mY = \mp \frac{(y'-y)F}{\rho}, \quad mZ = \mp \frac{(z'-z)F}{\rho},$$

$$m'X' = \mp \frac{(x-x')F}{\rho}, \quad m'Y' = \mp \frac{(y-y')F}{\rho}, \quad m'Z' = \mp \frac{(z-z')F}{\rho},$$

the sign is $-$ or $+$, according as F is repulsive or attractive; hence since

$$\rho^2 = (x-x')^2 + (y-y')^2 + (z-z')^2,$$

$$\rho d\rho = (x-x')(dx-dx') + (y-y')(dy-dy') + (z-z')(dz-dz'),$$

there will result

$$m(Xdx + Ydy + Zdz) + m'(X'dx' + Y'dy' + Z'dz') = \pm Fd\rho,$$

for the part of the second member of equation (a) which corresponds to the force F, and consequently, $\pm\ 2\int_a^u Fd\rho$ will be the variation of the living force of the system, which this force F will produce, while the distance $\rho$ changes from $\rho = a$ to $\rho = u$. As the superior sign has place when the action is repulsive, and as the quantity F is always positive, it is evident that there will be an increase or diminution of living force, according as $u > a$ or $u \angle a$, that is to say, according as $\rho$ is increased or diminished; it follows, for example, that the mutual action of the molecules of a gas which tends to increase their mutual distances, produces always an increase of living

force, in the system of which this fluid constitutes a part, when it is actually dilated, and a diminution when it is condensed. When the force F is attractive, the preceding quantity should be affected with the inferior sign, and it will produce opposite effects. It appears also that if a weight P be applied to a machine or any system of material points, it will produce an increase of living force expressed by the product $2\text{P}h$, when it descends through a vertical height $h$, and a diminution, likewise equal to $2\text{P}h$, when it is elevated the same height, whatever be the route which the body pursues in these two cases, whether a right line or a curve.

567. If the point $m$ is constrained to remain on a moveable surface, the equation of which is $\text{L} = 0$, then L is a given function of $x$, $y$, $z$, $t$. If the resistance of this surface, acting in the direction of one of the two parts of the normal, be denoted by N, and if for conciseness, we make

$$\text{V} = \left[\left(\frac{d\text{L}}{dx}\right)^2 + \left(\frac{d\text{L}}{dy}\right)^2 + \left(\frac{d\text{L}}{dz}\right)^2\right]^{-\frac{1}{2}},$$

then we shall have, for the components of this unknown force N,

$$m\text{X} = \text{NV}\frac{d\text{L}}{dx}, \quad m\text{Y} = \text{NV}\frac{d\text{L}}{dy}, \quad m\text{Z} = \text{NV}\frac{d\text{L}}{dz}.$$

Hence the part of the second member of equation (a), which corresponds to this force, will be

$$m(\text{X}dx + \text{Y}dy + \text{Z}dz) = \text{NV}\left(\frac{d\text{L}}{dx}dx + \frac{d\text{L}}{dy}dy + \frac{d\text{L}}{dz}dz\right);$$

and as by differentiating completely the equation $\text{L} = 0$, with respect to $t$, $x$, $y$, $z$, we obtain

$$\frac{d\text{L}}{dt}dt + \frac{d\text{L}}{dx}dx + \frac{d\text{L}}{dy}dy + \frac{d\text{L}}{dz}dz = 0,$$

it is evident that this part may be reduced to $-\text{NV}\frac{d\text{L}}{dt}dt$. Therefore in order to take the force N into account, in calculating the

living force of the system, we should add the double of the integral of this quantity to the second member of equation (b); consequently $-2\int \text{NV}\frac{d\text{L}}{dt}dt$ will be the variation of the living force, produced by the force N, during the time $t$, the integral being taken in such a manner that it may be cipher, when $t = 0(d)$.

This variation will be positive or negative, according to the sign of $\frac{d\text{L}}{dt}$, and according to that of V, which last will depend on the direction in which the force N acts. As the magnitude of this resistance N depends in part on the centrifugal force of the point $m$, in order to know it, and consequently to be able to calculate the value of the preceding integral, the velocity of the point $m$ and its trajectory must have been previously determined; this supposes that the problem with which we are occupied has been resolved, as far as concerns the point $m$. The variation of living force produced by this unknown force, will be no longer independent of the track which this point pursues in going from one position to another; and the principle of living forces, such as it has been announced above, will not have place; indeed its demonstration implies, that the equation $\text{L} = 0$ does not contain the time explicitly.

568. Neither will this principle have place, although the surface, of which $\text{L} = 0$ is the equation, may be immoveable, when the friction of the point $m$ against this surface is taken into account; as the variation of the living force produced by the friction, depends on the pressure, which is equal and contrary to the unknown force N, we cannot calculate *à priori* the magnitude of this variation; however it is easy to prove that the effect of the friction will be always to produce a diminution of living force.

In fact, as the friction is proportional to the pressure, that of the point $m$ against the surface, the equation of which is $\text{L} = 0$, may be represented by $f\text{N}$, in which $f$ denotes a given

fraction, which, as well as the unknown N, will be a positive quantity. Moreover, as the friction acts in the direction of a tangent to the trajectory of the moveable, and as this direction is contrary to that of its velocity, if the arc described by the point $m$ during the time $t$ be denoted by $s$, the components of the force $f$N parallel to the axes of $x$, $y$, $z$, will be

$$-f\text{N}\frac{dx}{ds}, \quad -f\text{N}\frac{dy}{ds}, \quad -f\text{N}\frac{dz}{ds};$$

consequently, as $dx^2 + dy^2 + dz^2 = ds^2$, the term of the second member of equation (a), which arises from this force, will be reduced to $-f\text{N}ds(e)$, and, there will occur in the second member of equation (b), a term $-2\int f\text{N}ds$, in which the integral should be taken in such a manner, that it may vanish with $s$, and this evidently indicates a diminution of living force.

This result will equally agree to the case, in which one body of a system slides on another; by assuming for $ds$ the element of the curve described by their point of contact in virtue of this sliding, for N the reciprocal pressure of these two bodies, and for $f$ a coefficient depending on the nature of their surface, the quantity $-2\int f\text{N}ds$ will still express the diminution of living force, which arises from this friction.

In the same manner it may be shown that the resistance of a medium produces constantly a diminution of living force, which will depend on the velocity of the moveables. Thus, the frictions of the parts of a system against each other, or against fixed obstacles, and the resistances of the medium which the moveables traverse, diminish continually the sum of the living forces of all these bodies; and it is in this manner, that these forces eventually reduce the entire system to a state of rest, if it has been put in motion, and then abandoned to itself, without being subjected to the action of other motive forces, which may reproduce continually the living forces destroyed by these resistances. This is what, for example, the force of

the spring effects in common time-pieces; its action restores, to the vibrating body, the living force, which, without this, it would have lost at each return to the vertical, and thus causes it to reascend constantly to the same height, notwithstanding the effect of friction and of the resistance of the air. In time-pieces moved by weights, the living force lost is restored to the pendulum by a weight which descends a very small space during each oscillation.

569. If the coordinates of the centre of gravity of a system $m, m', m''$, &c. of material points, be denoted, at any instant whatever, by $x_1$, $y_1$, $z_1$, and, if we make

$$x = x_1 + x_{,}, \quad y = y_1 + y_{,}, \quad z = z_1 + z_{,},$$

so that $x_{,}, y_{,}, z_{,}$, may be the coordinates of $m$ any point whatever, referred to this centre as the origin, we shall have

$$\Sigma m \frac{dx_{,}}{dt} = 0, \quad \Sigma m \frac{dy_{,}}{dt} = 0, \quad \Sigma m \frac{dz_{,}}{dt} = 0,$$

and because

$$v^2 = \frac{dx^2 + dy^2 + dz^2}{dt^2},$$

there will result($f$)

$$\Sigma mv^2 = \left(\frac{dx_1{}^2 + dy_1{}^2 + dz_1{}^2}{dt^2}\right) \Sigma m + \Sigma m \left(\frac{dx_{,}{}^2 + dy_{,}{}^2 + dz_{,}{}^2}{dt^2}\right),$$

or, what comes to the same thing,

$$\Sigma mv^2 = v_1{}^2 \, \Sigma m + \Sigma m v_{,}{}^2,$$

in which $v_1$ denotes the velocity of the centre of gravity, and $v_{,}$ the velocity of the point $m$, in its motion about this centre. Consequently, the sum of the absolute living forces of all the points of the system, will be obtained by adding the product of the square of the velocity of their centre of gravity and of the sum of their masses, to the sum of the living forces of all these same points in their relative motion about this centre.

It appears from this theorem, that if $m$ denotes the mass of one of the heavenly bodies, U the sum of the living forces of all its points in its motion of rotation about its centre of gravity, and $u$ the velocity of this centre in space, $U + mu^2$ will be the sum of the absolute living forces of $m$. Consequently, if equation (b) be applied to the solar system, we shall have

$$\Sigma U + \Sigma mu^2 = D + 2\phi(x, y, z, x', \&c.);$$

in which the sums $\Sigma$ comprehend the sun, planets, satellites, and even the comets, if their masses were known; D is an arbitrary constant, depending on the velocities and positions of all these bodies at a given instant, and $\phi$ denotes a function relative to their mutual attractions. We shall likewise have, in virtue of the same theorem,

$$\Sigma mu^2 = V^2 \Sigma m + \Sigma m\left(\frac{dx_i^2 + dy_i^2 + dz_i^2}{dt^2}\right),$$

in which V denotes the velocity of the centre of gravity of the solar system in space, and $x_1, y_1, z_1$ the coordinates of the centre of figure of $m$, referred to this centre of gravity as their origin. Consequently, the equation of living forces will become

$$\Sigma U + V^2 \Sigma m + \Sigma m\left(\frac{dx_i^2 + dy_i^2 + dz_i^2}{dt^2}\right) = D + 2\phi(x, y, z, x', \&c.) \quad (c)$$

In order to obtain the expression of the function $\phi$, it may be observed, that in consequence of the nearly spherical form of the heavenly bodies, and the smallness of their dimensions compared with their distances from each other, we may consider them as masses condensed into their centres of gravity (No. 242). Therefore, if the intensity of universal attraction at the unit of distance, and referred to masses taken to represent unity, be denoted by $f$, the masses of two of these bodies by $m$ and $m'$, and the distance of their centres of gravity by $\rho$, their mutual attraction will be expressed by $\frac{fmm'}{\rho^2}$, and the

value of the corresponding term of the function $\phi$ will be $\frac{-fmm'}{\rho}$; hence we shall obtain for its complete value,

$$\phi\,(x, y, z, x', \&c.) = -f\Sigma\frac{mm'}{\rho};$$

in which the sum $\Sigma$ extends to all the heavenly bodies, taken two by two.

Let it now be observed, in order to simplify equation (c), that if the action of the stars on the bodies of the solar system is not taken into account, the motion of its centre of gravity is uniform and rectilinear, and the velocity $\mathrm{V}$ is a constant quantity. Moreover, if the perturbations of the motion of rotation of each of the celestial bodies, which arise from the attractions of all the others on that part of the one in question, by which it differs from a sphere, be not taken into account, the quantity $\mathrm{U}$ is likewise constant for each body in particular (No. 419); hence, if the variable part of $\Sigma\mathrm{U}$ be neglected, equation (c) will become, by substituting another constant $\mathrm{C}$ in place of $\mathrm{D} - \Sigma m\mathrm{U} - \mathrm{V}^2\Sigma m$,

$$\Sigma m\left(\frac{dx_{\prime}^2 + dy_{\prime}^2 + dz_{\prime}^2}{dt^2}\right) = \mathrm{C} - 2f\Sigma\frac{mm'}{\rho} \qquad \text{(d)}$$

If the origin of the coordinates be transferred to the centre of the figure of the sun, and if $x$, $y$, $z$ be the coordinates of the centre of $m$ referred to this centre, and $g$, $h$, $k$ the coordinates of this centre referred to the centre of gravity of the solar system, then we must have

$$x_{\prime} = x - g, \quad y_{\prime} = y - h, \quad z_{\prime} = z - k,$$

for the coordinates of the centre of $m$, whose origin is at the centre of gravity of the system; there will result from these equations,

$$\frac{dg}{dt}\Sigma m = \Sigma m\frac{dx}{dt}, \quad \frac{dh}{dt}\Sigma m = \Sigma m\frac{dy}{dt}, \quad \frac{dk}{dt}\Sigma m = \Sigma m\frac{dz}{dt};$$

and because

$$\Sigma m\left(\frac{dx_{,}^2+dy_{,}^2+dz_{,}^2}{dt^2}\right)=\Sigma m\left(\frac{dx^2+dy^2+dz^2}{dt^2}\right)$$

$$-2\frac{dg}{dt}\Sigma m\frac{dx}{dt}-2\frac{dh}{dt}\Sigma m\frac{dy}{dt}-2\frac{dk}{dt}\Sigma m\frac{dz}{dt}$$

$$+\left(\frac{dg^2+dh^2+dk^2}{dt^2}\right)\Sigma m,$$

equation (d) will be changed, by eliminating the quantities $\frac{dg}{dt}, \frac{dh}{dt}, \frac{dk}{dt}$, into the following $(g)$:

$$\Sigma m\left(\frac{dx^2+dy^2+dz^2}{dt^2}\right)-\frac{1}{\Sigma m}\left[\left(\Sigma m\frac{dx}{dt}\right)^2+\left(\Sigma m\frac{dy}{dt}\right)^2\right.$$

$$\left.+\left(\Sigma m\frac{dz}{dt}\right)^2\right]=\text{C}-2f\Sigma\frac{mm'}{\rho}.$$

The sums $\Sigma$, with the exception of $\Sigma m$ and $\Sigma\frac{mm'}{\rho}$, will not contain the mass of the sum. However we can also separate from these two sums, the terms relative to this star, namely,

$$\text{M},\ \frac{\text{M}m}{r},\ \frac{\text{M}m'}{r'},\ \frac{\text{M}m''}{r''},\ \&\text{c.},$$

by denoting the mass of the sun by M, and the distances of the centres of $m, m', m''$, &c., from M by $r, r', r''$, &c. By this means the equation of living forces applied to the solar system, will finally become

$$\Sigma m\left(\frac{dx^2+dy^2+dz^2}{dt^2}\right)-\frac{1}{\text{M}+\Sigma m}\left[\left(\Sigma m\frac{dx}{dt}\right)^2+\left(\Sigma m\frac{dy}{dt}\right)^2\right.$$

$$\left.+\left(\Sigma m\frac{dz}{dt}\right)^2\right]=\text{C}-2f\text{M}\Sigma\frac{m}{r}-2f\Sigma\frac{mm'}{\rho};$$

in which the sums $\Sigma$ extend only to the planets, the satellites, and, if it is possible to estimate their masses, to the comets also; the origin of their coordinates being in this expression at the centre of the sun.

We may remark on this occasion, that the most direct means of determining whether the combined action of the comets has any sensible influence on the system of the world, would be to calculate at epochs separated by considerable intervals from each other, the value of the quantity c, deduced from this equation, by means of the relative velocities, the mutual distances, and the masses of the other celestial bodies, at these different epochs; if the values of c are found to be sensibly unequal, their variations must be ascribed to the action of the comets, as we suppose that the action of the stars is always neglected, and that no impact or explosion takes place in this interval; for we shall see immediately, that sudden changes of velocity alter the sum of the living forces of the system, and, consequently, produces a change in the value of the quantity c.

570. It is evident from No. 346, that the function denoted by $\phi$ in equation (b), is a maximum or a minimum for the values of the coordinates $x, y, z, x'$, &c., which belong to a position of equilibrium of the system; it follows, therefore, that the sum of the living forces of all its points ceases to increase or decrease as often as the system, during its motion, passes into a position in which it would remain in equilibrio, if its points were not actuated by any acquired velocities; and as these functions of the time must be alternately *maxima* and *minima*, it results also, that the positions of equilibrium through which the system passes will be alternately *stable* and *instable;* the latter corresponding to the *minima* of the function $\phi$, and the former to its *maxima.*

Nevertheless, as the distinctive character of the two states of equilibrium has been merely stated in No. 347, it remains for us to prove, that, in fact, the stability of equilibrium obtains when the function $\phi$ is a maximum, which we proceed to do by means of equation (b).

For this purpose, let $a, b, c, a', b', c'$, &c., be the coordinates of the points $m, m', m''$, &c., in a state of equilibrium of

the system, then let them be made to deviate by ever so small quantities from their positions, by impressing on them very small velocities $k, k', k''$, &c., so that at the end of the time $t$, the coordinates of the same points may be

$$x = a + p, \quad y = b + q, \quad z = c + r,$$
$$x' = a' + p', \quad y' = b' + q', \quad z' = c' + r',$$
&c.

It is proposed to show that the variables $p, q, r, p'$, &c., will always remain very small, if the quantity $\phi\ (a, b, c, a'$, &c.) is a *maximum*.

In fact, if $\phi\ (x, y, z, x'$, &c.) be developed according to the powers and products of $p, q, r, p'$, &c., then by the common property of *maxima* and *minima*, the sum of the terms depending on the first powers of these variables will be always cipher, whatever the number of independent variables, which occur in the question, may be. It is also demonstrated in the differential calculus, that the sum of the terms depending on the squares and products of $p, q, r, p'$, &c., that is to say, the sum of the terms of the second order, with respect to these quantities, may, in the case of a *maximum*, be decomposed into as many squares as there are independent variables, each of which is affected with the sign $-$. Hence, if the remainder of the series which includes the terms of the third and higher orders be represented by $\text{R}$, we shall have

$$\phi\ (x, y, z, x', \&c.) = \phi\ (a, b, c, a', \&c.) - (s^2 + s'^2 + s''^2 + \&c.) + \text{R};$$

$s, s', s''$, &c., being linear functions of $p, q, r, p'$, &c., which will vanish at the same time as these variables. If this value of $\phi$ be substituted in equation (a), we obtain

$$\tfrac{1}{2}\Sigma mv^2 = \tfrac{1}{2}\Sigma mk^2 - (s^2 + s'^2 + s''^2 + \&c.) + \text{R}.$$

Now, immediately at the commencement of the motion, the variables $p, q, r, p'$, &c., are very small; and, as long as this is the case, the quantities $s, s', s''$, &c., are equally so;

and, conversely, when the values of $s$, $s'$, $s''$, &c., are very small, those of $p$, $q$, $r$, $p'$, &c. must be so likewise. Moreover, for such values, each term of the second order is greater, without reference to the sign, than R, which only contains terms of an order superior to the second; consequently, however small the squares $s^2$, $s'^2$, $s''^2$, &c., may be, each of them surpasses the value of R.

This being established, we are justified in concluding that all the quantities $s$, $s'$, $s''$, &c., will continue very small, and that none of them will ever surpass $\sqrt{\frac{1}{2}\Sigma mk^2}$; for as these quantities vary by continuous degrees, this cannot take place before that $s$, the greatest of them, for example, becomes equal to $\sqrt{\frac{1}{2}\Sigma mk^2}$; and as this value of $s$ will continue very small, since by hypothesis all the quantities $k$, $k'$, $k''$, &c., are very small, we should have at the same time,

$$s^2 = \tfrac{1}{2}\Sigma k^2, \quad s'^2 > \mathrm{R}, \quad s''^2 > \mathrm{R}, \text{ \&c.},$$
$$\tfrac{1}{2}\Sigma mv^2 = -(s'^2 + s''^2 + \text{\&c.}) + \mathrm{R};$$

which, as $\frac{1}{2}\Sigma mv^2$ is essentially positive, would be absurd. Consequently the variables $p$, $q$, $r$, $p'$, &c., will continue always very small, and the system will only oscillate about its position of equilibrium, which will be therefore a stable equilibrium, as we proposed to demonstrate.

When the quantity $\phi\,(a, b, c, a', \text{\&c.})$ is a *minimum*, the sum of the terms of the second order in the development of $\phi\,(x, y, z, x', \text{\&c.})$, is a positive quantity; the equation of living forces may then subsist, though the variables $p, q, r, p'$, &c., be not always very small; but this is not sufficient to justify us, in concluding that they will in fact cease to be so, at the end of a certain time, however small they and the initial velocities $k$, $k'$, $k''$, &c., may be supposed to be at the commencement of the motion; and it is only by determining, in each problem, their values in functions of $t$, that we can be certain that they are not restricted to narrow limits.

571. As the initial velocities by which the points $m$, $m'$, $m''$, &c., are actuated at the commencement of the motion, and of which the components have been represented by $a$, $b$, $c$, $a'$, $b'$, $c'$, &c., in No. 535, satisfy necessarily the given conditions which connect the moveables with each other and with the other points of space; and since by hypothesis these conditions are expressed by equations, namely, by equations (2) of No. 531, it follows that if an infinitely small portion of time be denoted by $\varepsilon$, and if we assume

$$\delta x = a\varepsilon, \quad \delta y = b\varepsilon, \quad \delta z = c\varepsilon, \quad \delta x' = a'\varepsilon, \text{ \&c.}$$

the displacements of the points $m$, $m'$, $m''$, &c., which correspond to these increments of their coordinates, and also the contrary displacements, will satisfy the given conditions, that is to say, equations (3), which have been deduced from equations (2) in No. 531. We may therefore employ these values of $\delta x$, $\delta y$, &c., in equation (5) of No. 535; so that if at the commencement of the motion, the factor $\varepsilon$ common to all the terms be suppressed, we shall obtain the equation

$$\Sigma m\left[(\mathrm{A} - a)\,a + (\mathrm{B} - b)\,b + (\mathrm{C} - c)\,c\right] = 0, \qquad (e)$$

between $\mathrm{A}$, $\mathrm{B}$, $\mathrm{C}$, $\mathrm{A}'$, &c., the components of the velocities by which the points $m$, $m'$, $m''$, &c., would be actuated if they were free and detached, and those of the velocities by which they are actually actuated.

It is easy to verify this equation in the initial motion of rotation of a solid body about a fixed point. In fact, if $m$ be changed into $dm$, and $\Sigma$ into $\int$, and if we make

$$\int (a^2 + b^2 + c^2)\,dm = h;$$

so that $h$ may represent the sum of the living forces of all the points of the body, the preceding equation will become ($h$)

$$\int (\mathrm{A}a + \mathrm{B}b + \mathrm{C}c)\,dm = h. \qquad (f)$$

Besides, we shall have (No. 408)

$$a = qz_{,} - ry_{,}, \quad b = rx_{,} - pz_{,}, \quad c = py_{,} - qx_{,};$$

$x_{\prime}, y_{\prime}, z_{\prime}$, being the three coordinates of $dm$ referred to the principal axes of the body which intersect at the fixed point, and $p$, $q$, $r$ denoting the components of the velocity of rotation about these same axes. Moreover, if the principal moment, relative to the fixed point, of the quantities of motion impressed on all the points of the body, be denoted by $k$, and the angles which the axis of this moment makes with the axes of $x_{\prime}, y_{\prime}, z_{\prime}$ by $a$, $\beta$, $\gamma$, we shall have

$$\text{s}\,(\text{B}x_{\prime} - \text{A}y_{\prime})\,dm = k\cos\gamma,$$
$$\text{s}\,(\text{A}z_{\prime} - \text{C}x_{\prime})\,dm = k\cos\beta,$$
$$\text{s}\,(\text{C}y_{\prime} - \text{B}z_{\prime})\,dm = k\cos a\,;$$

for it is evident that the first members of these equations are the moments, with respect to the axes of $z_{\prime}, y_{\prime}, x_{\prime}$, of the quantities of motion, of which $k$ is the principal moment; so that the values of these integrals might be deduced from the value of $k$, by multiplying it by $\cos\gamma$, $\cos\beta$, $\cos a$ (No. 281). Now, if the values of $a$, $b$, $c$ be substituted in equation (f), there results, by taking into account these last equations($i$),

$$k\,(p\cos a + q\cos\beta + r\cos\gamma) = h\,;$$

consequently, if the component of the angular velocity of rotation about the axis of the principal moment be denoted by $\omega$, so that we may have

$$\omega = p\cos a + q\cos\beta + r\cos\gamma,$$

there will result

$$k\omega = h;$$

which agrees with the theorem of 419, according to which this component of the velocity of rotation is equal to the sum of the living forces, divided by the principal moment of the quantities of motion.

572. Now, if a sudden change should take place, during the motion, in the velocities of the moveables, we may assume for $\delta x$, $\delta y$, &c., in equations (5) of No. 535, the infinitely

small displacements of all the points of the system, which actually take place at any epoch whatever of this change, provided, as is explained in the following number, that at this instant the points of the parts of the system that are in contact, are actuated by the same velocity for the two adjacent parts, in the direction which is perpendicular to their common surface. This being so, there will be two distinct cases to consider:

1st. If the sudden change is produced by the meeting of two or more bodies of the system, or by the impact of these moveables against fixed obstacles, the condition in question will be satisfied at the instant of the greatest compression (No. 468). Hence if $a$, $b$, $c$, $a'$, &c., the components of the velocities of the moveables, be supposed to refer to this instant, and A, B, C, A$'$, &c., to the commencement of the impact, we may assume, as in the preceding number,

$$\delta x = a\varepsilon, \quad \delta y = b\varepsilon, \quad \delta z = c\varepsilon, \quad \delta x' = a'\varepsilon, \text{ \&c.};$$

and equation (e) will obtain between the components of the velocities at these two epochs, which will be those of the commencement and end of the stock, when the moveables are destitute of elasticity. Now, this equation (e) gives

$$\Sigma m\,(\mathrm{A}a + \mathrm{B}b + \mathrm{C}c) = \Sigma m\,(a^2 + b^2 + c^2);$$

and as

$$\Sigma m\left[(\mathrm{A} - a)^2 + (\mathrm{B} - b)^2 + (\mathrm{C} - c)^2\right]$$
$$= \Sigma m\,(\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2) + \Sigma m\,(a^2 + b^2 + c^2) - 2\,\Sigma m\,(\mathrm{A}a + \mathrm{B}b + \mathrm{C}c),$$

there results ($k$)

$$\Sigma m\,(\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2) - \Sigma m\,(a^2 + b^2 + c^2)$$
$$= \Sigma m\left[(\mathrm{A} - a)^2 + (\mathrm{B} - b)^2 + (\mathrm{C} - c)^2\right];$$

so that the excess of the sum of the living forces of all the points of the system before the impact, over the sum of the living forces after the impact, is a certain sum of living forces, and, consequently, a positive quantity. Hence, in the sudden changes of velocity, arising from the impact of bodies destitute of elasticity,

against each other, or against fixed obstacles, there is a loss of living force; this has been already observed in the case of the impact of two spherical and homogeneous bodies, whose centres move in the same right line (No. 361).

2ndly. When the sudden change is produced by internal explosions, which break in pieces one or more bodies of the system, the condition of No. 536 will then be satisfied by A, B, C, A′, &c., the components of the velocities at the commencement of the phenomenon, and not by $a, b, c, a'$, &c., the components of the final velocities; so that in this case, we can no longer employ the preceding values of $\delta x, \delta y$, &c., in equation (5) of No. 535. But if the infinitely small portion of time be denoted as before by $\varepsilon$, we may assume

$$\delta x = \text{A}\varepsilon, \quad \delta y = \text{B}\varepsilon, \quad \delta z = \text{C}\varepsilon, \quad \delta x' = \text{A}'\varepsilon, \ \&c.;$$

by means of which equation (5) will be changed in the following:

$$\Sigma m\left[(\text{A} - a)\,\text{A} + (\text{B} - b)\,\text{B} + (\text{C} - c)\,\text{C}\right] = 0;$$

hence we obtain

$$\begin{aligned}&\Sigma m\,(a^2 + b^2 + c^2) - \Sigma m\,(\text{A}^2 + \text{B}^2 + \text{C}^2)\\ &= \Sigma m\left[(a - \text{A})^2 + (b - \text{B})^2 + (c - \text{C})^2\right];\end{aligned}$$

from which it appears, that the sum of the living forces of all the points of the parts of moveables, after the explosion, is always greater than the sum of their living forces before the explosion. It is evident, in fact, that if the moveables be at rest before the separation of their parts, this separation will be always followed by an increase of living force; but in virtue of the theorem that has been just stated, whatever may be the motions of translation and rotation of a body, the sudden change produced by an internal explosion will always produce an increase of living force, and not a diminution, as in the case of the impact of bodies destitute of elasticity.

Without it being necessary to add anything to what has been stated in No. 469, respecting the impact of elastic bodies,

it is evident that the first part of this phenomenon, from the commencement to the termination of the greatest compression, is analogous to the first of the two preceding cases, and the second part, i. e., from this instant to the separation of the moveables, to the second of these two cases; there is, consequently, a loss of living force experienced during the first part, and an increase during the second. Moreover, if the moveables are perfectly elastic, so that they resume, when they separate, the same form which they had before the impact, and if the two parts of the phenomenon are perfectly similar, the increase of living force, during the second part, will be equal to the diminution which takes place during the first; consequently, the sum of the living forces of the system will be the same before and after the impact, agreeable to what has been observed in No. 565. This, however, implies, that no account is taken of the loss of living force, which will always take place in the case of the friction, or of the sliding of the bodies, the one against the other, during the continuance of their contact.

573. The *principle of the least action*, which it remains for us to consider, consists in this, that if in the motion of a system of bodies, for which the principle of living forces has place, the product of the velocity of each material point of the system, of its mass, and of the element of its trajectory be taken, and if the sum of similar products for all the moveables be taken and then integrated, from a given position of the system to another position likewise given, the value of this integral will be generally a *minimum*.

This theorem is an extension of that of No. 160, and may be demonstrated in the same manner, on that account, we shall, for the sake of conciseness, suppress the demonstration of it here. If the element of the trajectory of $m$ be denoted by $ds$ and its velocity by $v$, it will be the integral of $\Sigma mvds$, whose value will, in general, be a *minimum*; but in some cases, as in that of the motion of a material point on a surface which

returns into itself, the *minimum* may be replaced by *maximum*; and all that can be *demonstrated* on the subject is, that the infinitely small variation of $\int \Sigma mvds$ is always equal to cipher.' Because $ds = vdt$, if we make $\mathrm{v} = \Sigma mv^2$, the integral in question is the same thing as $\int \mathrm{v}dt$. Therefore, the principle of the least action implies, that the integral of the product of the living force of the system, and of the element of the time, is generally a *minimum*; so that, in nature, when a system of bodies is transferred from one position to another, the least quantity possible of living force is expended. When the moveables are not subject to the action of any motive force, the quantity v is constant (No. 565), and it is then the time of the transit which is a *minimum*. If the principle of least action be compared with the principles of living forces, of the conservation of the motion of the centre of gravity, and of the conservation of areas, it is evident that the first is merely a rule to enable us to form the differential equations of motion, which is now useless, since these equations may be obtained in a more direct and general manner, by means of formula (1) of No. 531; whereas the other principles, at the same time that they indicate important properties of motion, have also the advantage of furnishing the integrals of these differential equations, which, in the greater number of problems, is the only thing that can be known respecting them.

The principle of the conservation of the motion of the centre of gravity furnishes three integrals in finite quantities, namely ($m$),

$$\Sigma mx = a\Sigma m + \mathrm{A}t,$$

$$\Sigma my = b\Sigma m + \mathrm{B}t,$$

$$\Sigma mz = c\Sigma m + \mathrm{C}t;$$

$a$, $b$, $c$, A, B, C, being six arbitrary constants, the three first of which represent the coordinates of the centre of gravity of the system at the commencement of the motion, and the three others are the sums of the quantities of motion impressed, at

this epoch, on all the points of the system parallel to the axes of the coordinates.

The integrals which result from the principle of the conservation of areas are three integrals of the first order, namely,

$$\Sigma m\,(xdy - ydx) = cdt,$$
$$\Sigma m\,(zdx - xdz) = c'dt,$$
$$\Sigma m\,(ydz - zdy) = c''dt;$$

$c$, $c'$, $c''$ being three arbitrary constants which express the moments of the initial quantities of motion of all the points of the system, with respect to the axes of $x$, $y$, $z$, or double of the areas described, in the unit of time, about these same axes.

Finally, the principle of living forces furnishes only one integral, which is equation (b) of No. 564, and which may be written as follows,

$$\tfrac{1}{2}\,\Sigma m\left(\frac{dx^2 + dy^2 + dz^2}{dt^2}\right) = \mathrm{D} + \phi\,(x, y, z, x', \&c.);$$

D being an arbitrary constant.

# BOOK THE FIFTH.

## HYDROSTATICS.

### CHAPTER I.

#### PRELIMINARY NOTIONS.

574. HYDROSTATICS is the part of statics which treats of the equilibrium of fluids. A fluid is a collection of material points, which yield to the least effort that is made to separate them from one another. The fluids which we meet with in nature approach, in different degrees, to a state of perfect fluidity; the adherence which exists between the molecules of several of these substances, and which is termed their *viscidity*, prevents the separation of their parts; but in the theory which we now proceed to explain, we shall only consider the case of perfect fluids, and, with the exception of certain liquids whose viscidity is considerable, we shall find that the laws of equilibrium at which we shall arrive, may, without sensible error, be applied to all other fluids.

These substances are, like solid bodies, composed of detached molecules, and separated by empty spaces; but if a fluid be divided into parts of an insensible extent, each of which, nevertheless, contains an immense number of molecules, we may admit that the conditions of the equilibrium of each part are the same as if it was infinitely small, that it always retains its fluidity, and that its density is that of the body, such as it has been defined in No. 98. This comes, in

fact, to the same thing, as if the fluid was regarded as a continuous mass, whose density is constant, or variable by insensible degrees.

575. Fluids are divided into two classes, namely, *liquids* and *aeriform* fluids.

*Liquids* are also termed *incompressible* fluids, but, in point of fact, they are substances which can be compressed sensibly, only when they are subjected to very great pressures. If, for example, a vertical cylinder is filled with water to a certain height, and if we suppose that the only pressure on its upper surface, is that of a weight equal to the atmospherical pressure, it appears from observation, that in this case, if the cylinder retains the same diameter, and if its sides do not yield to the pressure transmitted to them through the liquid, the primitive height of the water is diminished only by 46 millionths. If the pressure on the upper surface be increased until it becomes equal to several hundreds of atmospherical pressures, the condensation of the water, as given by experiment, increases proportionably to this pressure. Mercury is still less compressible than water, and we have not succeeded by any effort hitherto made, in diminishing its volume in an appreciable manner.

Aeriform fluids, under which denomination the atmospheric air, and the different gases are supposed to be included, are compressible, and endowed with perfect elasticity; so that they can at once change the form and volume by compression, and exactly revert to their original form, when this compression ceases. They have been, in consequence, denominated *elastic fluids*.

*Vapours* are also elastic fluids; but for a given temperature, a given space can only contain a determinate quantity of vapour; so that if, when the vapour has attained this limit, either the space or temperature be diminished ever so little, a portion of the vapour liquefies. It appears from *experiment*, that this *maximum* of vapour is always the same at equal

temperatures, in space void of air, and in space filled with air more or less dilated or compressed. In general, the density of the vapour is inconsiderable, relatively to that of the liquid from which it arises; but if, when a liquid is contained in a vessel closed on every side, of which it occupies, for example, the third, or the half, its temperature be elevated to a very high degree, the entire liquid, after being dilated, is suddenly converted into a transparent vapour, whose density is a third, or one-half, of the primitive density of this same liquid.

The air and the gases are denominated permanent fluids, in contradistinction to vapours; but there is reason to believe that they may be liquefied by the application of a very great compression, or by means of a very great refrigeration, i. e. by reducing their temperature considerably, and this, in fact, has been verified in the case of several of them.

576. The characteristic property of fluids, which distinguishes them essentially from solids, and which is the basis on which the theory of their equilibrium is founded, is the faculty which they possess of transmitting equally, and in all directions, the pressures exerted on their surfaces. In the author's memoirs on the general equations of the equilibrium and motion of elastic fluids, inserted in the twentieth volume of the Journal of the Polytechnic School, it is shown how (this property arises from a mutual disposition of the molecules of the fluid, to which it reverts very rapidly,) when it has been compressed or dilated; and how the resultant of the molecular attractions and repulsions, which produces the interior pressures, may vary in a very high ratio, for the very small variations of distance of the molecules, which have place in the liquids. But, in this present treatise, we shall consider the property in question, as furnished by experiment, and as admitted by all philosophers and geometricians who have treated on hydrostatics, so that its accuracy cannot be questioned. In the same way, when the equilibrium of the elastic plate was dis-

cussed (No. 306), we set out from a secondary principle, instead of referring to the molecular actions from which it was derived.

577. In order to obtain an exact notion of the equality of pressure in all directions, we shall first consider the case of incompressible fluids.

Let a prismatic vessel, which is at right angles to its base, and rests on a horizontal plane, be filled up to EF with a liquid such as water, for example, and let ABCD (fig. 33) represent a vertical section of it, likewise let this vessel be supposed to be exactly closed by means of a horizontal piston. In order to simplify the question, we will not take into account the weight of the water, so that this fluid does not exercise of itself any pressure on the sides of the vessel. Finally, let a given weight P be laid on the piston, in which weight that of the piston itself is supposed to be included. It is evident that the horizontal base of the prism will be pressed in the same manner as if the weight P was laid immediately on this base, and uniformly distributed throughout its entire extent. All its points will experience equal vertical pressures, and the pressure that will result for $\alpha$, any portion whatever of this base, will be proportional to $\alpha$, and it will be equivalent to a vertical force applied to the centre of gravity of the area $\alpha$, and expressed by $\frac{P\alpha}{a}$, $a$ denoting the area of the entire base of the prism, which is also that of the base of the piston in contact with the liquid. Now, the principle of the equality of pressure in all directions, consists in this, that the pressure which the weight P exerts on the upper part of the water is transmitted by the intervention of the fluid, not only on the base of the vessel, but also on its lateral faces; all the points of the vessel are equally pressed in directions perpendicular to the sides; and an area $\alpha$, taken on one of the lateral faces of the prism, experiences the same pressure $\frac{P\alpha}{a}$, as if it constituted a part of its horizontal base.

Generally, if the form of the vessel is that of any poly-ron whatsoever, of which figure 34 represents a section, if this vessel is exactly filled with a liquid devoid of gra-, and then exactly closed; when one of the faces of this sel is removed, and replaced by a piston, to which a given e P is applied perpendicular to the surface of the adjacent id, the vessel and the fluid will remain at rest, and by the iciple just explained, the pressure which the force P exerts he adjacent surface, will be transmitted, by the intervention he liquid, on all the faces of the polyhedron. All the points he vessel, and also the points of the base of the piston, will equally pressed from within outwards, in directions perpen-ilar to the sides; and, relatively to an area $a$, taken on one hese sides, or on the surface of the piston, the pressure will a force perpendicular to its plane, applied to its centre of vity, and equal to $\frac{Pa}{a}$, $a$ being the entire area of the base he piston, in contact with the liquid.

This transmitted pressure acts in the same manner in the erior of the liquid; so that if we conceive a portion of the iid to be terminated by plane faces, or if a solid polyhedron plunged into it, any part such as $a$ of one of its faces will ewise experience a normal pressure equal to $\frac{Pa}{a}$, and acting n without inwards.

These results may, without difficulty, be extended to the e in which the pressed surface is no longer plane; it is y sufficient to decompose it into infinitely small elements, ich may then be regarded as the plane faces of an infini-mal polyhedron; and if $\omega$ denotes the area of one of these ments, $\frac{P\omega}{a}$ will be the normal pressure which it will expe-nce; $a$ being always the area of the piston, and P the per-idicular force applied to it. If the constant pressure which lane area equal to unity experiences be denoted by $p$, then

we shall have $\frac{P}{a} = p$, and the products $p\omega$ and $pa$ will express the pressures on the element $\omega$ and on the plane area equal to $a$.

If the liquid has a certain degree of viscidity, the property of pressing equally in every direction has still place, only in this case, the pressure is not transmitted laterally with the same rapidity as in the direction of the force P itself; but, after the lapse of a definite time, the lateral pressure becomes equal to the direct pressure; and, it is at this instant, that the equilibrium of the fluid is considered.

578. When the liquid contained in a vessel is heavy, it transmits the pressure exerted on its surface in the same manner as when it is devoid of weight, but it exerts besides, on the sides of the vessel, a pressure which arises from its weight, and is variable from one point to another: the same is the case when the points composing the liquid are solicited by the action of gravity and by other given forces, and it is in equilibrio in the vessel. If the sides of the vessel are necessary, in order to secure the equilibrium, so that if an opening be made, the liquid would immediately escape; it necessarily follows that the sides experience in each point a particular pressure directed from within outwards, along the normal to the surface of the vessel; for it is only in this direction that a surface can prevent a material point in contact with it from moving, and thus destroy, by its resistance, the motive force of this moveable.

The same thing has place in the interior of the liquid, both with respect to portions of the liquid itself, and also relatively to bodies plunged in it, as has been stated in the preceding number. The pressure on any point whatever is an unknown quantity, which we shall determine in the sequel, and which will depend on the position of this point and on the motive forces which act on the fluid. As in general, it changes from one point to another, it can only be supposed rigorously con-

stant for an area of an infinitely small extent; now, in order to measure the pressure on a determinate element of a surface, we suppose a plane area which is assumed to represent unity, and which experiences, through its entire extent, the same pressure as this element, then if $p$ be the total pressure which this area sustains, and $\omega$ the infinitely small extent of this element, the product $p\omega$ will be the pressure corresponding to this element, and normal to the surface of which it constitutes a part. The coefficient $p$ will be a function of the coordinates of this same element, which we shall term *the pressure referred to the unit of surface.*

This being established, if a plane portion of the surface of the vessel be taken away, and if it be replaced by a piston of the same extent, it is evident that when a force equal and contrary to that which this portion of the vessel experiences, is applied to this piston, the equilibrium will subsist as before. Moreover, if the vessel is closed on all sides, and is every where in contact with the liquid, and firmly secured, the equilibrium will not be disturbed, by increasing this first force by the addition of any other force such as $P$; for since the forces applied to the points of the fluid are in equilibrio, every thing takes place relatively to this force $P$, as if these forces had no existence, in like manner as in the preceding number. Consequently, the pressure exerted by this force $P$ on the surface of the liquid in contact with the piston, will be transmitted equally in every direction, by the intervention of the fluid, and the pressure $p$ referred to the unit of surface, will be increased in each point by a constant quantity equal to $\frac{P}{a}$; in which $a$ always denotes the area of the piston which is in contact with the liquid.

It is important to distinguish, as has been done here, the two descriptions of pressures which are exerted against the sides of a vessel that contains a liquid in equilibrio, or which the parts themselves of this liquid sustain; one of these pres-

sures, namely, that which arises from the weight and other motive forces that act on the fluid mass, varies from one point to another; the other, which arises from the forces applied to its surface, and is transmitted through its intervention, remains the same throughout the entire extent of the fluid. The combined effect of these two pressures at each point constitutes the total pressure.

579. In consequence of this property which fluids possess of transmitting equally in every direction the pressures exerted on their surface, an incompressible fluid contained in a vessel firmly secured, must be considered as a real machine; for a *machine* is in general an apparatus by means of which a force acts on points that are beyond its direction, and exerts on these points efforts which are greater or less than if it was immediately applied to it, and this is evidently the case of the force P, which has been considered in the preceding numbers.

The principle of virtual velocities has place in the equilibrium of this machine, as in that of all other known machines. In order to prove it, let us consider an immoveable vessel of any form whatever, which may have as many openings as we please, let a cylinder which extends indefinitely without the vessel be applied to each of these openings, then let this vessel be filled with any liquid, such as water, the given weight of which we shall not take into account; and let us suppose that the water rises in all the cylinders to a certain distance from their orifices, and that it is terminated by plane surfaces perpendicular to the lengths of the cylinders. Finally, let pistons be introduced into the cylinders which fit them exactly, and which at the same time are at liberty to slide without friction in the direction of their length. Let $a$, $a'$, $a''$, &c., be the bases of these pistons, which are likewise those of the cylinders; and let the forces P, P′, P″, &c., be applied to these bodies, in a direction perpendicular to their bases, and acting from without inwards, and finally, let the given forces which act on one another through the intervention of the water, be

in equilibrio. In this state, the pressure referred to the unit of surface must be the same on all the sides of the vessel and on the bases of the pistons (No. 577). If therefore it be denoted by $p$, the total pressures which the bases of the pistons sustain from within outwards, will be $pa$, $p'a$, $p''a'$, &c. In order that there may be an equilibrium, these pressures must be respectively equal to the forces P, P′, P″, &c.; consequently we shall have

$$P = ap, \quad P' = a'p, \quad P'' = a''p, \text{ \&c.} \qquad \text{(a)}$$

By means of one of these equations, the value of $p$ can be determined; and by substituting it in the others, the equations of the equilibrium of the system will be obtained, the number of which will be less by one than that of the pistons. Now if we conceive, agreeably to the definition of the principle of virtual velocities, that the parts of the system are displaced in such a manner that the pistons actually correspond to the sections CD, C′D′, C″D″, &c., of the cylinders. One set of these bodies will have advanced, and another set must have receded; let these displacements be denoted by $h$, $h'$, $h''$, &c., and let them be considered as positive or negative, according as the pistons have advanced or receded; then, in the figure, the distance $h$ comprised between the sections EF and CD is positive, and the distance $h'$ comprised between the sections E′F′ and C′D′ is negative. The volumes of water which issue from the cylinders, and flow into the vessel, correspond to the positive values of $h$, $h'$, $h''$, &c., and those which issue from the vessel to flow into the cylinders, correspond to their negative values. Both the one and the other will be expressed by the products $ah$, $a'h'$, $a''h''$, &c., no reference being made to the signs. Consequently, water being considered as incompressible, and the figure of the vessel as invariable, the sum of these positive or negative products must be cipher, and we shall have

$$ah + a'h' + a''h'' + \text{\&c.} = 0. \qquad \text{(b)}$$

If this equation be multiplied by $p$, there results, in consequence of equations (a),

$$Ph + P'h' + P''h'' + \&c. = 0, \qquad (c)$$

which is the equation resulting from the principle of virtual velocities, applied to the forces P, P′, P″, &c., and to $h$, $h'$, $h''$, &c., the displacements of their points of application The condition of the system, which in the present case is the invariability of the volume of the liquid, is expressed by equation (b). Not only do the displacements $h$, $h'$, $h''$, &c., satisfy this condition, but also the opposite displacements, $-h$, $-h'$, $-h''$, &c., as is required by the principle of virtual velocities (No. 331). The magnitudes of the quantities $h$, $h'$, $h''$, &c., may be finite, provided that none of the pistons enters into the vessel, or moves beyond the cylinder in which it ought always to be contained.

580. The principle of the equality of the pressure in every direction, belongs to elastic fluids as well as to liquids; but in the case of the first, in order that they may press against the sides of the vessels which contain them, it is not necessary that any motive forces should act on their molecules, or that any pressure should be exerted on their surfaces, the elasticity of these fluids, in virtue of which they continually endeavour to occupy a greater volume, is sufficient to produce this pressure. Hence, if we suppose a mass of air, of gas, or of any vapour to be contained in a vessel closed on all sides, and if the weight of the fluid is not considered, the sides of the vessel will sustain equal pressures in all their points, directed from within outwards, along the normals to these sides. The pressure referred to the unit of surface, will be the same throughout the entire extent of the vessel; in order to determine it, let an opening be made in any part whatever of the vessel, and let a piston be applied to this opening, then if the force necessary to maintain it in equilibrio be divided by the area of the base of the piston in contact with the fluid, the quotient will

express the required pressure, which will be always the same quantity in whatever part of the vessel the opening is made. If, for example, the vessel represented by figure 35, is filled with an elastic fluid, the forces P, P′, P″, &c., that should be applied to the pistons, which close the cylinders, in order to hinder them from sliding, will be proportional to the bases $a, a', a''$, &c.; the ratio of each force to the corresponding base will be the same for all the pistons; and equations (c) will still have place, but only for the motions of the system in which the total volume of the fluid undergoes no change.

This constant pressure, which an elastic fluid exerts on the sides of the vessel that contains it, depends on its matter, its density, and its temperature. It has been also termed *the elastic force of the fluid.* It appears from experiment, that for the same fluid, when the temperature is not changed, the elastic force is proportional to the density; so that if $p$ denote the measure of the elastic force, that is to say, the pressure referred to the unit of surface, and $\rho$ the density, we have in each fluid

$$p = k\rho;$$

$k$ being a coefficient which depends only on the matter and temperature of the fluid. When the gravity of the fluid is taken into account, or more generally, when its molecules are solicited by given forces, the pressure $p$ varies from one point to another of the vessel, according to a law which depends on these forces, and which we shall determine in the sequel.

## CHAPTER II.

### GENERAL EQUATIONS OF THE EQUILIBRIUM OF FLUIDS.

581. In order to discuss the question in the most general manner, let us consider a fluid mass ABCD (fig. 36), which may be either homogeneous or heterogeneous, compressible or incompressible, all whose material points are solicited by given forces, and let it be proposed to express the conditions of its equilibrium by equations.

Let $x$, $y$, $z$ be the coordinates of M any point whatever of this mass, parallel to the rectangular axes O$x$, O$y$, O$z$; we shall suppose for greater clearness, that the plane of the axes of $x$ and $y$ is horizontal, that the axis O$z$ is drawn in the direction of gravity, and that the mass ABCD is comprised below the plane of the axes of $x$ and $y$, in the solid angle, contained by the three planes of the positive coordinates. Let the fluid mass be distributed into parts, which, agreeably to what is stated above (No. 574), we shall consider as infinitely small elements; and let these elements be supposed to be comprised between planes infinitely near to each other, and parallel to those of the coordinates; so that these elements may be each of them rectangular parallelopipeds, the adjacent sides of which are parallel to the axes, and equal to the differentials of the coordinates, the two horizontal bases of that which corresponds to any point such as M, and which is represented in the figure, will be equal to $dxdy$, its vertical height MM′ will be equal to $dz$, and its volume will be $dxdydz$.

If the density of the fluid in this point, such as it has been defined in No. 98, be denoted by $\rho$, and the differential element

of the mass corresponding to this same point by $dm$, we shall have

$$dm = \rho\, dx\, dy\, dz.$$

In homogeneous liquids, if the small compressions that they experience, and which may be unequal in different points, be not taken into account, the factor $\rho$ will be a constant quantity; and it will be a known or unknown function of the coordinates $x, y, z$, in heterogeneous liquids, and also in elastic fluids, which are not equally compressed in every direction.

Let $\mathrm{X}dm, \mathrm{Y}dm, \mathrm{Z}dm$ denote the components of the motive force which acts on the element $dm$ resolved parallel to the axes of $x, y, z$, so that X, Y, Z may be the components of this force referred to the unit of mass, or of the accelerating force relative to the point M. Each of these three quantities will be a function of $x, y, z$, the values of which shall be regarded as positive or negative, according as the force which it represents tends to increase or diminish the coordinate to which it is parallel. Moreover, the element $dm$ will be pressed from without inwards, on its six faces, by the surrounding fluid, and, in order that it may remain in repose, these exterior pressures must be in equilibrio with the interior forces $\mathrm{X}dm$, $\mathrm{Y}dm$, $\mathrm{Z}dm$.

This being the case, if the vertical pressure which is exerted on the upper base $dxdy$, in the direction of gravity, be denoted by $pdxdy$, $p$ being the pressure which corresponds to the unit of surface on this infinitely small base (No. 577); this quantity $p$ will be an unknown function of $x, y, z$; and at the point M′, the coordinates of which are $x, y, z+dz$, it will become $p + \frac{dp}{dz}dz$, and it will express the vertical pressure, relative to the unit of surface, exerted on the inferior base of $dm$. Consequently, this second base will experience, in the direction of gravity, a pressure equal to $(p + \frac{dp}{dz}dz)\,dxdy$; the resistance of the fluid on which the

element $dm$ presses, is a force equal and contrary to this pressure; so that this element $dm$ is urged in a vertical direction by the two opposite forces $p\,dx\,dy$ and $(p + \frac{dp}{dz}dz)\,dx\,dy$, or by a force equal to their difference $\frac{dp}{dz}dx\,dy\,dz$, and directed upwards. Now, in order that this element $dm$ may be neither moved in the direction of gravity, nor in the contrary direction, this force must be equal to $\mathrm{Z}dm$, the vertical component of the motive force which acts downwards, consequently, we shall have,

$$\frac{dp}{dz}dx\,dy\,dz = \mathrm{Z}dm;$$

in like manner, if $q$ and $r$ denote the pressures, referred to the unit of surface, which correspond to the faces of $dm$ parallel to the planes of the axes of $x$ and $z$, and of $y$ and $z$, then, in order that the element $dm$ may not move either in the direction of the axis of $y$, nor in that of the axis of $x$, we should have

$$\frac{dq}{dy}dx\,dy\,dz = \mathrm{Y}dm, \quad \frac{dr}{dx}dx\,dy\,dz = \mathrm{X}dm.$$

Now if in these three equations, the preceding value of $dm$ be substituted, they become, by suppressing the common factor $dx\,dy\,dz$,

$$\frac{dp}{dz} = \rho\mathrm{Z}, \quad \frac{dq}{dy} = \rho\mathrm{Y}, \quad \frac{dr}{dx} = \rho\mathrm{X}. \qquad (1)$$

582. If the elements into which the mass ABCD is divided be solid, so that this mass may be regarded as a collection of solid rectangular parallellopipeds, in juxta-position with each other, it is not necessary that any relation should subsist between the pressures which each of these parallellopipeds experiences on those faces which are not parallel; the element $dm$ may, for example, experience a considerable pressure on

its horizontal bases, and none at all on its vertical faces; but as this infinitely small element must be considered as fluid, as well as all the parts of the entire mass, which should have a finite magnitude (No. 574), it follows from the fundamental property of fluids, that the three quantities $p, q, r$, must be equal to each other, or at least, if they differ, the difference can be only an infinitely small quantity, which may be neglected in equations (1).

In fact, the pressure which the surrounding fluid exerts on each of the faces of the parallelopiped $dx\,dy\,dz$, is transmitted on the other faces, by the intervention of the fluid, of which the element $dm$ consists; this transmission is made in the manner already explained, from which it follows, that if the pressure which has place from without inwards, on the upper horizontal base, be denoted by $p\,dx\,dy$, the pressures transmitted on the lateral faces, and which act from within outwards, will be represented by $p\,dx\,dz$ and $p\,dy\,dz$; moreover, we should add to these transmitted pressures, those which result from the motive force of the fluid $dm$, consequently, if the pressure due to this force, and exerted, for example, on the face $dy\,dz$, be denoted by $\gamma$, the entire pressure which has place from within, outwards, or from right to left, on this face $dy\,dz$, will be expressed by $p\,dy\,dz+\gamma$. On the other hand, the pressure arising from the surrounding fluid, and exerted from without inwards, or from left to right, on this face $dydz$, has been represented by $r\,dy\,dz$; this force is the resistance which the surrounding fluid opposes to the interior pressure $pdydz+\gamma$; consequently, we must have

$$r\,dy\,dz = p\,dy\,dz + \gamma.$$

Now, although the value of $\gamma$ may be unknown, we are nevertheless certain, that this quantity can only be an infinitely small one of the third order, like the motive force of $dm(a)$, from which it arises; hence if $\gamma$ be neglected relatively to $p\,dy\,dz$, we shall have $r=p$; in the same way it may be

also shown, that we must have $q = p$. The conclusion would be still the same, if the element $dm$, instead of being a rectangular parallelopiped, was any polyedron whatsoever, all whose sides were always infinitely small; and it may be demonstrated in the same manner, that the exterior pressure, exerted in a direction which is perpendicular to all the faces of the surrounding fluid, is proportional to their respective areas, and independent of the motive force of the polyedron. It follows, therefore, that all the elements of the surface which pass through the point M, experience the same pressure referred to the unit of surface, and that if $\omega$ be the area of one of them, the normal pressure which it sustains on one or other of its two sides, is equal to $p\omega$, whatever may be the direction of the plane to which it belongs.

In consequence of the condition $r = q = p$, equations (1) become

$$\frac{dp}{dx} = \rho \mathrm{X}, \quad \frac{dp}{dy} = \rho \mathrm{Y}, \quad \frac{dp}{dz} = \rho \mathrm{Z}; \tag{2}$$

and they are the general equations of the equilibrium of fluids, which it was proposed to find.

583. The conditions of equilibrium which they express, are reduced in each particular case, to our being able to find for $p$ a function of $x, y, z$, which satisfies at the same time these three equations. Now, if they be respectively multiplied by $dx, dy, dz$, and then added together, there results

$$dp = \rho\,(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz); \tag{3}$$

therefore, in order that the value of $p$ may be possible, the product of $\rho$, and of the formula $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$, should be an exact differential of a function of three independent variables $x, y, z$. Conversely, when this condition is satisfied, $p$ will be the integral of this product, and in this manner equations (2) will be satisfied.

If the coordinates of any point whatever of the surface of ABCD be substituted in place of $x, y, z$, in this value of $p$, the

pressure at this point on the side of the vessel in which this fluid mass is contained, will be had; this pressure will be always destroyed, provided that this side is fixed and susceptible of indefinite resistance; but in those parts, where the vessel is open, and where the fluid is entirely free, there is nothing to destroy the pressure $p$, consequently, its value must be cipher, for all the points of the free surface of a fluid mass in equilibrio; this gives for the differential equation of this surface

$$\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz = 0. \qquad (4)$$

This equation also obtains, when there is a constant pressure made on the free surface of the fluid; for then we must have $dp = 0$, for all its points, and as $\rho$ the density is not cipher, equation (4) results at once from formula (3). If by any means whatsoever, a pressure be made on the free surface of a fluid, which is variable from one point to another, and if this pressure, referred to the unit of surface, be represented by $f(x, y, z)$, the value of $p$ deduced from equation (4), should coincide, for all the points of the free surface, with the given function of $x, y, z$; and, in this case, the differential equation of this surface would be

$$\rho\,(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz) = df(x, y, z).$$

In the subsequent part of this treatise we shall always suppose that the exterior pressure is either cipher, or constant throughout the entire extent of the free surface of a fluid in equilibrio.

As the pressure $p$ is proportional to the density in elastic fluids (No. 580), it follows that this pressure can never be cipher in a fluid of this nature, as long as the density does not vanish, that is to say, as long as the fluid exists, and has not lost its entire elastic force by the effect of cold. Hence an elastic fluid cannot be in equilibrio, except when it is contained in a vessel which is closed on all sides, or, which is the same thing, when a pressure is made on its surface directed from without, inwards.

584. It follows from equation (4), that the resultant of the accelerating forces X, Y, Z, which act on each point of the free surface of a liquid in equilibrio, is perpendicular to this surface, both in the case where there is no exterior pressure, and also when there is exerted on this surface, a pressure which is constant from one point to another. In fact, if any curve whatever be traced on this free surface, and if $ds$ be the differential element of this curve corresponding to the point whose coordinates are $x, y, z$, so that $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, may be the cosines of the angles which the tangent to this curve, at this same point, makes with lines drawn parallel to the axes of the coordinates, and if R be the resultant of the forces X, Y, Z, since the cosines of the angles which its direction makes with these parallels, will be $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$, by dividing equation (4) by $Rds$, we shall have

$$\frac{X}{R}\frac{dx}{ds} + \frac{Y}{R}\frac{dy}{ds} + \frac{Z}{R}\frac{dz}{ds} = 0;$$

from which it appears, that the direction of the force R, and the tangent to the curve traced arbitrarily on the surface, are perpendicular, the one to the other, and consequently, this direction must coincide with the normal to the point which is considered. In general, this pressure will act from without inwards; but when the exterior pressure is not cipher, it may, on the contrary, be directed from within outwards.

If equation (4) be integrated, and if there be assigned to the arbitrary constant, introduced by the integration, any series of particular values, the determinate equations which will result will belong to as many different surfaces, and equation (4) will be the differential equation of each of these, and consequently, each of them will possess the property of being equally pressed in every direction throughout its entire extent, and of intersecting at a right angle, in all its points, the direction

of the resultant of the forces X, Y, Z. Such of these surfaces as, from the value of the arbitrary constant, must, when produced, lie in the interior of the fluid, are termed *surfaces of level.* If this arbitrary constant be made to increase by infinitely small degrees, the fluid mass will be divided into an infinite number of infinitely slender strata, and comprised between two consecutive surfaces of level, which, on this account, have been termed *strata of level.*

The value of the constant which belongs to the exterior surface will be determined in each case, by means of the given volume of the liquid; so that the exterior pressure will have no influence, either on the figure of equilibrium, or on the dimensions of this fluid considered as incompressible. If the liquid should be reduced to a state of solidity, the equilibrium would not be deranged; hence it follows that a constant normal pressure acting from without, inwards, on all the elements of the surface of a liquid or solid body, is destroyed of itself, and cannot impress on this body any motion either of translation or rotation. For a liquid, this equilibrium of exterior pressures results from the characteristic property of fluids, of transmitting equally in every direction, the pressures exerted on their surface (No. 577); in the sequel it will be shown that this is true, independently of this property, and that it equally obtains for a solid body of any form whatever.

585. Let us now suppose that the fluid in equilibrio is composed of homogeneous matter, and that it has every where the same density and temperature. As the quantity $\rho$ is constant, it follows from equation (3), that the formula $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$ must be an exact differential of a function of three independent variables. If this is not the case, the equilibrium is impossible in the fluid mass, whatever form may be given to it, even if it be contained in a vessel closed on every side.

But the condition of the integrability is always satisfied,

with respect to forces which are either attractive or repulsive, and whose intensities vary in functions of the distances from the centres from which they emanate (No. 158). Consequently, the equilibrium of a homogeneous liquid, subject to the action of similar forces, will be *possible;* and in order that it may actually have place, there should be such a form given to the fluid, as that its free surface may intersect at right angles, throughout its entire extent, the resultant of these forces, whether they be attractive or repulsive ($b$).

If, for example, the fluid mass be supposed to be entirely free, and that a constant pressure is exerted on its surface, then when the only force which acts on its particles is directed to a fixed centre, the figure of the mass ABCD in equilibrio about this point, will be that of a sphere whose centre is this point, and radius, a line depending on the given volume of this mass. If the force directed towards the fixed centre be supposed to be attractive, and to vary in the inverse ratio of the square of the distance, and if the intensity of this accelerating force at the surface of the liquid be denoted by $g$, its radius by $a$, and the exterior pressure by $\Pi$, $\frac{ga^2}{r^2}$ will be the attraction at the distance $r$, and it follows from equation (4), that $p$ the pressure at the same distance is equal($c$),

$$p = \Pi + \frac{g\rho a^2}{r} - g\rho a.$$

This will also have place, if the fixed centre be replaced by a solid sphere, all whose points attract those of the liquid in the inverse ratio of the square of the distance; but in this case, if the radius of this sphere be $c$, the value of $p$ is furnished by the preceding equation only for values of $r$ comprised between $r = c$ and $r = a$. When the attraction is changed into a repulsive force, it is only necessary to change the sign of $g$; so that we shall have

$$p = \Pi + g\rho a - \frac{g\rho a^2}{r}.$$

The least value of $p$ belongs to $r = c$, and will be

$$p = \Pi - \frac{g\rho a(a-c)}{c}.$$

It must be positive, in order that the liquid stratum may not detach itself from the solid body and be dispersed in space; consequently, the external pressure $\Pi$ must surpass the quantity $\frac{g\rho a(a-c)}{c}(d)$. In general, it is necessary in the equilibrium of a fluid, that the value of $p$ the pressure, should be positive throughout the entire extent of its mass, in order that the contiguous parts may press against one another, and the fluid be not separated.

When the radius $c$ is very great, the attractive forces directed towards the centre of the sphere are sensibly parallel, and the surface of the liquid is, for an inconsiderable extent, sensibly plane and perpendicular to the direction of this force. This is the case of a heavy liquid, which will be particularly discussed in the following chapter.

586. If, whatever be the forces of attraction or repulsion directed towards fixed centres, which act on all the points of any fluid mass ABCD, we make

$$\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz = d\phi;$$

in which $\phi$ denotes a function of the coordinates $x$, $y$, $z$, depending on the laws of these forces in functions of the distances, equation (3) will then become

$$dp = \rho d\phi.$$

In order that it may subsist when the density $\rho$ is variable, this density should be a function of the quantity $\phi$; and conversely, when this condition is fulfilled, there is always a value of $p$ which satisfies this equation of equilibrium. Now, by equation (4), the quantity $\phi$ is constant throughout the entire extent of each stratum of level; therefore in a heterogeneous

fluid and in a compressible fluid in equilibrio, it is necessary that the density should be constant throughout the entire extent of the same stratum; and as the superficial stratum, subjected to a given constant pressure, is a stratum of level, it is likewise necessary that it should have the same density throughout its entire extent(*e*).

If the fluid is incompressible, the density $\rho$ may be any function whatever, either continuous or discontinuous, of the quantity $\phi$; when it is given, the value of $p$ in a function of $\rho$, may be obtained by integrating the formula $\rho d\phi$, and determining the arbitrary constant, by means of the constant and given magnitude of the external pressure.

In the case of a heterogeneous liquid, subjected to the action of a central force, it is necessary, in order that there may be an equilibrium, that its mass should be composed of concentrical spherical strata, whose density must be the same throughout the entire extent of each of them, though it may vary arbitrarily from one stratum to another(*f*). In like manner, if several heavy liquids are contained in a vessel, it is necessary, in order to an equilibrium, that each horizontal and infinitely slender stratum should consist of only one liquid; this condition will be satisfied, if the upper surface which is supposed to be subjected to the action of a constant pressure, and the surfaces which separate two consecutive liquids, are all plane and horizontal. Moreover, in order that the equilibrium may be stable, it is necessary that the densities of the superimposed liquids should decrease from the lowest to the highest stratum, in order that the centre of gravity of this system of bodies should be the lowest possible (No. 348).

587. In an elastic fluid, the density is connected with the pressure (No. 580), and cannot be arbitrarily assigned, as in the case of an incompressible and heterogeneous fluid. By dividing equations $dp = \rho d\phi$, and $p = k\rho$, the one by the other, there results

$$\frac{dp}{p} = \frac{d\phi}{k}. \tag{5}$$

If the temperature be every where the same, $k$ will be a constant quantity, and, by integrating, we shall have

$$p = \Pi e^{\frac{\phi}{k}}, \quad \rho = \frac{\Pi}{k} e^{\frac{\phi}{k}}, \tag{6}$$

which expresses the laws of the density and pressure in the state of equilibrium of the fluid; $e$ denoting the base of the Naperian system of logarithms, and $\Pi$ an arbitrary constant, which expresses a certain pressure, that may be determined from knowing the pressure in a given point($g$). If the temperature varies from one point to another, $k$ will likewise vary; but in order that equation (5) may subsist, it is necessary that this quantity should be a function of $\phi$, which may be arbitrarily assigned. Consequently, the temperature must be also a function of $\phi$, and therefore it is constant throughout the entire extent of each stratum of level of an elastic fluid in equilibrio. This condition being satisfied, equations (6) should be replaced by the following,

$$p = \Pi e^{\int\frac{d\phi}{k}}, \quad \rho = \frac{\Pi}{k} e^{\int\frac{d\phi}{k}}.$$

When the centrifugal force and want of sphericity of the earth are not taken into account, the direction of the weight of the molecules of the air is towards the centre of the earth, and the strata of level are spherical and concentrical. Therefore, in order that the atmosphere may remain in equilibrio, it is necessary that the temperature should be every where the same at the same height above the surface of the earth, and that it should only vary with the elevation of the concentrical strata. Now, this is never the case; for the heat of the sun acts unequally on the different points of the surface of the earth and of each atmospherical stratum. As the temperature depends on the latitude, it is not possible that an equilibrium could have place, and this is the cause of those permanent winds which are, in fact, observed near to the equator.

Moreover, the condition of the equilibrium of the atmospherical strata can throw no light on the variation of temperature in the vertical direction; for equation (5) obtains whatever may be the value of $k$ in a function of $\phi$, and consequently, whatever may be the law of this variation.

When the mass ABCD is composed of several gases of different natures, the conditions of equilibrium may be satisfied in two different ways; when these gases are completely mixed together, so as to constitute a perfectly homogeneous fluid, and when they are, on the contrary, disposed in strata, resting the one on the other, so that their surfaces of separation are all faces of level. The first case obtains in an atmosphere of which the composition is the same at all heights. This state of perfect mixture is that of the most stable equilibrium; and when two different gases are placed the one over the other, in a vessel closed on all sides, they will become eventually perfectly mixed together, unless we take care to secure the vessel which contains them, from the slightest agitations.

588. The centres of the attractive or repulsive forces, which act on each point such as M of the fluid mass ABCD, may be all the other material points of this mass. In this case, X, Y, Z, the components of the total accelerating force acting on the point M, will consist of an infinite number of terms, and if we suppose that the natural law of action equal and contrary to reaction, obtains in their mutual attractions and repulsions, and that besides, all these points are subjected to the action of the same extraneous forces, this will not prevent them from being certain functions of $x$, $y$, $z$.

In the case of nature, these mutual actions are of two different kinds; 1° the one varies in the inverse ratio of the square of the distance, and 2° the intensities of the others are expressed by functions which decrease with extreme rapidity, so that their values are only then sensible, when the distances are insensible. X, Y, Z, the total components of the forces of the first species, are computed by distributing the mass ABCD into infinitely

small elements (No. 98), and then determining by the integral calculus, the sums of the attractions or repulsions of all these points, along each direction. With respect to the actions of the second species, which are properly termed *molecular* forces, and which are attractive or repulsive, according as the attraction of the ponderable matter is greater or less than the calorific repulsion, they ought not to be taken into account in the calculation of the forces X, Y, Z, relative to an interior point M; for it is precisely these molecular forces, which produce the pressure $p$, that is equal in every direction about M, and which was already considered in forming the equations of equilibrium.

It results from this last consideration, that equations (2) of No. 582 are the necessary and sufficient conditions of equilibrium of all the forces, the molecular actions among the rest, which act on $dm$ any element of the fluid mass; so that the equilibrium has certainly place, when there is such a value of $p$ as satisfies these equations for all the points of the fluid, which coincides with the value of the pressure at the free surface, that is given directly, and which should not become negative in any point, in order that the parts of the fluid may remain contiguous.

If the law of the molecular forces, in a function of the distance, was given, and if we could deduce from these forces the expression of the quantity $p$ in a function of the mean interval of the molecules (No. 98), then by substituting this expression in equations (2), one of them would determine the magnitude of this interval, about the point M, in the state of equilibrium, and the two others would express the conditions of this equilibrium. The numerical value of $p$ would then result from that of the mean interval, or from the corresponding value of the density; and in the memoir cited above (No. 576), the author has explained how this pressure $p$ may vary in a very high ratio, for the very small variations of the density which are observed in liquids. But as the direct determination of

the pressure $p$ is impossible, its value must be deduced from the conditions themselves of equilibrium, or from formula (3), which is a consequence of them.

When the point M is situated on the surface of the fluid, or is less distant from it than the radius of activity of the molecular forces, these forces, and the rapid variation of the superficial density, should be taken into account in the calculation of the components X, Y, Z, and, consequently, of the value of $p$ deduced from formula (3). There results from this, an influence of molecular forces on the figure of the fluid in equilibrio, which, in general, is not sensible, and which only becomes so in capillary spaces. In this treatise, these are not taken into account; and for every thing concerning the phenomena of capillary forces, the reader is referred to the new theory of capillary action lately published by the author.

589. If a homogeneous, or heterogeneous liquid turns uniformly about a fixed axis, we can by means of the preceding formulæ determine the conditions which are necessary and sufficient to be satisfied, in order that it may retain a permanent figure, and move like a solid body. For this purpose it will suffice to join to the components X, Y, Z, those of the centrifugal force which results from this rotation.

Let then the axis of rotation be that of the coordinates of $z$, $r$ the distance of any point M from this line, so that we may have

$$r^2 = x^2 + y^2,$$

let $a$ denote the constant angular velocity, which is common to all the points of the fluid, $ra$ will be the absolute velocity of the point M, and since it describes a circle whose radius is $r$, the value of the centrifugal force will be $ra^2$ (No. 174). As this force acts in the direction of the production of $r$, its components parallel to the axes of $x$ and $y$ will be obtained by multiplying it by $\frac{x}{r}$ and $\frac{y}{r}$; this gives, for their respective values,

$xa^2$ and $ya^2$, which should be added to the forces X and Y; and as the force Z is not changed, formula (3) will become

$$dp = \rho\,(\text{X}dx + \text{Y}dy + \text{Z}dz + a^2xdx + a^2ydy). \qquad \text{(a)}$$

The quantity comprised between the parentheses will be still an exact differential, namely, the differential of the function $\phi$ of No. 586, increased by $\frac{1}{2}a^2(x^2+y^2)$ or by $\frac{1}{2}a^2r^2$. Consequently, the permanent form will be possible; and if the free surface of the liquid experiences a constant pressure throughout its entire extent, the equation common to this surface, and to all the surfaces of level, will be

$$\text{X}dx + \text{Y}dy + \text{Z}dz + a^2(xdx + ydy) = 0. \qquad \text{(b)}$$

In the case of a homogeneous liquid, it will be sufficient to determine the free surface by the integral of this differential equation, and its arbitrary constant can be determined by knowing the entire volume of the liquid, as we shall see immediately by an example. In the case of an heterogeneous liquid, it is moreover necessary that it should be composed of homogeneous strata, whose figures may likewise be determined by the integral of this same equation, and which will only differ from the exterior figure in the values of the arbitrary constant.

590. Let equation (b) be applied to the case of a heavy homogeneous liquid, subject to the action of gravity; let it be supposed to turn about a vertical axis, and to be contained in an open vessel. If the gravity be denoted by $g$, and if the coordinates of the positive $z$s be estimated in the contrary direction to this force, we shall have

$$\text{X} = 0, \quad \text{Y} = 0, \quad \text{Z} = -g,$$

consequently, equation (b) will become

$$gdz = a^2(xdx + ydy);$$

hence, by integrating and denoting the arbitrary constant by $c$, there results

$$z = \frac{\alpha^2}{2g}(x^2 + y^2) + c,$$

from which it appears, that the free figure of the liquid is that of a paraboloid of revolution, whose axis will be that of the rotation.

In order to determine the constant $c$, let us suppose that the vessel is a vertical cylinder with a circular base, and let its axis of figure be that of the coordinate $z$, or of rotation; let its radius be $a$, and $h$ the height through which a body should fall in order to acquire the absolute velocity of the surface, namely $\alpha a$, so that we may have $\alpha^2 a^2 = 2gh$, and, consequently, $z = \frac{hr^2}{a^2} + c$; likewise, let $b$ be the height of the water before the commencement of the motion, $\pi a^2 b$ will be the volume of the liquid, which remains the same during the rotation; now, if the paraboloid be divided into infinitely slender cylindrical strata, whose common axis is that of the axis of $z$, then $2\pi r dr$ will be the base, and $2\pi z r dr$ the volume of the stratum of which the radius is $r$, and the thickness $dr$, therefore, the entire volume will be obtained by integrating $2\pi z r dr$ from $r = 0$ to $r = a$; hence it follows that

$$a^2 b = 2\int_0^a z r dr.$$

By substituting for $z$ its value, and performing the integration, we obtain for the value of $c$ ($h$),

$$c = b - \tfrac{1}{2}h.$$

The equation of the upper surface will therefore be

$$z = \frac{hr^2}{a^2} + b - \tfrac{1}{2}h.$$

The least and greatest values of $z$ which belong to $r = 0$, and $r = a$, will be $b - \frac{1}{2}h$, and $b + \frac{1}{2}h$, so that the depression of the liquid about the axis, and its elevation at the circumference, which are produced by the rotation, will be the same, and their

value will be half the height through which a body should fall to acquire the velocity of the circumference.

591. When the forces of which X, Y, Z are the components, arise from the attractions of all the points of the liquid, in the inverse ratio of the squares of the distances, or according to any other laws, the total values of X, Y, Z, depend, in general, on the form of the liquid and on its strata of level, and, conversely, this form depends on the values of these components. This mutual dependence of the attractions of the fluid and of its figure, renders the determination of the latter, by means of equation (b), extremely difficult. Even in the case, when the fluid is homogeneous, the problem cannot be resolved in the ordinary case of the attraction varying in the inverse ratio of the square of the distance, except on the supposition, that the centrifugal force is so inconsiderable, that the fluid differs very little from the spherical form, which it would assume if this force was cipher, that is to say, if the fluid was at rest. It might, in this case, be demonstrated by an analysis founded on the consideration of series, but which cannot be introduced here, that the figure of the fluid is necessarily that of an ellipsoid of revolution, whose compression may be determined by means of the magnitude of the centrifugal force at the equator, compared with the attraction of the fluid at this same point.

But it is easy to verify the above statement, namely, that the elliptic figure always satisfies equation (b), when the velocity $a$ does not pass a certain limit, and that then there are two ellipsoids of revolution, which correspond to the same value of this velocity of rotation. In fact, if the equation of the surface of the fluid, in its permanent state, is

$$\frac{z^2}{c^2} + \frac{x^2 + y^2}{c^2(1+\gamma^2)} = 1, \qquad \text{(c)}$$

which is that of an ellipsoid of revolution, whose axis of figure and equatorial diameter are respectively $2c$ and $2c\sqrt{1 + \gamma^2}$,

and if $\mathrm{X}$, $\mathrm{Y}$, $\mathrm{Z}$ are the components of the accelerating force arising from the total attraction of this body on the point of its surface whose coordinates are $x, y, z$, estimated along the productions of these coordinates, that is to say, in a direction opposite to that of the components, the expressions of which were given in No. 106; then, by changing the signs of these expressions, and observing that the mass of the ellipsoid is expressed by $\frac{4}{3}\pi\rho c^3(1+\gamma^2)$, we shall have(*i*)

$$\mathrm{X} = \frac{2\pi f\rho x}{\gamma^3}[\gamma - (1+\gamma^2)\,\mathrm{arc}\,(\mathrm{tang} = \gamma)],$$

$$\mathrm{Y} = \frac{2\pi f\rho y}{\gamma^3}[\gamma - (1+\gamma^2)\,\mathrm{arc}\,(\mathrm{tang} = \gamma)],$$

$$\mathrm{Z} = \frac{4\pi f\rho(1+\gamma^2)z}{\gamma^3}[\mathrm{arc}\,(\mathrm{tang} = \gamma) - \gamma];$$

in which $f$ expresses, as in the number just cited, the intensity of attraction at the unit of distance, and between masses respectively equal to unity. It will be, therefore, necessary to prove, that these values, joined with equation (c), satisfy equation (b). Now, if they be substituted in this equation, and if all its terms be multiplied by $\gamma^3$, which implies that $\gamma$ is not cipher, by making, for conciseness,

$$\frac{\alpha^2}{4\pi f\rho} = \varepsilon,$$

there results

$$[\tfrac{1}{2}\gamma - \tfrac{1}{2}(1+\gamma^2)\,\mathrm{arc}\,(\mathrm{tang} = \gamma) + \varepsilon\gamma^3]\,(xdx + ydy)$$
$$+ (1+\gamma^2)\,\mathrm{arc}\,(\mathrm{tang} = \gamma) - \gamma]\,zdz = 0;$$

by differentiating equation (c), we obtain

$$xdx + ydy + (1+\gamma^2)\,zdz = 0;$$

and, in order that this differential equation may coincide with the preceding, it is necessary, and it suffices, that we should have

$$\tfrac{1}{2}\gamma - \tfrac{1}{2}(1+\gamma^2)\,\mathrm{arc}\,(\mathrm{tang} = \gamma) + \varepsilon\gamma^3 = \mathrm{arc}\,(\mathrm{tang} = \gamma) - \gamma,$$

or by reducing($k$)

$$\frac{3\gamma + 2\varepsilon\gamma^3}{3 + \gamma^2} - \text{arc}\,(\text{tang} = \gamma) = 0; \qquad \text{(d)}$$

so that it only remains to ascertain whether this equation has real roots, and to determine their number.

For this purpose, let its first member be represented by $\beta$, and let a curve be supposed to be traced, of which the corresponding values of $\gamma$ and $\beta$ are the abscissa and ordinate; this curve will cut the axis of the abscissa at the origin; however, the root $\gamma = 0$, does not belong to the question, as long as the velocity $a$, and consequently $\varepsilon$, is not cipher. The other roots of equation (d) are equal two by two, and of contrary signs; but it will be sufficient to consider its positive roots, since equation (c) only contains the square of $\gamma$. This being so, if the differential of $\beta$ be put equal to cipher, we obtain($l$)

$$\varepsilon\gamma^4 + 2\,(5\varepsilon - 1)\,\gamma^2 + 9\varepsilon = 0, \qquad \text{(e)}$$

by means of which the abscissæ corresponding to the *maxima*, or *minima*, of this ordinate, may be determined. Now, as this equation is of the second degree with respect to $\gamma^2$, it follows that there can be only one *maximum*, or *minimum*, on each side of the origin of the abscissæ, hence it is evident, that the curve can only cut the axis of the positive abscissæ, *beyond* this origin, in two points, so that, at most, there will be only two positive and real roots of equation (d). Moreover, we may observe, that if equations (d) and (e) have place for the same value of $\gamma$, the curve will($m$) touch the axis of the abscissæ in a point which corresponds to a double root of equation (d). Now, from equation (e) we can deduce

$$\varepsilon = \frac{\gamma^2}{(1 + \gamma^2)(9 + \gamma^2)};$$

and if this value be substituted in equation (d), there results($n$)

$$\frac{7\gamma^5 + 30\gamma^3 + 27\gamma}{(1 + \gamma^2)(3 + \gamma^2)(9 + \gamma^2)} = \text{arc}\,(\text{tang} = \gamma);$$

an equation which can have only one positive root, besides $\gamma = 0$. This root really exists, and its approximate value is found by trial to be

$$\gamma = 2{,}5293.$$

The corresponding value of $\varepsilon$ is

$$\varepsilon = 0{,}1123;$$

from which it follows that for values of $\varepsilon$ less than this fraction, there are two distinct intersections of the axes of the positive abscissæ, and two unequal roots of equation (d); that for this value of $\varepsilon$, these intersections coalesce into a contact, and the two roots become equal; and that finally, for greater values of $\varepsilon$, equation (d) has no real roots, and the intersections have not place. It is certain that these roots correspond to the less values of $\varepsilon$, and not to the greater; for when $\varepsilon = \infty$, equation (d) has no value different from cipher; and on the contrary, when $\varepsilon$ is a very small fraction, the two real roots of this equation may be easily determined($o$).

When by means of equation (d), the two approximate values, which answer to a given value of $\varepsilon$ less than the preceding fraction, shall have been determined, the preceding values of X, Y, Z will make known the attraction of the fluid in any point whatever of its interior or of its surface, and the values of $c$ can be deduced from the volume of the fluid, which is also given. When the value of $\varepsilon$ surpasses this fraction, we are not justified in concluding that a permanent figure of the liquid is impossible, but only that it cannot be an ellipsoid of revolution; for, with the exception of the case in which this figure is supposed to differ little from that of a sphere, it has not been yet demonstrated that the elliptic figure of revolution is the only one which suits the equilibrium of the centrifugal forces and of the mutual actions of the molecules; it has not been even proved that the sphere is the only figure which a fluid mass at rest can assume, when its molecules

mutually attract each other, however natural it may appear to be.

592. If the quantity $\varepsilon$ is a very small fraction, equation (d) may be satisfied by making $\gamma$ a very small quantity. We then have

$$\operatorname{arc}(\operatorname{tang} = \gamma) = \gamma - \tfrac{1}{3}\gamma^3 + \&c.;$$

$$\frac{1}{3+\gamma^2} = \tfrac{1}{3} - \frac{\gamma^2}{9} + \&c.;$$

and if these values be substituted in equation (d), we obtain by suppressing the factor $\gamma$, which is common to all the terms, and then neglecting powers of $\gamma$ superior to the first($p$),

$$\gamma^2 = \frac{15\varepsilon}{2};$$

which corresponds to an ellipsoid very little compressed. As the two semiaxes are $c$ and $c\sqrt{1+\gamma^2}$, the compression must be very nearly equal to $\frac{1}{2}\gamma^2$; $a^2c$ may be taken to express the centrifugal force at the equator, and $\frac{4}{3}\pi f\rho c$ the attraction in a point of the surface; which would be the exact values of these two forces, if the body was exactly spherical. The ratio of the first to the second is $3\varepsilon$; consequently when a homogeneous fluid turns about a fixed axis, and differs very little from the spherical figure, its compression is equal to five times the centrifugal force at the equator divided by four times the attraction at the surface($q$). It may be also demonstrated that if the fluid is composed of strata that are very little compressed, whose densities decrease from the centre to the surface, the compression will be always less than in the case of homogeneity, but still greater than the two-fifths of that which answers to this case($r$).

In the motion of rotation of the earth, the ratio of the centrifugal force to the gravity, or to the terrestrial attraction, is about $\frac{1}{288}$ (No. 177) at the equator. Therefore if the earth was a homogeneous fluid mass, its compression would be $\frac{1}{232}$,

the two-fifths of which is $\frac{1}{580}$; $\frac{1}{290}$ the value which results from observation, is comprised between these two limits, as in the case of a fluid mass, whose density decreases from the centre to the surface.

By making

$$z = c, \quad \sqrt{x^2 + y^2} = c\sqrt{1+\gamma^2},$$

in the values of z and $\sqrt{\mathrm{X}^2 + \mathrm{Y}^2}$, and then developing according to the powers of $\gamma$, we find (*s*)

$$\mathrm{Z} = -\frac{4\pi f\rho c}{3}\left(1 + \frac{4\gamma^2}{10} + \&c.\right),$$

$$\sqrt{\mathrm{X}^2 + \mathrm{Y}^2} = -\frac{4\pi f\rho c}{3}\left(1 + \frac{3\gamma^2}{10} + \&c.\right),$$

for the attractions which have place at the poles and at the equator. By adding to the second the centrifugal force $a^2c$, whose value is, by what precedes, $4f\pi\rho c\varepsilon$, or the product of $\frac{4}{3}\pi f\rho c$ and $\frac{4\gamma^2}{10}$, there results

$$\sqrt{\mathrm{X}^2 + \mathrm{Y}^2} + a^2c = -\frac{4\pi f\rho c}{3}\left(1 - \frac{\gamma^2}{10} + \&c.\right),$$

for the weight at the equator. If Z the weight at the pole be taken from this expression, and if it be then divided by Z, we obtain (*t*)

$$\frac{\sqrt{\mathrm{X}^2 + \mathrm{Y}^2} + a^2c - \mathrm{Z}}{\mathrm{Z}} = \tfrac{1}{2}\gamma^2 - \&c.;$$

so that if the square of $\gamma^2$ be neglected, this ratio is equal to the compression $\frac{1}{2}\gamma^2$, and consequently, the sum of these two quantities is equal to five times the centrifugal force divided by twice the weight at the equator, conformably to the theorem cited in No. 193(*u*).

In this same case of a very small value of $\varepsilon$, equation (d) may be also satisfied by means of a very great value of $\gamma$. For such a value of $\gamma$, we have the identical equation

$$\text{arc}(\text{tang} = \gamma) = \tfrac{1}{2}\pi - \text{arc}\left(\text{tang} = \frac{1}{\gamma}\right);$$

therefore we shall have in a converging series

$$\text{arc}(\text{tang} = \gamma) = \tfrac{1}{2}\pi - \frac{1}{\gamma} + \tfrac{1}{3}\frac{1}{\gamma^3} - \tfrac{1}{5}\frac{1}{\gamma^5} + \&\text{c.};$$

in like manner we have

$$\frac{1}{\gamma^2+3} = \frac{1}{\gamma^2} - \frac{3}{\gamma^4} + \&\text{c.};$$

and if these developments be substituted in equation (d), a value of $\gamma$ may then be obtained, arranged according to the increasing powers of $\varepsilon$, namely,

$$\gamma = \frac{\pi}{4\varepsilon} - \frac{8}{\pi} + \&\text{c.},$$

which will be the second real root of this equation($v$).

For more details on this important theory, and on its application to the figure of the earth, the reader is referred to the second and fifth volumes of the *Mechanique Céleste*.

593. There is an essential difference between the surfaces of level traced in the interior of a liquid, subject to the mutual action of all its points, and those described in a fluid the points of which are only solicited by extraneous forces, that is to say, by attractions or repulsions, which emanate from fixed centres, and are functions of the distances from these centres. Let ABCD (fig. 37) be the free surface of a liquid at rest, or for greater generality, turning about a fixed axis. Let EFGH be a surface of level traced in its interior, and let R be the resultant of all the forces which act on M any point whatever of this surface. In the two cases adverted to above, this force will act in the direction of NMP the normal at this point; now, since, in the second case, its magnitude and direction do not at all depend on any action of the points of the fluid, it will continue to be perpendicular to the surface EFGH, though the stratum of liquid comprised between EFGH and ABCD should be taken

away, so that after this abstraction, the liquid terminated by EFGH will still remain in equilibrio; but in the case of the mutual actions of the points of the system, the force R will depend on the action of this interior liquid, and on that of the exterior stratum; in general, its magnitude and direction will be changed, when the stratum comprised between ABCD and EFGH is suppressed, and the equilibrium of the fluid terminated by EFGH will no longer have place. In order that it should be reestablished, the form of the surface EFGH should be changed, so as to become perpendicular in each point to that part of the force R which remains.

The action of the exterior stratum comprised between ABCD and EFGH, will be nothing on all the points of the interior fluid and of the surface EFGH, when the entire mass of the fluid being homogeneous, it deviates very little from the spherical figure, and its points are only solicited by their mutual attractions in the inverse ratio of the square of the distances, and by the centrifugal force. In fact, all the surfaces of level are then *similar* ellipsoids, and, consequently, the stratum comprised between ABCD and EFGH, two of these surfaces, does not exercise any action on the points situated in the interior space (No. 105). But this nullity of action of a stratum terminated by two surfaces of level on the interior fluid, is *not a condition* of the equilibrium of fluids; for if the forces be such as have been supposed above, it has no longer place, for example, when the liquid is heterogeneous; this renders the surfaces of level dissimilar, though they are still elliptical, and such that the ellipticity of any surface whatever as EFGH depends on the thickness and constitution of the exterior stratum. See *Mechanique Celeste*, tom ii. p. 85, and following pages.

In the case of homogeneity, the elliptic stratum comprised between ABCD and EFGH, may be taken away or replaced at pleasure, without deranging the equilibrium or changing the form of the interior fluid, provided that the velocity of rotation remains always the same. But there are also other strata

which may be added to the fluid terminated by EFGH, without deranging its equilibrium, although their attraction on the points of this liquid may not be cipher. It is evident, that the exterior surface of the additive stratum may be an ellipsoid similar to EFGH, and having its centre in a point of the axis of rotation different from the point O. The exterior surface of this stratum may also have its centre in this point, and be an ellipsoid whose compression is different from that of the interior surface. In order to make this appear, let ABCD and A′B′C′D′ (fig. 38) be the two different ellipsoids which satisfy the condition of equilibrium of the same homogeneous fluid, turning about a fixed axis with a given angular velocity (No. 591); likewise let EFGH and E′F′G′H′ be two surfaces of level traced in the interior of these ellipsoids, respectively similar to the exterior surfaces, having the same centre O as these, and intersecting at the point M; we can, without deranging the equilibrium of the liquid terminated by EFGH, add to it the stratum comprised between the two dissimilar and concentrical surfaces EFGH and A′B′C′D′. It should be observed, that not only the action of this additive stratum on the points of the interior liquid, and of the surface EFGH, is not cipher, but that this action on each point of the surface is not even directed along the normal. Thus, at the point M, the action of the stratum comprised between the surfaces EFGH and A′B′C′D′ is not directed along NMP the normal to the first surface; for the action of the interior liquid terminated by this surface, is already directed along NMP, and if the action of the additive stratum had also this direction, the action of the entire mass, terminated by the surface A′B′C′D′, would be still directed along this normal, while we know it should be directed along N′M′P′, the normal to E′F′G′H′, the other surface of level.

Whatever may be the nature of the forces which act on a heterogeneous or homogeneous fluid mass turning about a fixed axis, we should not forget that the sole condition of equilibrium is the existence of a quantity $p$ which satisfies

equation (a), and which must be cipher or constant at the free surface of the liquid. All other conditions which we may wish to add to this, are already comprised in it, or if not, the equilibrium cannot have place.

594. Among the different laws of attraction, there is one which is not that of nature, but which possesses some remarkable properties. This law is that of a mutual action in the direct ratio of the distance, and one of the properties to which we refer is, that the resultant of the actions of all the points of a body on any point whatever, is independent of the form and constitution of this body, whether homogeneous or heterogeneous, and is the same as if the entire mass was condensed into its centre of gravity ($x$).

In fact, if $x$, $y$, $z$ be the coordinates of the attracted point, $x'$, $y'$, $z'$, those of an attracting point, $\mu$ the mass of this second material point, $u$ the distance of the two points, $k\mu u$ the accelerating force directed from the first point towards the second; $k$ being a constant coefficient, the components of this force in the direction of parallels to the axes of the coordinates, drawn through the attracted point, will be $k\mu(x'-x)$, $k\mu(y'-y)$ $k\mu(z'-z)$, for the cosines of the angles which its direction makes with these lines are the differences $x'-x$, $y'-y$, $z'-z$, divided by $u$. Consequently, if the total components of the accelerating force of the attracted point be X, Y, Z, we shall have

$$\text{X} = k\Sigma\mu x' - kx\Sigma\mu,$$
$$\text{Y} = k\Sigma\mu y' - ky\Sigma\mu,$$
$$\text{Z} = k\Sigma\mu z' - kz\Sigma\mu;$$

in which the sums $\Sigma$ extend to all the points of the attracting body. Now, if the entire mass of this body be denoted by $m$, and the three coordinates of its centre of gravity by $x_1$, $y_1$, $z_1$, we shall have

$$\Sigma\mu = m,$$
$$\Sigma\mu x' = mx_1,$$
$$\Sigma\mu y' = my_1,$$
$$\Sigma\mu z' = mz_1;$$

consequently, there will result,

$$X = km(x_1 - x),$$
$$Y = km(y_1 - y),$$
$$Z = km(z_1 - z);$$

equations which evidently contain the proposition to be proved.

If these values of X, Y, Z be substituted in equation (b), and if we make, for conciseness, $\frac{a^2}{km} = \varepsilon$, there results

$$(x_1 - x)\,dx + (y_1 - y)dy + (z_1 - z)dz + \varepsilon(xdx + ydy) = 0;$$

hence, by integrating and denoting the arbitrary constant by $c$, we obtain

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 - \varepsilon(x^2 + y^2) = c.$$

This equation will be that of the surfaces of level in a liquid turning about the axis of $z$, and whose points are attracted in the direct ratio of the distance; it shows that all these surfaces are concentrical, and of the second degree. Moreover, if the origin of the coordinates be transferred to their common centre, that is to say, to the centre of gravity of the fluid, the first powers of the corresponding coordinates must disappear, which cannot be the case unless $x_1 = 0$, $y_1 = 0$, $z_1 = 0(y)$. The preceding equation will be therefore reduced to

$$z^2 + (1 - \varepsilon)(x^2 + y^2) = c; \qquad \text{(f)}$$

consequently, the surfaces of level are ellipsoids, or hyperboloids of revolution, according as $\varepsilon \angle 1$, or $\varepsilon > 1$; and, in each case, they have the same axis of figure, which is the axis of rotation. When the volume of the liquid is given, the hyperboloid is not possible, except when the fluid is contained in a vessel, and then equation (f) is solely applicable to the free part of its surface. Therefore, when $\varepsilon > 1$, the permanent figure of a liquid which is free on all sides is impossible, in the case of such forces as we have just considered. If

$\varepsilon \angle 1$, all the surfaces of level are ellipsoids, which differ from each other in the values of $c$. In order to determine the value of this quantity, which corresponds to the exterior surface, the volume of the ellipsoid, the expression of which is $\frac{4\pi c\sqrt{c}}{3(1-\varepsilon)}$ should be put equal to the given volume of the liquid. It is remarkable, that in this example, the law of the densities of the strata has no influence on its exterior figure, or on that of its strata of level($z$).

# CHAPTER III.

## OF THE EQUILIBRIUM OF FLUIDS ACTED ON BY GRAVITY.

595. If in a vessel ABCD open at its upper surface, and of which the horizontal base AB rests on a fixed plane, a heavy homogeneous liquid be poured to the height A′B′, then, in order that this liquid should be in equilibrio, it is necessary that the free surface A′B′ should be horizontal or perpendicular to the direction of gravity, and this equilibrium will not be deranged if a constant pressure of any magnitude whatever be made on this surface.

The pressure referred to the unit of surface, will be the same throughout the entire extent of each horizontal section of the liquid. If it be denoted by $p$ at the depth $z$ below A′B′, and if the constant density of the liquid be denoted by $\rho$, and the gravity by $g$, then by equation (3) of No. 583, we shall have $dp = \rho g dz$. Hence, by integrating, we shall obtain

$$p = \rho g z + \Pi;$$

$\Pi$ being the exterior pressure, which will be generally the atmospherical pressure corresponding to $z = 0$. This constant pressure will be transmitted, without alteration, on all the elements of the sides of the vessel, and of the bodies plunged in the liquid; and it should be added, in each point, to the variable pressure due to the gravity of the liquid. It will be therefore always easy to take it into account; and, for greater simplicity, we can suppose it equal to cipher, and thus reduce the preceding equation to

$$p = \rho g z.$$

Let $b$ denote the area of the base AB of the vessel, P the entire pressure exerted on this base, $h$ the distance of A′B′ from this same base, or the height of the liquid, then we shall have at the same time

$$z = h, \quad P = bp = \rho gbh;$$

from which it appears, that the pressure exerted on the horizontal base of a vessel, is equal to the weight of a cylinder filled with the liquid, whose base is equal to that of the vessel, and height that of the liquid, so that the mass and volume will be consequently $\rho bh$ and $bh$.

This pressure P is therefore independent of the form of the vessel; so that if three vessels represented by fig. 40, having equal bases, and placed on the same horizontal plane, be filled with the same liquid to an equal height, the pressures exerted on their bases are equal, although one of the vessels may be a right cylinder, another a vessel of the form of a truncated cone resting on its lesser base; another a similar cone resting on its greater base. The common pressure on each of the three bases, is the weight of the liquid contained in the cylindrical vessel; this remarkable result is fully confirmed by experiment.

In the case of several liquids placed one over the other in a vessel, it will be sufficient and necessary, in order to insure their equilibrium, that the surface of separation of two consecutive liquids should be horizontal (No. 586); and in fact, if this is the case, each new liquid will exert on all the points of *its* base a constant pressure, which will not derange the equilibrium of the inferior liquid. The total pressure exerted on the bottom of the vessel may be determined in the following manner.

596. Let a new liquid, the density of which is $\rho'$, be poured on the fluid in equilibrio in the vessel ABCD (fig. 39), and then if its superior surface A″ B″ be horizontal, as that of the first, these two fluids will be in equilibrio in the

vessel, and if $h'$ be height of A″B″ above the level of A′B′, and $b'$ the area of A′B′, which is the base of the second fluid, it will exert on this base a pressure equal to $\rho' g h' b'$. This pressure will be transmitted, through the intervention of the inferior liquid, on AB the bottom of the vessel, and the area of AB being denoted by $b$, there results from it a pressure on this plane surface, expressed by $\rho' g h' b$ (No. 577); consequently, the entire pressure exerted by the two fluids on the horizontal base of the vessel, will be $\rho g h b + \rho' g h' b$.

If a third liquid, whose surface A‴B‴ is also horizontal, be poured into the vessel, the equilibrium will not be deranged; $\rho''$ being its density, $h''$ the height of its level A‴B‴ above A″B″, the upper surface of the second liquid, and $b''$ the area of this last surface, this third liquid will exert on its base A″B″ a pressure equal to $\rho'' g h'' b''$, which will be transmitted by the second liquid, and become $\rho'' g h'' b'$, on A′B′, or $b'$, the upper surface of the inferior liquid; this pressure will be transmitted in like manner, by the intervention of the inferior liquid, and become $\rho' g h'' b$ on the bottom of the vessel; consequently, the three superimposed liquids will exert on the bottom of the vessel, a pressure equal to $\rho g h b + \rho' g h' b + \rho'' g h'' b$, or to $(\rho h + \rho' h' + \rho'' h'') g b$.

By continuing in this manner, it is evident that when any number whatever of liquids of different densities are superimposed and in equilibrio in the same vessel, the pressure which they exert on the horizontal base of this vessel, depends only on the extent of this base, on the thicknesses of the different fluids, and on their densities. In the case of a cylindrical and vertical vessel, it will be equal to the sum of the weights of all the fluids; and it will not change with the form of the vessel, provided that neither the extent of its base, nor the thickness or density of each liquid undergoes any change.

As this result is independent of the thickness of the horizontal strata, it subsists even when these strata are infinitely † slender, that is to say, when the density of the fluid mass

varies by continuous degrees in the vertical direction, and consequently corresponds to the case of compressible fluids. It is equally true when the gravity varies from one stratum to another with the density; this is the case when the height of the liquid cannot be neglected relatively to the radius of the earth. This also may be inferred from the equation $dp = \rho g dz$, which obtains in the equilibrium of all fluids, whether compressible or not, and in which the gravity $g$, and the density $\rho$, may be supposed to be functions of the vertical ordinate $z$.

597. Let us now consider the equilibrium of liquids subject to the action of gravity contained in several vessels, and which may flow through lateral openings from the one to the other. If all these openings are closed at once, the equilibrium will not be deranged; it is therefore, in the first place, necessary that the liquids should be arranged in each vessel, in horizontal strata; but this condition is not sufficient, and it is also necessary that when the openings are not closed, a certain relation should subsist between the elevations of the liquids in the different vessels, which will depend on the ratio of their densities.

If the same liquid is contained in the different communicating vessels, the level of this liquid must be the same in all these vessels. For if a homogeneous liquid is contained in two vessels, for example, which communicate laterally by the canal EF (fig. 41), and which are placed on fixed horizontal planes; and if this liquid rises to AB in one of the vessels, and to CD in the other, if these two horizontal sections do not exist in the same plane, so that when the plane of CD, the section of one of the vessels, is produced, it may intersect the other vessel in a section $\alpha\beta$, situated lower than AB, by the distance $\delta$, then if in this state the equilibrium subsists, it will not be deranged by substituting for the open section CD, a fixed plane; the fluid contained between AB and $\alpha\beta$ will exert on $\alpha\beta$ a pressure equal to $\rho g \gamma \delta$, $\rho$ denoting its density, and $\gamma$ the area of this section $\alpha\beta$; this pressure will be

transmitted by the intervention of the liquid contained in the two vessels to the plane CD; and there will result from it, a pressure on this horizontal plane directed upwards, and expressed by $\rho g\gamma c$, $c$ denoting the area of CD. Consequently, if the plane CD be removed, the equilibrium will not have place, unless $\delta$ the difference of level of the liquid in the two vessels be cipher; which was proposed to be demonstrated.

The two sections AB and CD being comprised in the same plane, if a liquid be poured on AB to a height A′B′, and if its density be denoted by $\rho'$, it will exert on AB a pressure equal to $\rho' gbh$; $b$ being the area of AB, and $h$ the distance comprised between the horizontal sections AB and A′B′. This pressure will be transmitted on CD, where it will be equal to $\rho' gch$, and it will act from below upwards; in order to destroy its effect, the vessel should be closed by a fixed plane at CD, or a fluid should be poured on CD, the pressure of which on CD should be equal and contrary to $\rho' gch$. In this last case, if the fluid rises to C′D′, and if its density be denoted by $\rho_{,}$, and the distance comprised between CD and C′D′ by $k$, the pressure exerted by this fluid on CD, will be $\rho_{,} gkc$; and so that in order to an equilibrium, we should have $\rho_{,} k = \rho' h$.

Therefore, it appears, that when different liquids contained in communicating vessels, are in equilibrio, their densities should be in the inverse ratio of their heights above the sections of these vessels made by the same horizontal plane. If new liquids be poured over those which have been now considered, it will appear in the same manner, that denoting the thicknesses of the liquids in one of these vessels by $h, h', h''$, &c., and their densities by $\rho', \rho'', \rho'''$, &c., and representing the corresponding quantities in the other vessel, by $k, k_{,}, k_{,,}$, &c., $\rho_{,}, \rho_{,,}, \rho_{,,,}$, &c., the equation

$$\rho' h + \rho'' h' + \rho''' h'' + \&c. = \rho_{,} k + \rho_{,,} k_{,} + \rho_{,,,} k_{,,} + \&c.,$$

must obtain; hence it results, that the pressures referred to the unit of surface, will be equal on the two upper sur-

faces AB and CD, of the homogeneous liquid which communicates between the vessels, and it may be either that whose density is $\rho'$, or that of which the density is $\rho_{,}$, or more generally, any other liquid whatsoever, provided that the two surfaces AB and CD which terminate it, exist in the same horizontal plane.

It may observed, that infinitely thin strata comprised in different vessels, and contained between the same horizontal planes, will experience the same pressure referred to the unit of surface; but, above the plane which terminates the inferior liquid, they may contain *different* liquids; so that the properties of strata of level, or those which are perpendicular to the direction of gravity, have place *as to equality of pressures, but not as to the homogeneity of the liquid* (No. 586), when these strata are separated by fixed planes.

598. The laws of the equilibrium of heavy fluids in communicating vessels, are susceptible of a great number of applications, we shall restrict ourselves to the consideration of those that are most common.

That which first presents itself, and which we shall merely state, is the theory of levellings, and of instruments that have been termed *levels*.

In the *syphon*, the two branches of which are open at their upper part, and which contains water, or any other liquid, the equilibrium has place when the two extremities of the liquid are comprised in the same plane, whatever the magnitude of the atmospherical pressure at these two points may be. The equilibrium may also exist in the inverted siphon, provided that then the pressure of the atmosphere has a suitable magnitude.

Let ABC (fig. 42) be this inverted tube, B its highest point, E and F the points at which the liquid is sustained in these two branches, and which are situated in the same horizontal plane. If $\rho$ denote the density of the liquid, and $h$ the height of the point B above this plane, the pressure on the unit of

surface, exercised by the liquid on these points E and F will be equal to $\rho gh$; and if $\Pi$ be the pressure with which the atmospheres urges each of these points upwards, we should have $\Pi > \rho gh$, or at least $\Pi = \rho gh$, in order that this pressure may prevent the liquid from flowing out. In the second case, the pressure on the point B will be cipher; in the first, it will be equal to $\Pi - \rho gh$; if $\Pi \angle \rho gh$, the pressure at the point B will be negative, the liquid will be separated at this point, and will flow out through the two branches of the syphon. Moreover, in the inverted syphon, the equilibrium of the liquid is but instantaneous, and can only obtain in consequence of the adherence of its molecules among one another, or to the interior of the tube; from the instant that its extremity F is ever so little below or above its extremity E, the excess of the atmospherical pressure above that of the liquid is greater or less at the point E than at the point F, and the liquid flows through the branch BC or the branch BA of the syphon. In the common application of this inverted tube, the shortest branch BA is plunged in a vessel H, containing the liquid that rises to the point D of the tube, then a vacuum is made by exhausting the air contained in this tube, the liquid rises above its primitive level until it reaches B the summit of the tube, and it then descends to the point C, and thus flows out at this extremity of the tube. The flowing out of the liquid stops when the point D is so depressed in the branch BA that it becomes lower than the point C; this, however, never can be the case, since AB is supposed to be the shorter of the two branches of the tube ($a$).

The hydraulic press, the invention of which is attributed to Pascal, consists of a prismatic vertical vessel H (fig. 43), open at its upper part, and filled with water to AB. A lid cover placed on AB, while it fits the vessel accurately, can slide along its sides; and below AB there is an opening C, to which a bent tube CDE is fitted, whose vertical branch DE is open at its upper part E. The water in the vessel flows through

the orifice C, and, on account of the weight of the cover placed on AB, this liquid rises in the tube DE to a point F, situated above the production of this horizontal section AB. This being so, if an additional weight X be added to the weight of the cover, the liquid will descend to A′B′ in the vessel H, and rise to F′ in the tube DE. By this addition of X, the pressure on the unit of surface will be greater on A′B′ than on AB, by the quantity $\frac{X}{b}$, $b$ denoting the area of the horizontal section of H; at the same time, the pressure on the unit of surface and due to weight of water contained in the vertical tube DE, will be increased by the quantity $\rho g x$, $\rho$ denoting the density of the liquid, and $x$ the elevation of the point F′ above the point F. Therefore, in order that the equilibrium may subsist, we must have

$$X = \rho g b x;$$

an equation, by means of which, when $x$ is determined, the weight X will be known. Care should be taken to make the surface $b$ very considerable, in which case small elevations of water in the vertical tube may correspond to very great loads on the moveable cover, and thus enable us to measure them. The horizontal section of the tube is very small with respect to $b$, consequently, the depressions of the cover in the chest H are very small relatively to the elevations of the liquid in the tube; for if $y$ be the distance comprised between AB and A′B′, and $c$ the horizontal section of the tube, we shall have $by = cx$, because the entire volume of the water must be invariable. In fine, it is not necessary that the tube DE should be either vertical or cylindrical; and by the preceding formula, the value of X can be always obtained, provided that $x$ is the distance comprised between the two levels of the liquid in F and F′.

A *barometer* is, in general, a tube ABC, (fig. 44), whose branches BA and BC are vertical, and it is closed at A the extremity of BA, and open at C the extremity of BC. A per-

fect vacuum is made in this tube, and then mercury is poured into it, which rises to D in the branch AB, and to a less height E, in the open branch CB. If through the point E a horizontal plane be drawn intersecting the branch AB in F, the mercury situated below this plane will be in equilibrio of itself; and in order that this state may continue, the pressures on the unit of surface which are exerted at F by the mercury FD, and at E by the atmosphere, must be equal to each other. Hence, if $\Pi$ denote the pressure of the atmosphere, $m$ the density of the mercury, and $h$ the vertical height of the point D above the point F, that is to say, the difference of the levels D and F of the fluid in the two branches of the barometer; $mgh$ will be the value of the pressure of the mercury at the point F, and consequently, we shall have

$$mgh = \Pi.$$

If the branch BC be supposed to be prolonged vertically to the extremity of the atmosphere, the equilibrium will not be deranged; hence then the atmospherical pressure which is in equilibrio with that of the mercury, is in fact the weight of the air contained in a vertical cylinder, extending indefinitely into the atmosphere, and having for its base the unit of surface: it depends on the decrease of the gravity according as we ascend above the surface of the earth, on the density and temperature of the strata of the air, and on the quantities of aqueous vapours which they may contain. As this weight varies in the same part of the earth, $h$ the height of the barometer varies also, its magnitude changes also in consequence of the action of the parts of the winds whose direction is vertical, and which render the pressure of the atmosphere greater or less than it would be, if the air was in a state of repose; at Paris the mean value of $h$ is $0^m,76$. If any other liquid be substituted in the barometer in place of mercury, the height $h$ would change in the inverse ratio of the density of this liquid, compared with that of mercury, it being always supposed that there is a perfect vacuum above the

level D, in the closed branch of the barometer. In the case of water, this elevation $h$ is about $10^m, 4$, and it is also the greatest height to which water can be raised in a pump above its exterior level. When there is a stratum of air between the surface of the liquid and the piston, this air is dilated, and it exerts a less pressure than that of the atmosphere on the interior liquid; it therefore renders the ascent of the water less than it would be if there was a vacuum below the piston; the actual diminution will be determined in the next chapter.

599. We now proceed to calculate the pressures exerted by heavy liquids on the inclined or curved sides of the vessels which contain them, and on the surfaces of the solid bodies that are plunged into them.

The pressure which a homogeneous liquid exerts on the side of a vessel that is inclined to the horizon, is equal to the weight of a prism of this liquid, whose base is this side, and whose height is the distance of its centre of gravity from the level of the liquid. In fact, if $\omega$ be an element of this side, and $z$ its distance from the level of the liquid, the pressure on this element will be $p\omega$, or, by substituting for $p$ its value $\rho g z$, given in No. 595, $\rho g z \omega$; and as the pressures on all the elements are perpendicular to the plane side, the value of the resultant of these parallel forces will be the product of $\rho g$ and of the integral of $z\omega$, extended to the entire side; now, it is evident, that if $b$ be the area of this side, and $z_1$ the distance of its centre of gravity from the level of the liquid, this integral is equal to $z_1 b$, consequently, the pressure on the inclined plane will be $\rho g b z_1$, agreeably to the statement of the theorem given above.

In the case of several liquids superimposed in the vessel, the pressures exerted or transmitted by each of these liquids, on the inclined side, should be determined separately, and then the total pressure sustained by this surface will be equal to their sum. In consequence of the pressure of the atmosphere

represented above by $\Pi$, this total pressure will be increased by a quantity equal to $b\Pi$.

As all the points of the horizontal base of the vessel experience equal pressures, the resultant of these parallel forces passes through the centre of gravity of this base; but in the case of an inclined side, the inferior elements experience a greater pressure than those which are nearer to the surface; and the point where the total resultant of all these pressures meets the side, and which may be denominated *the centre of pressure*, will be always lower than the centre of gravity of this same surface. When a plane surface plunged in a homogeneous liquid turns about its centre of gravity, the magnitude of the pressure which it experiences will not be changed, but the point of application of this constant normal force will change its position on this surface($c$).

600. For an example of the determination of the centre of pressure, let the plane surface be a trapezium ABCD (fig. 45), whose two bases AB and CD are horizontal. If this surface be divided into elements parallel to these bases, and of an infinitely small height, each of these elements will experience the same pressure throughout its entire length, and its centre of pressure will be at its middle point; now if AB and CD the sides of the trapezium be produced until they meet in the point K, and if then KH be drawn from this point to H the point of bisection of AB, this line will bisect CD in G, and all the elements of the trapezium, therefore, the required centre of pressure will be in this line, and it is only necessary to determine its distance from AB. Let $x'$ be this distance, $x$ that of any element whatever from the same base AB, $u$ the length MN of this element, $z$ its distance from the level of the fluid, $h$ the height of the trapezium, $udx$ and $\rho gzudx$ will respectively denote the area of this element, and the pressure which it experiences; the value of the total pressure will be $\int_0^h \rho gzudx$; and by the theory of the moments of parallel forces, we shall have

$$x' \int_0^h \rho gzudx = \int_0^h x\rho gzudx.$$

In like manner, if $c$ denote the distance of AB from the level of the liquid, $\alpha$ the angle comprised between the vertical plane drawn from this line to the level of the liquid, and the production of the plane of the trapezium, we shall have

$$z = c + x \cos \alpha;$$

and if this value of $z$ be substituted in the preceding equation, there will result, by suppressing the constant factor $g\rho$, which is common to its two members,

$$x' \left( c \int_0^h udx + \cos \alpha . \int_0^h xudx \right)$$
$$= c \int_0^h xudx + \cos \alpha . \int_0^h x^2udx.$$

Let the lengths of the two bases AB and CD be denoted by $a$ and $b$, and the perpendicular from the point K on CD by $k$, then the perpendicular from the same point on AB or $a$, will be $k+h$, and on MN or $u$, it will be $k + h - x$; and as these lines $a, b, u$ are parallel, we shall have

$$u : b :: k + h - x : k,$$
$$a : b :: k + h : k.$$

If the value of $k$ be deduced from the second proportion, and substituted in the first, there results ($e$)

$$k = \frac{bh}{a-b}, \quad u = \frac{ah - (a-b)x}{h}.$$

By substituting this value of $u$ in the preceding equation, and then integrating, we obtain ($f$)

$$x' = \frac{2hc(a+2b) + h^2(a+3b)\cos \alpha}{6c(a+b) + 2h(a+2b)\cos \alpha}.$$

Consequently, if EF be drawn at this distance $x'$, parallel to AB, the point P where it cuts the line GH drawn from the

middle of AB to that of CD, will be the centre of pressure of the trapezium.

In figure 45, AB the upper base is supposed to be the greater, but if the contrary was the case, it is evident that it would be only necessary to substitute for the angle $a$, its supplement, which is the same thing as if the sign of cos $a$ was changed in the formula. When $a = 90°$, the surface is horizontal, and the formula becomes

$$x' = \frac{h(a+2b)}{3(a+b)};$$

which in fact coincides with the distance of the centre of gravity of the trapezium from its base $a$.

Whatever be the value of the angle $a$, if this base is on a level with the water, $c = 0$, and the general value of $x'$ becomes

$$x' = \frac{h(a+3b)}{2(a+2b)};$$

so that in this case it is independent of the inclination of the surface. If $b = a$, the trapezium becomes a parallellogram, and we have then

$$x' = \tfrac{2}{3}h.$$

If $b$ or $a$ is cipher, the trapezium becomes a triangle, and we shall have either

$$x' = \tfrac{1}{2}h, \text{ or } x' = \tfrac{3}{4}h.$$

In the first case, the base of the triangle is on a level with the water, and $x'$ is the distance of the centre of pressure from this base; in the second case, $x'$ is the distance from the summit which is at the surface of the liquid ($g$).

601. The pressures on a portion of a curved surface, may be determined by resolving the normal pressure on each element, into three forces parallel to the axes of the coordinates, and then calculating by double integrals, the total com-

ponents in these three directions, which components may always be reduced to two forces, that for the most part are not reducible to a single resultant (No. 264). But in the case of pressures exerted on the entire surface of a body immersed in a fluid, the reduction to a single force will be always possible, and the direction of this unique resultant will be vertical, as we now proceed to show.

Let AMB (fig. 46) be the body in question, $x$, $y$, $z$ the coordinates of M any point whatever of its surface, and let the level of the fluid be the plane of the axes of $x$ and $y$, the axis of $z$ vertical, and drawn in the direction of the gravity. Let $\omega$ be the differential element of the surface, and $p$ the pressure on the unit of surface at the point M, so that the pressure exerted on this element, and acting in the direction of the interior normal MN, may be equal to $p\omega$. The value of $p$ will be the same for all points, which are at the same distance $z$ from the level of the liquid, whether this stagnant fluid be homogeneous, or only composed of horizontal strata, whose density varies from one stratum to another. Likewise, let $\alpha$, $\beta$, $\gamma$ be the angles which the normal MN makes with parallels to the axes of $x$, $y$, $z$, drawn through the point M in the interior of the body. Finally, let $\omega$ be projected on the three planes of the coordinates, and let its projection on the plane of the axes of $y$ and $z$ be denoted by $a$, on that of the axes of $z$ and $x$ by $b$, on that of the axes of $y$ and $x$ by $c$, so that as $\alpha$, $\beta$, $\gamma$ are the inclinations of the tangent plane at M on these three planes, we shall have

$$a = \omega\cos\alpha, \quad b = \omega\cos\beta, \quad c = \omega\cos\gamma;$$

and if these equations be multiplied by $p$, there will result

$$pa = p\omega\cos\alpha, \quad pb = p\omega\cos\beta, \quad pc = p\omega\cos\gamma;$$

from which it appears that the products $pa$, $pb$, $pc$, are the components of the normal pressure $p\omega$ resolved parallel to the axes of $x$, $y$, $z$; so that the component perpendicular to each

plane of the coordinates, and, generally, to any plane whatever, may be deduced from $p\omega$, by substituting for the element $\omega$, its projection on this plane.

This being established, as the body AMB is bounded on all sides, there is, at least, a second element of its surface, which has the same projection on each given plane, as the element $\omega$. Thus, if from the point M, a perpendicular MP is let fall on the plane of the axes of $y$ and $z$, this perpendicular or its production will meet the surface of the body in a point M$'$, and the projection of the element $\omega'$, which corresponds to this point on this plane, will be the same as that of $\omega$, and equal to $a$. As the two elements are situated at the same distance from the level of the liquid, $p\omega$ and $p\omega'$, the normal pressures which they sustain will be to each other as the areas $\omega$ and $\omega'$, of which the ratio may be any magnitude whatever. But their components parallel to the axis of $x$, will have a common value $pa$, and as the forces $p\omega$ and $p\omega'$ act along the interior normals MN, M$'$N$'$, these equal components will evidently act in opposite directions, the one to the other, that is from M towards M$'$ at the point M, and from M$'$ towards M at the point M$'$. Consequently, the component parallel to the axis of $x$, of the pressure exerted on $\omega$, will be destroyed by the component acting in the same direction, of the pressure exerted on the other element $\omega'$. In the same manner, it will appear, that the component of $p\omega$ parallel to the axis of $y$, will be also destroyed by the component in this direction, of the pressure relative to a third element, which corresponds to the point where the perpendicular let fall from the point M on the plane of the axes of $x$ and $z$, meets the surface of the body a second time. Therefore, we may infer, that these horizontal components of the pressures exerted on the elements of the surface of the immersed body, mutually destroy each other, in each of the infinitely slender horizontal sections, and, consequently, on its entire surface. Hence it follows also, that all these pressures are reducible to one sole force, which

is the resultant of their vertical components, and which arises from the preponderance of the value of $p$ in the lower part of the body.

If $p$ results from a pressure exerted at the surface of the liquid, its value would be constant throughout the entire extent of the surface AMB; and the components of the pressures will be mutually destroyed two by two, in the vertical as well as in the horizontal directions. Therefore whatever be the form of a fluid or solid body, a constant normal pressure exerted on all the points of its surface, cannot produce any motion, either of translation or rotation, as has been already stated (No. 584).

602. In order to determine the resultant of the vertical pressures exerted on AMB, let a perpendicular be let fall from M any point whatever on the horizontal plane of the axes of $x$ and $y$, meeting this surface AMB in $M_1$. The elements $\omega$ and $\omega'$ which correspond to the points M and $M_1$, will have the same projection $c$ on this plane; but the pressures on the unit of surface will be different, and if they be denoted by $p$ and $p_1$, the filament of the body which is terminated by these two elements, and whose length is MN, will be urged vertically, from below upwards, by a force $pc - p_1c$. For greater simplicity, the liquid in which the body is immersed is supposed to be homogeneous. If its density be denoted by $\rho$, and the length of $MM_1$ by $l$, we shall have $p - p_1 = \rho g l$, and the vertical pressure $\rho g l c$ will be the weight of $lc$, the volume of the liquid, that is to say, the weight of the volume of the liquid, whose place is occupied by this filament of the body. If the body be decomposed into infinitely slender vertical filaments, each of these filaments will be urged upwards by a similar force; hence it follows, that the resultant of all the vertical pressures will only differ from the weight of the fluid filaments which are replaced by those of the immersed body, in the direction of its action, so that it will be equal to the total weight of the volume of the fluid, which this solid body displaces, applied

to the centre of gravity of this volume, in a direction opposite to that of gravity; which centre of gravity will coincide with that of the body itself, when this last is homogeneous.

If the body is not entirely immersed in the liquid, it is not necessary, in calculating the pressure which it experiences, to take into account the part of it which is situated above the level of the liquid; the point $M_1$ will then appertain to the section of the body made by the production of the plane of this level, and by taking $p_1 = 0$, this case will come under the preceding. The value of the resultant of the vertical pressures, which will be always that of all the pressures, will be then the weight of the volume of the liquid displaced by the immersed part of the *floating* body, and its point of application will be the centre of gravity of this same volume.

603. These results likewise have place when the liquid is composed of horizontal strata. We may also arrive at them by an indirect consideration, which it is useful to point out. When the equilibrium is established in this liquid, it evidently will not be deranged if any part whatever of this liquid was to become solid, in which case, this part becomes a floating or immersed body. Now, in order that the normal pressures exerted by the surrounding liquid on the surface of this body, may be in equilibrio with the weight of this solid part, they should be reducible to one sole force, equal and directly contrary to its weight. Moreover, if the part of the liquid which was supposed to become solid was replaced by another body which had exactly the same surface, it is evident that the pressures of the surrounding fluid are not altered; consequently the pressures on the surface of a body entirely or partly immersed in a stagnant liquid, whether homogeneous or heterogeneous, are always reducible to an unique force, equal to the total weight of the successive horizontal strata of the liquid, of which this body occupies the place, and applied to the centre of gravity of these same strata, in a direction opposite to that of gravity.

It follows from this, that in order that a body entirely im-

mersed in a liquid may be in equilibrio, it is necessary that its mean density should be equal to that of the liquid, the place of which it occupies, and that its centre of gravity and that of this portion of the liquid should exist in the same vertical. The second condition is always satisfied when the body and also the liquid are homogeneous. With respect to bodies which are only partly immersed, and which float on the surface of the liquid, the conditions of their equilibrium will be considered in the following chapter.

604. The hydrostatical principle which has been just demonstrated, is commonly expressed by stating, that a body immersed in a liquid loses a part of its weight equal to the weight of the fluid which it displaces (No. 191). It appears from this that in order to obtain the true weight of a body, it should be weighed in a vacuo. Two bodies weighed in the air, in water, or in any other fluid, and which are in equilibrio when weighed in an exact balance, have really different weights, unless their volumes are equivalent. That body, whose volume is the greater, has the greater weight, since, though it experiences a greater loss in the fluid, it is still in equilibrio with the other.

If the same body is weighed in a vacuo and in water, and if $P$ be its weight in the vacuo, and $P'$ its weight in water, then $P$ and $P - P'$ will be the absolute weight of this body and of the same volume of water; they are therefore to each other as the densities of these two substances (No. 60). If the density of water be taken as the unit, and that of the body be denoted by $D$, we shall have

$$D = \frac{P}{P - P'}.$$

It is by this formula that the densities of bodies, which can be weighed in water without being dissolved, are determined by means of the hydrostatic balance.

605. The demonstration of No. 601 is equally applicable to the lateral surfaces of a vessel, which contains a liquid ($h$); and

it follows from it, that the horizontal components of pressures exerted from within outwards, on the entire interior surface of the vessel, mutually destroy each other's effect two by two, so that if the bottom of the vessel is placed on a fixed horizontal base, the action of the fluids which it contains cannot put it in motion; this also results from the principle of the conservation of the motion of the centre of gravity (No. 553). But if an opening be made in one of the lateral surfaces, below the level of this liquid, it will flow out through this orifice; and as there is no pressure on the part of the surface which is taken away, that which is exerted on the opposite part of the vessel will be no longer destroyed; consequently this vessel will be put in motion in a direction opposite to that in which the fluid flows out. This is the principle on which several machines, that are moved by the reaction of a fluid, are constructed, and on which is founded the proposition of D. Bernoulli, to move vessels without the aid either of oars or wind(*i*).

In like manner, it may be shown by the same reasoning as in No. 602, that the *entire pressure exerted on the bottom of the vessel and on its lateral surfaces*, is always equal to the weight of the fluid it contains, applied at the centre of gravity of this fluid, in the direction of gravity. Each vertical filament of the fluid, which may be continued without interruption, from its level to any point whatever of the base, exerts on this point a normal pressure, the component of which is equal to the weight of this same filament. That which meets the interior of the vessel in two points, namely, in the bottom and one of the sides, exerts in these two points pressures whose vertical components act in opposite directions. The component which corresponds to the inferior point, acts in the direction of gravity, and exceeds the other by a quantity equal to the weight of this filament; and, in this manner it appears, that the resultant of the vertical pressures of all these fluid filaments, is the same thing as the weight itself of the fluid in question.

This pressure should be carefully distinguished from that which acts solely on the bottom of the vessel (No. 595), and which is equal to the weight of the fluid only when the vessel is a right cylinder or prism. It is less than this weight, when the vessel enlarges from the bottom to the top, like the frustum of a cone that rests on its lesser base, because the vertical filaments of the fluid which issue from its *level*, and are intercepted by the lateral surfaces, do not press on the bottom of the vessel; on the other hand, it is greater than the weight of the liquid when the vessel is like a frustum placed on its greater base, since the vertical filaments which issue from the *bottom* of the liquid, and are intercepted by its lateral sides, exert, nevertheless, the same vertical pressure on the bottom of the vessel, as if they extended to the surface of the liquid, as the deficiency in weight of each of these incomplete filaments is compensated by the resistance of the surface by which they are terminated(*k*).

## CHAPTER IV.

### OF THE EQUILIBRIUM AND MOTION OF FLOATING BODIES.

606. In order that a heavy body may be in equilibrio on the surface of a fluid at rest, its weight should be less than that of an equal volume of this fluid; nevertheless, there are cases in which a species of vacuum of small extent is formed about a floating body, and which, as it should be added to its volume, diminishes, consequently, its mean density, so that bodies of small volume may float, though their proper density surpasses that of the liquid($a$). This circumstance, which is connected with the theory of capillary phenomena, will not be taken into account in this treatise (No. 588).

The density of the solid body, if it is homogeneous, or its mean density, if it is not so, being less than that of the liquid, the body sinks in the liquid, until the weight of the displaced fluid becomes equal to *its* entire weight; and when this equality obtains, the body remains in equilibrio, provided that its centre of gravity and that of the displaced fluid exist in the same vertical, for the pressure of the liquid which should be in equilibrio with the weight of the body, is equal to the weight of the displaced liquid, and it acts at its centre of gravity in an opposite direction (No. 602).

If the floating body is homogeneous, as well as the liquid, the centre of gravity of the displaced liquid coincides with that of the immersed portion of the body. In the state of equilibrium, the volume of this part of the body is to that of the entire body, as the density of the body is to that of the liquid, and the determination of the positions of equilibrium of a floating body becomes a geometrical problem, which may be

stated in the following manner. To cut a body by a plane, in such a manner that the volume of the segment may be to that of the body, in a given ratio, and that the centres of gravity of the segment and of the body may exist in the same perpendicular to the cutting plane. Let then the section of the body which satisfies these two conditions, be placed on the level of the liquid, in such a manner, that the segment whose volume has been thus determined, may be situated underneath, and therefore immersed in the fluid, and one of the positions of equilibrium of the floating body will be obtained. In each particular case, these two conditions will be expressed by equations, the complete solution of which will make known all the positions of equilibrium of this body. Sometimes their number will be infinite, as in the case of solids of revolution, whose axis is horizontal; in other cases, this number is finite and determinate; but it would be difficult to demonstrate, *a priori*, that whatever be the form of the body, it has always a position of equilibrium.

607. Let the case of a triangular prism, whose edges are horizontal, be selected for an example of the problem that has been just stated. The cutting plane will be evidently parallel to these edges; moreover, its direction will be independent of the distance comprised between the two bases. Hence then we need not take into account the length of the prism, but solely determine the intersection of the cutting plane and of one of these two bases, so that the problem becomes one of plane geometry, which will have place equally in the case of a prism or of a horizontal cylinder with any base whatever. Let ABC (fig. 47) be one of the bases of the given prism. It may happen that two angles of this triangle are immersed in the liquid, or only one of them. If the case of only one immersed angle be first considered, we shall then see how the other case is reducible to this. Let therefore C be this immersed angle, MN the intersection of the cutting plane and of the base ABC, which it is proposed to determine, and which will represent the level of the liquid. Let $a$, $b$, $c$ be the given

sides of this triangle ABC, which are respectively opposite to the angles A, B, C, and let $x$ and $y$ denote the unknown sides CM and CN of the triangle MNC, so that we may have

$$\text{BC} = a, \quad \text{AC} = b, \quad \text{AB} = c, \quad \text{CM} = x, \quad \text{CN} = y.$$

The area of any triangle is equal to half the product of two of its sides and of the sine of the included angle; hence we shall have

$$\text{ABC} = \tfrac{1}{2}\, ab \sin \text{C}, \quad \text{MNC} = \tfrac{1}{2}\, xy \sin \text{C}.$$

As the weight of the entire prism is equal to the weight of a prism of the fluid equal in volume to the part immersed, we shall have

$$\text{MNC} : \text{ABC} : r : 1;$$

$r$ being a quantity less than unity, which represents the ratio of the density of the floating body to that of the liquid. This proportion gives, by substituting for ABC and MNC their preceding values,

$$xy = rab. \qquad (1)$$

Now, if CD be drawn from C to the point of bisection of the base AB, and if on this line DG be taken $= \frac{1}{3}$ DC, the point G will be the centre of gravity of the triangle ABC. In like manner, E being the middle point of MN, if on CE a part EF be taken $= \frac{1}{3}$ CE, the point F will be the centre of gravity of MNC. Therefore the line GF must be perpendicular to MN($b$): but as the lines CD and CE are cut into proportional parts in the points G and F, the lines DE and GF are parallel; consequently, the line DE which joins the middle points of the two bases AB and MN, is also perpendicular to MN, from which it follows that the two lines DM and DN are equal. Conversely, if DM $=$ DN, the line DE will be perpendicular to MN, as is also its parallel GF; therefore, in order that DE the right line which joins the two centres of gravity G and F may be perpendicular to the intersection MN, it is necessary and sufficient that the values of DM and DN should be equal.

This being so, if we make CD $= h$, and denote DCB and

DCA, the two parts of the angle ACB, by $\beta$ and $\alpha$ respectively, we shall obtain, by considering the triangles DCM and DCN,

$$\text{DM}^2 = h^2 + x^2 - 2hx\cos\alpha,$$
$$\text{DN}^2 = h^2 + y^2 - 2hy\cos\beta;$$

and by putting these two values equal to each other, there will result

$$x^2 - 2hx\cos\alpha = y^2 - 2hy\cos\beta. \qquad (2)$$

By means of equations (1) and (2), $x$ and $y$ can be determined. We obtain by the elimination of $y(c)$,

$$x^4 - 2hx^3\cos\alpha + 2hrabx\cos\beta - r^2a^2b^2 = 0. \qquad (3)$$

The value of $x$ can be deduced from this equation, which is of the fourth degree, and then as $y = \frac{rab}{x}$, the corresponding value of this quantity can be determined.

As equation (3) is of even dimensions, and has its last term negative, it must have at least one positive and one negative root. The other two roots may be either real or imaginary. If they are real, it is evident, from the rule of Descartes, that equation (3) will have three positive roots and one negative root. For if we suppose a term $= \pm\, ox^2$ to be added to it, whether this coefficient is $+$ or $-$, we have always three *variations* and one *continuation*. As the unknown quantities $x$ and $y$, which are the sides of the triangle MNC, must be positive quantities respectively less than CA and CB, the sides of the triangle ABC, we should reject, as being inapplicable to the question, the negative root of equation (3), the values of $x$ which are greater than $a$, and also those which give a value greater than $y$. Thus, there are, at most, only three positions of equilibrium when only one angle C is immersed in the liquid.

608. If this angle be without the fluid, and the two points A and B below the level MN, the problem will be the same as in the preceding case, with this sole difference, that $1 - r$ should be substituted in place of $r$ in equation (1) and (3).

In fact, as the centres of gravity of the triangle ABC, and of its two parts MNC and MNBA exist on the same line, and as the centres of gravity of ABC and of the second part must be situated on a perpendicular to MN, the centres of gravity of the triangles ABC and MNC must always exist on this perpendicular; so that we have at once, without any change, equation (2), which expresses this condition. Moreover, the proportion

$$\text{MNBA} : \text{ABC} :: r : 1,$$

which should now have place, may be changed into the following,

$$\text{MNC} : \text{ABC} :: 1 - r : 1;$$

in consequence of which, equation (1) is changed into

$$xy = (1 - r)ab.$$

If by means of this, $y$ be eliminated from equation (2), we will light again on equation (3), in which the ratio $r$ will be replaced by $1 - r$, that is to say, we shall have

$$x^4 - 2hx^3 \cos\alpha + 2h(1-r)abx\cos\beta - (1-r)^2a^2b^2 = 0. \quad (4)$$

By reasoning in the same manner as before, it appears that there are at most but three positions of equilibrium, when the two angles A and B are immersed in the fluid.

If the three angles A, B, C be successively considered, and if we examine the case for each angle, in which it is solely immersed in the fluid, and solely without the fluid, all the horizontal positions of equilibrium of the given prism will be determined; and from what precedes, it results that this number can never exceed eighteen($d$).

609. When the triangle ABC is isosceles, we may dispense with equation (3) or (4), and resolve the equations in $x$ and $y$ directly. If, for example, $b = a$, the triangles CAD and CBD will be equal and rectangular, and we shall have

$$\beta = \alpha, \quad h^2 = a^2 - \tfrac{1}{4}c^2, \quad a\cos\alpha = h;$$

and equations (1) and (2) will become

$$xy = ra^2, \quad y^2 - x^2 - \frac{(4a^2 - c^2)}{2a}(y - x) = 0. \tag{5}$$

They may be satisfied at once, by making

$$x = y = a\sqrt{r};$$

which values, as $r \angle 1$, and $a\sqrt{r} \angle a$, are evidently admissible. There results from it a position of equilibrium, in which the triangle MNC is isosceles, and in which AB the base of the triangle ABC will be parallel to MN or horizontal. By changing $r$ into $1 - r$, a second position will be obtained, in which the point C will be situated without the liquid, and the base AB still horizontal. But there may be also other positions of equilibrium, for which this base will be inclined.

In fact, if the factor $y - x$ of the second equation (5) be suppressed, there arises

$$y + x = \frac{4a^2 - c^2}{2a}.$$

As this and the first equation (5) give the sum and product of the two unknown quantities $x$ and $y$, it follows that these quantities will be the two roots of the same equation of the second degree, and the expressions for these roots will be($e$)

$$\frac{1}{4a}\left[4a^2 - c^2 \pm \sqrt{(4a^2 - c^2)^2 - 16ra^4}\right].$$

If each of them be taken successively for $x$, and the other for $y$, there will result, when they are respectively real, and less than $a$, two new positions of equilibrium, in which the base AB is situated without the liquid. By substituting $1 - r$ in place of $r$, and always supposing the real roots less than $a$, two other positions of equilibrium will be obtained in which this base is immersed. When the two preceding roots are equal, the base AB is horizontal, and its new positions of equilibrium must coincide with the preceding, and, in point of fact, we have then $4a^2 - c^2 = 4a^2\sqrt{r}$, from which we obtain($f$)

$$y = x = a\sqrt{r}.$$

610. In the case of an equilateral triangle, $c$ becomes equal to $a$, in the preceding formulæ. The equal values of $x$ and $y$ will not be changed; their unequal ones will become($g$)

$$\frac{a}{4}(3 \pm \sqrt{9 - 16r}),$$

in the case of only one immersed angle, and

$$\frac{a}{4}(3 \pm \sqrt{16r - 7}),$$

in the case of only one angle without the liquid. It will be therefore necessary to know what the fraction $r$ ought to be, in order that its values may be real, and less than $a$. Now, if $r \angle \frac{9}{16}$ and $> \frac{1}{2}$, the first formula will be real, and its two values will be less than $a$; without these limits, this formula will be either imaginary, or one of its values will surpass $a$; and, in the same manner, in order that the values of the second formula may be real, and less than $a$, it is necessary and sufficient that we should have $r \angle \frac{1}{2}$ and $> \frac{7}{16}$. It appears, therefore, that from $r = \frac{9}{16}$ to $r = \frac{1}{2}$, the first formula is solely admissible, and that, on the contrary, it is the second that is solely admissible from $r = \frac{1}{2}$ to $r = \frac{7}{16}$; and that for $r > \frac{9}{16}$ or $r < \frac{7}{16}$, the two formulæ should be rejected.

As in the case of an equilateral triangle, every thing relatively to the three summits is similar, the number of positions of equilibrium will be always a multiple of three, and this number may be six or eighteen according to the value of $r$($h$).

611. Besides their *horizontal* positions of equilibrium, homogeneous prisms and cylinders may float in positions of equilibrium, in which their edges are vertical, and their bases parallel to the level of the liquid, and there are two for each body, since either of the two bases may be immersed in the liquid. The centres of gravity of a vertical prism, and of its immersed part exist on the same perpendicular to the level; the ratio of their volumes is the same as that of their heights ($i$), and, consequently, the height of the immersed prism is to that

of the entire prism as the density of the body is to that of the liquid, which is sufficient to enable us to determine the depth to which the body sinks in its state of equilibrium.

Solids of revolution, and in general, all bodies which are symmetrical about an axis, have likewise two positions of equilibrium, in which this line is vertical, and which may be easily determined. Let us suppose, for example, that the body was a homogeneous ellipsoid, whose three semiaxes are $a, b, c$; let the axis $2c$ be vertical, and let $u$ be the distance of the two other axes from the section, which coincides with the level of the water, $u$ is an unknown quantity, which is positive or negative, according as this section is below or above the centre of the ellipsoid. Likewise, let $\mathrm{z}$ be the area of the horizontal section of this body made at any distance such as $z$, from its centre; then as the volume of the semi-ellipsoid is $\frac{2\pi}{3}abc$, that of the immersed segment will be obtained by subtracting from $\frac{2\pi}{3}abc$, the integral $\int_0^u \mathrm{z}\,dz$, which expresses the value of the segment comprised between the level of the water and the horizontal section, made through the centre of the body, and which has the same sign as $u$. Therefore, in the case of equilibrium, we shall have($k$)

$$\frac{2\pi}{3}abc - \int_0^u \mathrm{z}\,dz = \frac{4\pi}{3}abcr;$$

$r$ being always the ratio of the density of the floating body to that of the liquid. As this body is an ellipsoid, we shall have (No. 89)

$$\mathrm{z} = \frac{\pi ab}{c^2}(c^2 - z^2);$$

and the equation of equilibrium will become

$$u^3 - 3c^2u - 2(2r-1)c^3 = 0.$$

It will have always one real root comprised between $\pm c$, and it will be positive or negative, according as $r > \frac{1}{2}$ or $r \angle \frac{1}{2}$.

In the extreme cases of $r = 0$ and $r = 1$, this root will be $u = c$ and $u = -c$.

The parts immersed in different liquids of the same body symmetrically arranged about a vertical axis, are in the inverse ratio of the densities of these fluids. This is the principle of the construction of the hydrometer, by means of which the densities of different fluids can be compared together.

612. Among the different positions of equilibrium of the same body floating on the surface of a liquid, there are some which are stable, and others which are not so; and from No. 570, it appears that if this body be made to turn about an axis, which, for greater clearness, we shall suppose to be horizontal, its successive positions of equilibrium will be alternately stable and instantaneous. It is of consequence to distinguish carefully the first from the last, which would not continue sufficiently long to be observed, but for the slight adhesion of the floating body to the liquid with which it is in contact. In the first place, let the floating body be supposed to be perfectly symmetrical, as to the form and density of its parts, on each side of a vertical section ABCD (fig. 48). Let G be its centre of gravity, which must exist in this section. Likewise, let AC be the line in which this section meets the level of the liquid when the body is in equilibrio in it, and let H be the centre of gravity of the volume of the liquid that is displaced by the body; this point will likewise exist on the section ABCD, on some point of BGK drawn from the point G, perpendicular to AC. When the body is homogeneous, the point H will be beneath G, as is represented in the figure; but when the floating body is *ballasted*, that is to say, when the density of its inferior part is increased, the point G may fall beneath the point H. This being so, if the floating body is caused to deviate a little from its position of equilibrium, by making it to turn about an axis *perpendicular* to ABCD; and if it be then remitted to itself without impressing on it any initial velocity, whatever may be the motion which the body

will assume, the section ABCD will always remain vertical, and G, the centre of gravity, will constantly exist on it.

In this new position, let A′C′ (fig. 49) be the line which indicates the level of the liquid, and that intersects AC in E, in which case, the segment of the body which belongs to AEA′, will be immersed in the liquid, and that which belongs to CEC′, will be raised out of it. As we have supposed that the volumes of these two segments are equal, it follows that the volume of the liquid displaced by the body will not be changed; consequently, the weight of this volume of the fluid will be still equal to that of the body, as in the state of equilibrium. Now, as G the centre of gravity of the floating body ought to move as, if the mass of this body being concentrated in it, the weight of this body and the pressure of the fluid were applied to it (No. 438); and as these two vertical forces act in opposite directions, and are, by hypothesis equal, it will not be necessary to consider the motion of the point G.

If H′ be the centre of gravity of the volume of displaced liquid, after the body has been made to deviate from the position of equilibrium, this point, like the centre of gravity G, will appertain to the section ABCD, but they will be, *in general*(*l*), no longer situated in the same vertical; consequently, the pressure of the fluid will cause the body to turn about a line passing through the point G, and perpendicular to the section ABCD; and the question will be to determine whether this motion tends to make the body revert to its position of equilibrium, or to cause it to deviate more and more from it, and eventually upset it.

Now, if through the point H′ the vertical H′M be drawn meeting the line BGK perpendicular to AC, at the point M, it is evident that the pressure of the fluid, which is exerted upwards in the direction H′M, will tend to bring back the line BGK to its vertical position, which is that of the equilibrium, or to cause it to deviate more from it, according as the point M is situated above or below the point G. In the first case,

the equilibrium will be stable, in the second case it will not be so. When the point M coincides with the point G, the body will still continue in equilibrio, in any position near to the first, in which it may have been placed. If G the centre of gravity of the body falls below that of the volume of the displaced liquid, which was H, in the state of equilibrium, that is to say, if this point G exists between the points B and H, on the line BK, the point M will be above G, and the equilibrium will be necessarily stable. If, on the contrary, the point G is above H, as in the case of a homogeneous body, the point M may be above or below G, and the equilibrium may be either stable or unstable. The point M, the consideration of which enables us to distinguish the two states of equilibrium of a floating body, is symmetrically situated with respect to a vertical section, and is termed the *metacentre*. We now, however, propose to give another rule, deduced from the principle of living forces, by means of which the stability of the equilibrium may, in all cases, be ascertained.

613. For this purpose, let us consider a body of any form whatever, homogeneous or heterogeneous, in equilibrio in the water. Let ABCD (fig. 50) be the intersection of this body with the level of the water, G the centre of gravity of the moveable, H that of the volume of water displaced by the part immersed of this body, V the volume of this part, and M the mass of the entire body; since it is supposed to be in equilibrio, the line GH is perpendicular to the plane ABCD, and the mass of displaced water is equal to that of the body, so that if the density of the water be denoted by $\rho$, we have

$$M = V\rho.$$

Let us suppose that the section ABCD is elevated above, or depressed below the level of the water (fig. 51), by a very small quantity, and that at the same time, the plane of this section is inclined by ever so small an angle; and also, for

greater generality, that small velocities are impressed on the points of the moveable. The equilibrium will be deranged, and to determine the stability, it will be necessary to consider whether, in consequence of the motion that the body acquires, the section ABCD, which is fixed in the interior of the body, deviates more and more from the level of the water, or tends to revert to it, by oscillating on each side of this level. During the motion which ensues, the natural level of the water cuts the floating body in a section which is variable in its interior, and which is termed *the plane of floatation*. Let A′B′C′D′ be this section at any instant whatsoever, AB″CD″ another variable section of this body made by a horizontal section which passes through the centre of gravity of the section ABCD; AC the intersection of ABCD and AB″CD″, which is variable on ABCD, $\theta$ the mutual inclination of these two sections, $\zeta$ the distance of AB″CD″ from the plane of floatation; which distance we shall consider to be positive or negative, according as this section exists above or below the level of the water. The variable quantities $\theta$ and $\zeta$ are supposed to be very small at the commencement of the motion, and the question will be to determine whether they remain very small during its entire continuance.

614. If the variable velocity of $dm$ any element whatever of the mass of the moveable, be denoted by $u$, the sum of the living forces of all its points will be given by the integral $\int u^2dm$ extended to the entire mass, and the equation which results from the principle of living forces, will be of the form (No. 564),

$$\int u^2dm = c + 2\phi; \qquad \text{(a)}$$

$c$ being an arbitrary constant, and $\phi$ a function depending on the forces which are applied to the points of the moveable.

These forces are the gravity which acts on all its points, and the vertical pressures which the fluid exerts on the immersed part of the surface of the body; now we can substitute

for these pressures, motive forces which act on all the elements such as $dm$ of its mass, that are situated below the level of the water, by taking for each element a force, acting in a direction opposite to that of gravity, and equal to the weight of the volume of the water which it replaces; for by No. 602, the magnitude, direction, and point of application of the resultant of these motive forces, will be the same as that of the vertical pressures. In this manner, if the gravity be denoted by $g$, the element of the volume of the moveable, which corresponds to $dm$ the element of its mass, by $dv$, the motive force of $dm$ will be $gdm - g\rho dv$, if this material point exists below the level of the water, and $gdm$ if it be above it. Moreover, if $z$ be the variable distance of $dm$ from the plane of floatation, which will be positive or negative according as $dm$ is below or above this plane, it follows from the general value of the function $\phi$, given in No. 564, that in the present question, we should have

$$\phi = \int zgdm - \int zg\rho dv;$$

in which equation, the first of these two integrals extends to the entire mass of the floating body, and the second solely to the part of its volume that is immersed.

If from G the centre of gravity of the mass M, a perpendicular GE be let fall on the plane A′B′C′D′, and if GE be made $= z_1$, we shall have at once, for the value of the first integral,

$$\int zgdm = g\text{M}z_1.$$

If the second be divided into two parts, the one relative to V the volume, situated below ABCD in the state of equilibrium, the other relative to the volume comprised between the sections ABCD and A′B′C′D′, then $g\text{V}\rho z'$ will be the value of the first part, $z'$ being the variable distance of H the centre of gravity of the volume V, from the plane of floatation, that is to say, the length of HF the perpendicular let fall from the point H on the plane A′B′C′D′. Therefore, if for one moment

$k$ be the value of the integral $\int z dz$ extended to all the elements $dv$ of the volume comprised between ABCD and A′B′C′D′; $g\rho k$ will be the second part of the second integral comprised in the expression $\phi$, and we shall have

$$\phi = g\text{M}z_1 - g\rho\text{V}z' - g\rho k,$$

for the complete value of this quantity. But as the right line GH is perpendicular to the plane ABCD, the angle which it makes with the vertical is $\theta$, the inclination of the plane ABCD to the horizontal plane; if, therefore, the constant length of GH be denoted by $a$, we shall have

$$z_1 = z' \pm a \cos \theta;$$

in which the superior sign will have place when the point G is below H, and the inferior sign in the contrary case. By substituting this value of $z_1$ in that of $\phi$, and observing that $\text{M} = \text{V}\rho$, there results

$$\phi = \pm g\rho a\text{V} \cos \theta - g\rho k;$$

and it will only remain to determine the value of the integral represented by $k$.

615. In order to obtain it, let the area of the section ABCD be decomposed into infinitely small elements, and let them be projected on the plane of flotation A′B′C′D′, the volume comprised between those two sections of the body will, by this means, be divided into an infinite number of vertical cylinders whose bases are the horizontal projections of the elements of ABCD. Let then any one of these cylinders be cut by an infinity of horizontal planes, and let $dv$ the element of the volume which is considered, be the part of this cylinder comprised between the two consecutive planes, whose distances from the plane of floatation are $z$ and $z + dz$, so that this element may be equal to the base of the cylinder multiplied by $dz$. Now as $d\lambda$ is the differential element of the section ABCD, the horizontal projection, or the base of the corres-

ponding cylinder will be $d\lambda \cos\theta$, because $\theta$ is the inclination of the plane of $d\lambda$ on the plane of projection, therefore, we shall have

$$dv = dz d\lambda \cos\theta;$$

consequently, $\int z dv$, the integral relative to one of the vertical cylinders, will be the product of $d\lambda \cos\theta$ and of $\int z dz$, or equal to ($m$) $\frac{1}{2} y^2 \cos\theta\, d\lambda$, $y$ being the height of this cylinder, or the perpendicular let fall from $d\lambda$ on the plane of floatation; therefore, the quantity denoted by $k$ will be

$$k = \tfrac{1}{2} \cos\theta \int y^2 d\lambda;$$

in which the integral is supposed to extend to the entire area of ABCD. The perpendicular $y$ consists of two parts, the one comprised between the two parallel planes A′B′C′D′ and AB″CD″, and which is denoted by $\zeta$, the other comprised between $d\lambda$ and the second plane, which will be equal to $l \sin\theta$, $l$ denoting the distance of this element from AC the intersection of the two planes ABCD and AB″CD″; we shall therefore have

$$y = \zeta + l \sin\theta,$$

in which $l$ will be regarded as positive or negative, according as $d\lambda$ is below or above the second plane. By substituting this value in the preceding equation, and observing that $\zeta$ and $\theta$ are constant in the integration adverted to above, there results ($n$)

$$k = \tfrac{1}{2}\zeta^2 \cos\theta \int d\lambda + \zeta \sin\theta \cos\theta \int l d\lambda + \tfrac{1}{2} \sin^2\theta \cos\theta \int l^2 d\lambda.$$

If $b$ denotes the area of the section ABCD, or the value of $\int d\lambda$, then as the line AC contains by hypothesis, the centre of gravity of this section, the integral $\int l d\lambda$ is cipher; and if we make

$$\int l^2 d\lambda = b\gamma^2,$$

so that $\gamma$ may be a line depending on the figure and extent of ABCD, we shall have finally

$$k = \tfrac{1}{2} b \cos\theta \,(\zeta^2 + \gamma^2 \sin^2\theta).$$

This formula does not give the exact value of $k$; in order that it should, the volume comprised between the sections A′B′C′D′ and ABCD should be the part of a vertical cylinder, cut off by the plane of the section ABCD; but whatever its form may be, we may suppose that the exact value of $k$ differs very little from the preceding, as long as $\zeta$ and $\theta$ are very small quantities, and it is easy to be satisfied that the difference of these values is a quantity of the third order with respect to $\zeta$ and $\theta$ (o). By substituting this approximate value of $k$ in the expression of $\phi$, and also making

$$\sin\theta = \frac{\theta}{1} - \frac{\theta^3}{1.2.3} + \&c., \quad \cos\theta = 1 = \frac{\theta^2}{1^2} + \&c.,$$

we shall obtain, when all the terms of the third order, with respect to these variables $\theta$ and $\zeta$, are neglected,

$$\phi = \pm g\rho \mathrm{v} a \mp g\rho \mathrm{v} a\theta^2 - \tfrac{1}{2} g\rho b\,(\zeta^2 + \gamma^2\theta^2),$$

by means of which equation (a) is changed into the following(p),

$$\int u^2 dm + g\rho\,[b\zeta^2 + (b\gamma^2 \pm \mathrm{v}a)\,\theta^2] = c, \qquad \text{(b)}$$

the term $\pm\, 2g\rho\mathrm{v}a$ being supposed to be comprised in the arbitrary constant $c$.

616. The value of this constant will be determined when the values of $u$, $\zeta$, $\theta$, which are supposed to be very small, are known at the commencement of the motion; and it appears, moreover, from equation (b), that its value is positive, if the coefficient $b\gamma^2 \pm \mathrm{v}a$ is positive, when the motion commences. If this coefficient remains positive during the entire continuance of the motion, it follows from this equation, by the same reasoning as has been employed in No. 570, that the variables $\zeta$ and $\theta$ will remain constantly very small, so that at any instant whatever, we shall have

$$\theta \angle \sqrt{\frac{c}{g\rho\,(l\gamma^2 \pm \mathrm{v}a)}}, \quad \zeta \angle \sqrt{\frac{c}{g\rho b}};$$

from which it appears, that the stability of equilibrium of a

floating body depends on the sign of the quantity $b\gamma^2 \pm \text{v}a$, and that this equilibrium will be stable when this quantity is positive at the commencement and during the continuance of the motion ($q$).

The integral $\int l^2 d\lambda$, which is denoted by $b\gamma^2$, must be a positive quantity, since all its elements are positive. The term $\pm \text{v}a$ must be affected with the sign $+$ when the point G is lower than the point H, therefore, in this case, the coefficient $b\gamma^2 \pm \text{v}a$ is positive, and the equilibrium is stable($r$). When, therefore, the centre of gravity of the entire mass of a floating body is lower than that of the volume of water, which it displaces in its position of equilibrium, we may be certain, that this equilibrium is stable for all the small motions that can be impressed on the body.

If, on the contrary, the point H is lower than the point G, the term $\pm \text{v}a$ must be affected with the sign $-$; therefore, we must have $b\gamma^2 > \text{v}a$, in order that the coefficient $b\gamma^2 \pm \text{v}a$ may be positive and the equilibrium stable. Now, the magnitude of the line $\gamma$ varies with the position of the line AC, which always passes through the centre of gravity of the section ABCD, and turns about this point during the continuance of the motion; hence if the line AC be made to make an entire revolution, it is evident that there is one position in which the line $\gamma$ will be less than in any other; if, therefore, this least value of $\gamma$ be calculated, and if then we find $b\gamma^2 > \text{v}a$, it is certain that the coefficient $b\gamma^2 \pm \text{v}a$ cannot become negative, and, consequently, that the equilibrium is stable.

In a ship, for example, it is easy to perceive that the line AC, to which the *minimum* of the integral $\int l^2 d\lambda$ corresponds, is the line drawn from the *prow* to the *poop*, consequently, if the area of the section on a level with the water, be divided into infinitely small elements, and if then the sum of all these elements, multiplied respectively by the square of their distances from this line, be determined, provided that this integral surpasses the product of the volume of the water

displaced by the vessel, and of the distance of the centre of gravity of this volume from that of the vessel, we may be certain that the equilibrium is stable, with respect to all the small motions of the vessel, even when the second centre of gravity is higher than the first.

617. Having discussed the question of the equilibrium and stability of floating bodies, we now proceed to determine the motion by which they are actuated, when they are made to deviate a little from a position of stable equilibrium. In order to solve the question completely, we should consider both this motion and also that of the liquid at the same time; this the author proposed to do in another treatise, at present the motion of the fluid will not be taken into account, and, in order to simplify the problem, with respect to the solid body, it will be assumed, that it is symmetrical on each side of a plane, which continues to be vertical during the entire motion.

This plane contains G and H the centres of gravity of the moveable and of the volume of fluid, which it displaces in its state of equilibrium. In this state, the line GH is vertical; let it be inclined by making it to turn in this plane about the point G, then let this line be elevated or depressed in this same plane, parallel to itself; and afterwards let the moveable be remitted to the action of gravity, and of the pressures of the surrounding fluid, without impressing any initial velocity on it; it is evident, that the section of the moveable made by the plane in question, and which divides it into two symmetrical parts, will continue constantly vertical. AC the intersection of the sections ABCD and AB''CD'' will remain always perpendicular to this vertical plane; and as by hypothesis, the line AC contains the centre of gravity of ABCD ($s$), it follows, that this centre will be the point K where it intersects the same plane, and that the line AC will always meet the contour of ABCD in the same points A and C. Independently of the symmetry of the body with respect to the plane perpendicular to AC, let us suppose besides, in order to simplify the question

still more, that the line GK is perpendicular to the plane of the section ABCD; this will be the case, when, for example, the plane passing through the two lines GK and AC, likewise divides the moveable into two symmetrical parts.

At the end of any time $t$, reckoned from the commencement of the motion, let $z$ denote GE the distance of the point G from the fixed plane A′B′C′D′, $\zeta$ the mutual distance of the two horizontal planes AB″CD″ and A′B′C′D′, $\theta$ the angle KGE comprised between the line GK and the vertical GE, $y$ the distance of $d\lambda$, any element whatever of the section ABCD, from the plane A′B′C′D′, $x$ its distance from the vertical plane drawn through the point G and parallel to the line AKC. Likewise, let $l$ be the constant distance of this element from this line, and $h$ the given length of GK. It is easy to perceive, that we shall have ($t$)

$$z_1 = \zeta + h\cos\theta, \quad y = \zeta + l\sin\theta, \quad x = l\cos\theta + h\sin\theta,$$

in which $l, h, \zeta$ are to be considered as positive or negative, according as the element $d\lambda$ is to the right or left of the line AKC, as the line GK is to the right or left of the vertical GE, and the plane AB″CD″ below or above A′B′C′D′. Finally, if the area of the section ABCD be denoted by $b$, and if $\gamma$ denotes the same line as before in No. 615, we shall likewise have

$$\textstyle\int d\lambda = b, \quad \int l d\lambda = 0, \quad \int l^2 d\lambda = b\gamma^2; \qquad (1)$$

in which the integrals are supposed to extend to all the elements of $b$.

The variables $\zeta$ and $\theta$ determine the position of the moveable at each instant. As the initial velocities of all the points of the body are supposed to be cipher, we shall have for $t = 0$,

$$\theta = \alpha, \quad \zeta = \beta, \quad \frac{d\theta}{dt} = 0, \quad \frac{d\zeta}{dt} = 0,$$

$\alpha, \beta$ denoting very small given quantities. The problem will consist in determining the values of $\theta$ and $\zeta$ in functions of $t$,

on the supposition that these variables remain very small during the entire continuance of the motion; in consequence of which we are permitted to neglect the square of $\theta$, and the product of $\theta$ and $\zeta$, and to consider the volume comprised between ABCD and A′B′C′D′, as a truncated cylinder.

618. The centre of gravity G will move as if M the mass of the moveable was condensed into it, and M$g$ the weight of the body and the resultant of the pressures of the fluid were applied to it. This resultant acts in a direction contrary to that of M$g$; it is equal to $(\text{V} + \text{U})\rho g$, V denoting the volume of the displaced water in its state of equilibrium, and V + U this volume at the end of the time $t$. Therefore, the motive force of the point G acting in the direction of gravity, will be $\text{M}g-(\text{V}+\text{U})\rho g$ or simply $-\text{U}\rho g$, because $\text{M} = \text{V}\rho\,(u)$; as its initial velocity is cipher, it will not deviate from the vertical in which it exists at the commencement of the motion, and the equation of its motion on this line will be

$$\text{M}\frac{d^2z_1}{dt^2} = -\rho g\text{U}.$$

U is the volume comprised between the two sections ABCD and A′B′C′D′; so that if this volume be decomposed into vertical cylinders as before, we shall have

$$\text{U} = \int y \cos\theta d\lambda\,;$$

in which the integral is supposed to extend to all the elements of $b$. By substituting for $y$ its preceding value, we obtain $(v)$

$$\text{U} = b\zeta\cos\theta.$$

If, therefore, $\cos\theta$ be assumed equal to unity in this value and in that of $z_1$, and if these values be then substituted in the equation of the motion of the point G, and if V$\rho$ be put also in place of M, there results

$$\frac{d^2\zeta}{dt^2} + \frac{gb\zeta}{\text{V}} = 0. \qquad (2)$$

At the same time, the moveable will turn about the point

G, as if it was fixed, and the forces which solicit it were not changed (No. 438). Therefore, this motion of rotation will have place about the fixed axis drawn through the point G, and perpendicular to the plane which divides the moveable into two symmetrical parts; and because the weight of the body passes constantly through this point, it will be due solely to the pressures of the surrounding fluid; consequently, the equation of this motion will be (No. 392) ($x$)

$$\text{M}k^2\frac{d\omega}{dt} = \mu.$$

$\text{M}k^2$ denoting the moment of inertia of the body with respect to the axis of rotation, $\omega$ its angular velocity of rotation at the end of the time $t$, and $\mu$ the total moment of the pressures at the same instant, with respect to the same axis. The velocity $\omega$ will be regarded as positive or negative, according as the motion has place in the direction indicated by the sagitta $s$, or in the opposite direction; so that we shall have

$$\omega = -\frac{d\theta}{dt}, \quad \frac{d\omega}{dt} = -\frac{d^2\theta}{dt^2};$$

and, this being the case, the moments of the pressures in the value of $\mu$ should be affected with the sign $+$ or $-$, according as they tend to produce a rotation in the direction of the sagitta $s$, or in the opposite direction.

Now, the total moment $\mu$ may be divided into two parts, the one relative to the constant volume V, the other relative to the variable volume U. The part of the pressure corresponding to the first volume is equal to $\text{V}\rho g$ applied to the point H, and acting in the direction HF; the value of the perpendicular let fall from the point G on this line produced, if necessary, is $a \sin\theta$, in which $a$ denotes the constant distance GH, hence the first part of the moment $\mu$ will be $\pm \text{V}a\rho g \sin\theta$, in which the superior or inferior sign ought to be taken, according as the point H is higher or lower than the point G ($s$).

With respect to the part of $\mu$ which corresponds to $\mathrm{U}$, if this volume be always decomposed into vertical prisms, and if the pressure corresponding to $y \cos \theta d\lambda$ any prism whatever, be considered, which pressure is always equal and contrary to the weight of the volume of the displaced fluid, we shall have $\rho g x y.\cos \theta\, d\lambda$ for the moment of this pressure; and, consequently, $\rho g \int x y \cos \theta d\lambda$ for the second part of $\mu$, in which the integral extends to the entire area $b$. By substituting for $x$ and $y$ their values, this quantity will become, by having regard to equations (1) ($y$)

$$(\gamma^2 \cos \theta + h\zeta)\, \rho g b \cos \theta \sin \theta;$$

consequently, the complete value of $\mu$ will be

$$\mu = (b\gamma^2 \cos^2 \theta \pm \mathrm{v}a + bh\zeta \cos \theta)\, \rho g \sin \theta.$$

If, in this value, unity and $\theta$ be substituted in place of $\cos \theta$ and $\sin \theta$, and if the product of $\zeta$ and $\theta$ be neglected, and if then, it and also the values of $\mathrm{M}$ and $\omega$ be substituted in the equation of the motion of rotation, it becomes

$$\frac{d^2\theta}{dt^2} + \frac{(b\gamma^2 \pm \mathrm{v}a\, g\theta}{\mathrm{v}k^2} = 0. \tag{3}$$

The problem, therefore, depends on the two differential equations (2) and (3); and as the variables $\theta$ and $\zeta$ are separated in them, it follows, that the motion of rotation of a floating body, and that of its centre of gravity, are independent of each other; a circumstance which obtains, because the line GK drawn from the centre of gravity of the moveable to that of ABCD is supposed to be perpendicular to this section ($z$). By integrating these two equations, and determining the arbitrary constants, by means of the initial values of $\zeta$, $\theta$, $\frac{d\zeta}{dt}$, $\frac{d\theta}{dt}$, we obtain ($a'$)

$$\zeta = \beta \cos t \sqrt{\frac{gb}{\mathrm{v}}}, \quad \theta = a \cos\left[\frac{t}{k}\sqrt{\frac{g\,(b\gamma^2 \pm \mathrm{v}a)}{\mathrm{v}}}\right].$$

As the vertical ordinate of the centre of gravity is equal to $h + \zeta$, when the square of $\theta$ is neglected, it follows, that the motion of this point is the same as that of a simple pendulum whose length is $\frac{\text{v}}{b}$. In order that the value of $\theta$ may not increase indefinitely, that is to say, in order that the equilibrium, from which the floating body has been made to deviate, may be stable, the quantity $b\gamma^2 \pm \text{v}a$ must be positive; which agrees with the theorem of No. 616. When this condition is satisfied, the oscillations of the line GK on each side of the vertical GE will be the same as those of a simple pendulum, the length of which is $\frac{\text{v}k^2}{\beta\gamma^2 \pm \text{v}a}$.

If the floating body is not symmetrical on each side of the plane passing through the lines GK and AKC, then when the perpendicular let fall from the point G on the section ABCD is not equal to the line GK, the variables $\zeta$ and $\theta$ will be no longer separated in equations (2) and (3), the first would contain a term multiplied by $\frac{d^2\theta}{dt^2}(y)$, and the second a term which has $\zeta$ for a factor; the motions of rotation and of the point G will be no longer independent of one another; and the moveable may perform four descriptions of simple oscillations, into which its motion of rotation can be always decomposed, agreeably to the principle of the coexistence of small oscillations, but which may be reduced to two in the particular case which we have considered ($b'$).

# CHAPTER V.

## OF THE MEASUREMENT OF HEIGHTS BY OBSERVATIONS OF THE BAROMETER.

619. It appears from No. 598, that the equation of equilibrium between the mercury contained in the closed branch of the barometer, and the pressure of the atmosphere in the open branch, is

$$mgh = \Pi; \qquad (1)$$

in which $\Pi$ denotes this pressure on the unit of surface, $g$ the gravity, $m$ the density of the mercury, and $h$ the difference of level of this fluid in the two branches of the barometer; in this equation it is assumed that there is no sensible pressure above its level in the closed branch.

As the equilibrium ought not to be deranged by supposing that the open branch of the barometer is extended to the limits of the atmosphere, it follows that the pressure $\Pi$ is the weight of a vertical cylindrical column of the atmosphere, whose base is the unit of surface, and the height that of this fluid. This weight is therefore equal to that of a cylinder of mercury, having the same base, and of which the height is about $0^m.76$; and it results from this, that the pressure of the atmosphere on each square metre of the surface of the earth is very nearly ten thousand kilogrammes($a$).

According as we ascend above this surface, the height and weight of the column of the air which presses on the mercury of the barometer, diminish more and more; consequently the height $h$ must also diminish, and there exists a relation between this height and that through which we have ascended, which will make one known by means of the other. This de-

termination will be the special object of this chapter; but it is necessary previously to deduce some consequences of importance to our subject from equation (1), and to explain the laws of the pressure of the air or of any gas whatever, relatively to its density and temperature.

620. If the total surface of the earth expressed in square metres be denoted by s, the mass of the atmosphere will be equal to $ms(0^m,76)(b)$; $m$ being always the density of the mercury; and if $r$ denote the radius of the earth, and $\delta$ its mean density, its mass is $\frac{1}{3}\delta s r$. If therefore $f$ represents the ratio of the first mass to the second, we shall have

$$f = \frac{3m(0^m,76)}{\delta r};$$

and as

$$2\pi r = 40000000^m, \quad m = 13^m,5975, \quad \delta = 5^m,50,$$

it follows that

$$f = 0,0000008854;$$

so that the mass of the atmosphere is a little less than a millioneth of that of the earth.

If the density of the air was the same throughout the entire extent of the atmospherical column, the height of this column, and $h$ the height of the mercury in the barometer, would be in the inverse ratio of the densities of the air and of the mercury; so that if $l$ denote the height of the atmosphere in this hypothesis, and $\rho$ the density of the air at the temperature of zero, and under a pressure of the barometer equal to $0^m,76$, we shall have

$$l = \frac{m}{\rho}(0^m,76);$$

and because (No. 61)

$$\frac{m}{\rho} = 10462,$$

there results very nearly

$$l = 7950^m.$$

It is evident that the atmosphere extends much higher than this(*c*), since the density and weight of its strata diminish according as their distance from the surface of the earth increases. A limit to which it cannot attain, is obtained by determining the height at which the centrifugal force is equal to the gravity, for beyond this, the centrifugal force would disperse its molecules in space. This limit is less at the equator than at any parallel (*d*). Now, in this place, the centrifugal force is $\frac{g}{289}$ (No. 178) at the surface of the earth; at a height $z$ above this surface it becomes $\frac{g(r+z)}{289r}$, and the intensity of gravity is $\frac{gr^2}{(r+z)^2}$, $r$ being the radius of the earth; therefore the limit in question will be determined by the equation

$$\frac{r+z}{289r} = \frac{r^2}{(r+z)^2};$$

from which we obtain, for this limit (*e*),

$$z = r(\sqrt[3]{289} - 1);$$

But there is reason to think that the air long before it reaches to so great a height is liquefied by the cold, which increases rapidly according as we ascend in the atmosphere(*f*). The law of this increase, when the air is free from the influence of surrounding objects, is altogether unknown to us, for it should not be confounded with that which is observed on the tops of mountains, where the temperature of the air and that of the ground mutually influence each other; the only datum which we have on this subject is that which results from observations made by M. Gay-Lussac in a balloon, in which he ascended to a height of $6980^m$; the simultaneous temperatures of the air at the surface of the earth and at this height were about 30,75, and − 9,50 of the centrigrade thermometer; which, on the supposition that the heat decreases

uniformly, gives about a diminution of one degree for every $175^m$ of elevation.

621. If the open branch of the barometer be closed at C (fig. 44), or, more generally, if this branch be placed in communication with a vessel H closed on all sides (fig. 52), the equilibrium of the system will not be deranged; and it is the pressure exerted at E by the air contained in this vessel, which is in this case in equilibrio with that of the column of mercury, DF, suspended in the closed branch, above E its level in the open branch. Therefore, the measure of this pressure on the unit of surface, or what is termed the elastic force of the air, will be $mgh$, which is the pressure of the mercury, and it will be equivalent to $\Pi$, the weight of the corresponding column of the atmosphere. A barometer which thus opens into a closed vessel, and by this means enables us to measure the elastic force of the air, or of any fluid whatever, is then termed a *manometer*; $mgh$ the weight of the mercury, depends on the nature, the density, and the temperature of this elastic fluid.

If when the *manometer* is transferred from one place to another, the density and the temperature of the air contained in the vessel H remain the same at these two places, the height of the mercury must vary in the inverse ratio of the gravity, in order that the weight of the column of mercury may continue to be the same. Hence by measuring this height at different latitudes, the variations of gravity can be determined; but in order that this measurement may be made exactly, the variation of the volume occupied by the air contained in H, which results from a corresponding variation of the height of the mercury in the closed branch, should be taken into account.

Thus, if $g$ and $h$ denote the gravity and the height DF of the mercury in a given latitude, then if, when the manometer is transferred to another parallel of latitude, we suppose that, the temperature being the same, the mercury rises in the closed branch from D to D′, and that, at the same time, it falls in the open branch from E to E′, if through the point E′,

a horizontal plane be drawn, cutting the closed branch in F', and if D'F' the difference of level of the two branches be denoted by $h'$, and the corresponding gravity by $g'$, the barometrical pressures in the two places will be as $gh$ and $g'h'$, and they will be proportional to the densities of the fluid contained in H, and consequently, in the inverse ratio of the volumes which it occupies in this vessel; therefore, if these volumes be denoted by V and V', we shall have

$$\frac{g'h'}{gh} = \frac{\text{V}}{\text{V}'}.$$

Now, if $c$ denotes the area of the horizontal section of the tube at the point D, the volume of mercury comprised between D and D' will be $(h' - h)c$; but, in consequence of the incompressibility of this fluid, it is necessary, that, whatever be the form of the vessel H, the volume V' should surpass V by this quantity $(h' - h)c$; therefore we shall have

$$\text{V}' = \text{V} + (h' - h)c;$$

and by substituting this value of V' in the preceding equation, we obtain for the ratio of the intensities of gravity at the two places

$$\frac{g'}{g} = \frac{\text{V}h}{[\text{V} + (h' - h)c]h'}.$$

But however carefully the quantities which occur in this formula are measured, this process is not susceptible of the same degree of precision, as that founded on experiments made with the pendulum.

622. If now the open branch of the barometer be closed at C (fig. 44), and the branch that was closed at A be opened, the pressure of the atmosphere will be added to that of the mercury, the air contained in EC will be compressed, and consequently the level of the mercury will ascend in this branch and descend on the other. If an additional quantity of mercury be poured into this branch, so that the difference of level may

be still equal to $h$, as before, and if, in this state, the mercury ascends from D to D′ and from E to E′, then when a horizontal plane is drawn through the point E′, cutting the other branch in F′, we will have D′F′ = DF. At the point F′, the pressure of the mercury will be $mgh$; by adding to it $\Pi$ the pressure of the atmosphere at the level D′, we shall have the entire pressure equal to $mgh + \Pi$ or $2mgh$; consequently, the elastic force of the air contained in CE′, which acts on the level E′, and which is in equilibrio with the total pressure, will be double of that which has place when the air occupies the space CE. Now, it appears from experiment that the space CE′ is half of the space CE; it likewise appears that if the pressure be tripled by a suitable addition of mercury, the space occupied by the air is reduced to a third, and, generally, the volume of the fluid is found to vary inversely as the pressure which it experiences, or in other words, the density increases in the same ratio as the elastic force.

This proportionality has been termed *the law of Mariotte*, from the name of the philosopher who first deduced it from observation. It implies that the fluid does not experience any change of temperature; so that in order to observe it exactly, the air contained in CE should get time to lose the increase of temperature, which it acquires by compression, and thus revert to its primitive temperature. This law obtains for all gases, and also for vapours, provided that in this last case, the pressure is less than that by which they are reduced to a liquid state. Finally, it also has place when different elastic fluids are mixed together; and if, for example, two gases have the same temperature and volume v, and if $p$ be the elastic force of one, and $p'$ that of the other, then when they are so intermixed as that their sum may occupy the same space v, the temperature of the mixture will be still the same, and the pressure or elastic force exerted on the unit of surface will become $p + p'$.

623. It is easy, by means of the law of Mariotte, to calculate the quantity by which water ascends in a pump, when

there is some air between this liquid and the piston: (it has been already stated (No. 598), that when the water is in contact with the piston, the space through which the water then rises, is $10^m,4$.) For this purpose, let ABCD (fig. 53) be the vertical cylindrical tube of a pump immersed as far as EF in water, and GH the horizontal base of the piston. The atmospherical pressure which is exerted on the exterior level of the water, that coincides with the interior level EF, is equal to $gl$ on each unit of surface, the density of the water being supposed to be unity, and $l$ equal to $10^m,4$ the height of the column of water which is in equilibrio with it; as the space contained between EF and GH is filled with air, whose elastic force is in equilibrio with the exterior pressure, $gl$ is also the measure of this force. In this state, let $a$ be the given height of GH above EF; if then the piston ascend to G′H′, and if $c$ denote the given height of G′H′ above GH, the water will ascend in the interior of the pump to a height E′F′ above EF, which we shall denote by $x$, and it will fall outside to the level of $E_{,}F_{,}$, a section of the pump situated at a distance $y$ below EF. If the horizontal section of the pump be denoted by $b$, and that of the reservoir in which it is immersed by $\beta$, we shall have at once, on account of the incompressibility of the liquid ($g$),

$$y = \frac{bx}{\beta};$$

and the question will be reduced to the determination of the value of $x$.

But the air which did occupy the space EFGH, will now occupy the space E′F′G′H′, and as this last is to the other in the ratio of $a + c - x$ to $a$, it follows, that the elastic force $gl$ will be diminished in the inverse ratio, and will become $\frac{gla}{a+c-x}$. This will be the pressure on the unit of surface at the section E′F′; and if it be added to the pressure of water

contained between $E'F'$ and $E_{,}F_{,}$, the value of which is $g(x+y)$, we will obtain the entire pressure exerted on the interior level $E_{,}F_{,}$, which should constitute an equilibrium with the exterior pressure $gl$; this requires that when $g$, the common factor, is suppressed, and the preceding value is substituted for $y$, we should have

$$\frac{la}{a+c-x}+x+\frac{bx}{\beta}=l.$$

This equation is, when reduced, of the second degree, and may be written as follows,

$$x^2-x(lf+a+c)+clf=0,$$

in which, we suppose for conciseness, that

$$\frac{\beta}{\beta+b}=f.$$

We obtain, by solving this equation, two real and positive values for $x$; but it is easy to perceive that one of them is always inadmissible. In fact, $x+y$, the elevation of the water above the exterior level, cannot exceed $l$; and as $x+y=\frac{x}{f}$ we must therefore have $x \angle fl$(h); moreover, it is evident that we must always have $x \angle a+c$. Now as the sum of the two roots of the preceding equation is $fl+a+c$, if one of them is less than $a+c$, the other will be greater than $fl$, or if one of them is less than $fl$, the other will be greater than $a+c$; consequently only one of the two roots will be admissible, and the other should be rejected as not belonging to the question(i).

In virtue of the equation of equilibrium, $\frac{gla}{a+c-x}$ the elastic force of the air contained between $E'F'$ and $G'H'$, is equal to $gl-g(x+y)$. There results from this a pressure on the lower base of the piston acting in a direction opposite to that of gravity, and equal to $glb-g(x+y)b$. The upper base of this body is urged in an opposite direction by the at-

mospherical pressure, which is equal to $glb$; therefore the load which the piston sustains, or the excess of this second force over the first, is $g\,(x+y)b$, which is the weight of the water contained between $E_{,}F_{,}$ and $E'F'$, and raised above the exterior level; this indeed may be considered as evident *à priori.*

624. It remains for us now to consider the law of the elastic force of the air, relatively to its temperature.

It appears from observation that if the air and different gases are all of them subjected to the same constant pressure, and placed in an enclosed receiver, the temperature of which varies at each instant, they are equally dilated. If one of them, as the air, for example, be taken as a thermometer, that is to say, if its total dilatation be divided into equal parts, which will indicate the degrees of the temperature, it results from this that the increments of volume of all these gases will be the same for equal increments of temperature, and proportional to these increments. It appears likewise from observation, that for a very considerable range, the indications of this aerial thermometer differ very little from those of the mercurial thermometer; so that, within this range, the dilatation of any gas whatever is proportional to its increase of temperature, as indicated by the degrees of the common thermometer. Finally, M. Gay-Lussac has found, that from zero to 100°, that is to say, from the temperature of melting ice, to that of boiling water, the volume of air subjected to a constant pressure, and consequently that of any gas whatever, increases in the ratio of unity to 1,375, which gives a dilatation of 0,00375 for each degree of the centrigrade thermometer ($k$).

Hence if $V$ be the volume of any gas whatever at a temperature equal to zero, $\Pi$ its elastic force, and $D$ its density, and if when the number of degrees of temperature becomes $\theta$, $\Pi$ the pressure on the unit of surface remains the same, we shall have ($l$)

$$V' = V(1 + a\theta),$$

v′ being what v becomes when the temperature is increased $\theta$ degrees, $\alpha$ being supposed equal to 0,00375; likewise if D′ be what D becomes in the same case, we shall also have, as the density varies in the inverse ratio of the volume,

$$D' = \frac{D}{1 + \alpha\theta}.$$

Now, if the pressure be supposed to vary while the temperature $\theta$ remains the same, and if $p$ and $\rho$ be what the pressure $\Pi$ and the density D′ become simultaneously, we shall have

$$p = \frac{\rho\Pi}{D'},$$

by the law of Mariotte; and by making

$$\frac{\Pi}{D'} = k,$$

there will result $(m)$

$$p = k\rho(1 + \alpha\theta) \qquad (2)$$

for the expression of the elastic force of any gas whatever, in a function of its density and temperature.

625. This formula is applicable to gases, vapours, and their mixtures. If the temperature be indicated by the mercurial thermometer, it has place for negative values of $\theta$ to about $-36°$, or a little less than the temperature at which this fluid congeals. It has also been verified for temperatures which are considerably greater than 100°; and the difference between the laws of the dilatation of air and mercury does not commence to become at all considerable until $\theta$ ascends to about 300°. This has been established in a memoir of MM. Petit and Dulong, inserted in the eighteenth Number of the Journal of the Polytechnic School.

The coefficient $k$ is different for different fluids. Relatively to the atmospheric air, MM. Biot and Arrago have found, at the Observatory of Paris,

$$\frac{m}{\mathrm{D}} = 10462,$$

for the ratio of the density of mercury to that of the air, when the atmospherical pressure is $0^m,76$, and the temperature zero (No. 61). Therefore, if we make at the same time($n$),

$$\Pi = m\mathrm{G}h, \quad h = 0^m,76,$$

the value of $k$ will be

$$k = (7951,12)\,\mathrm{G};$$

and, in this value, G should be taken equal to the gravity of the place at which the ratio $\frac{m}{\mathrm{D}}$ has been determined, that is to say, at the latitude of Paris, for which we have

$$\mathrm{G} = 9^m,80896.$$

The coefficient of G has been determined on the supposition that the air is perfectly dry; if it was humid, its density would be less under the same pressure and temperature, and the value of $k$ would vary in the inverse ratio of this density. If, for example, $\delta$ be the ratio of the density of the air at the *maximum* of humidity, that is, when it is saturated with moisture, to the density of the air, when perfectly dry, under a pressure of the barometer equal to $0^m,76$, and at the temperature zero, we shall have, as will be seen farther on,

$$\delta = \frac{0^m,76 - 0^m,00508}{0^m,76} + \frac{10}{16}\cdot\frac{0^m,00508}{0^m,76} = 0,99749;$$

consequently, if the preceding value of $k$ be divided by this value of $\delta$, the value of $k$ which corresponds to the maximum of humidity will be obtained, namely,

$$k = (7971^m,09)\,.\mathrm{G}.$$

626. We are now in a condition to form, without any difficulty, the different equations of equilibrium of the atmospherical column. Let us suppose that this column is a vertical cylinder, which extends to the limits of the atmosphere,

and having its horizontal base A resting on the surface of the earth. Let this column be divided into an infinite number of slender horizontal strata, and let the extent of the surface A be supposed to be such that neither the density nor temperature can vary from one point to another of any stratum. Let $p$, $\rho$, $\theta$ denote the elastic force, density, and temperature of the air, at the distance $z$ from the surface of the earth; likewise let $g'$ be the gravity at this same distance; $\mathrm{A}g'\rho dz$ will be the weight of the stratum of which the thickness is $dz$, and whose two faces correspond to $z$ and $z - dz$. The pressure which it experiences on its upper face will be $\mathrm{A}p$, that which acts against its inferior face will be $\mathrm{A}(p - \frac{dp}{dz}dz)$. But the pressure ought to increase by the weight $\mathrm{A}g'\rho dz$, in passing from the first to the second face; therefore we should have

$$\mathrm{A}\,(p - \frac{dp}{dz}\,dz) = \mathrm{A}p + \mathrm{A}g'\rho dz,$$

or simply (o)

$$dp = -g'\rho dz; \tag{3}$$

which coincides with the equation that might be deduced from formula (3) of No. 583, by making

$$\mathrm{X} = 0, \quad \mathrm{Y} = 0, \quad \mathrm{Z} = -g'.$$

If $\rho$ be eliminated by means of equation (2), there results

$$\frac{dp}{p} = -\frac{g'dz}{k\,(1 + \alpha\theta)}.$$

Let $r$ be the radius of the earth, and $g$ the gravity at its surface, we shall have

$$g' = \frac{gr^2}{(r + z)^2},$$

at the height $z$, and if the variation of the centrifugal force be neglected, and also the action of the mass of air comprised between the two concentrical spherical surfaces, whose radii are $r$ and $r + z$, (which action does not enter in the value

of $g$, and should be added to that of $g'$ (No. 101)), we shall have

$$\frac{dp}{p} = -\frac{gr^2dz}{k(1+\alpha\theta)(r+z)^2}.$$

In order to integrate this formula, the expression of $\theta$ in a function of $z$ should be known; but as the law of the temperatures is unknown, $\theta$ should be considered as a constant quantity; and its value in each case should be assumed to be the mean of the temperatures which belong to the extreme points of the column of air to which this equation is applied. Its integral is then

$$\log p = \frac{gr^2}{k(1+\alpha\theta)(r+z)} + \mathrm{c};$$

c being an arbitrary constant. If $\Pi$ denote the atmospherical pressure which has place at the surface of the earth, we shall have at the same time

$$z = 0, \quad p = \Pi, \quad \log \Pi = \frac{gr}{k(1+\alpha\theta)} + \mathrm{c};$$

and there will result from it at $z$, any height whatever above this surface $(p)$,

$$\log \frac{p}{\Pi} = -\frac{grz}{k(1+\alpha\theta)(r+z)}. \qquad (4)$$

In virtue of this equation and of formula (2), we shall have

$$p = \Pi e^{\frac{-grz}{k(1+\alpha\theta)(r+z)}}, \quad \rho = \frac{\Pi}{k(1+\alpha\theta)} e^{\frac{-grz}{k(1+\alpha\theta)(r+z)}},$$

for the expressions of the elastic force and density of the air, from the surface of the globe, to the height at which this fluid loses, by the effect of cold, all its elasticity.

From what has been stated in No. 619, it follows that the weight of a column of the atmosphere, the base of which is the unit of surface, must be equivalent to $\Pi$ the pressure relative to the lowest point; and, in fact, this weight is given by the

integral $\int_0^\infty \rho g' dz$, of which it is evident that, if the expressions of $\rho$ and $g'$ be substituted for these quantities, the value is $\Pi(q)$.

627. The motive force of a balloon which ascends *vertically* in the atmosphere, is the excess of the weight of the air which it displaces at each instant, over its own weight. Therefore, if its mass be denoted by $\mu$ and its volume by v, this force will be $\mathrm{v}\rho g' - \mu g'$ at the height $z$ above the earth; consequently, we shall have,

$$\mu \frac{d^2z}{dt^2} = \mathrm{v}\rho g' - \mu g',$$

for the differential equation of the vertical motion of this body at the end of $t$ any time whatever. If its mean density be denoted by $c$, in which case we have $\mu = c\mathrm{v}$, and if their preceding values be substituted for $\rho$ and $g'$, this equation will become

$$c\frac{d^2z}{dt^2} = \frac{\Pi g r^2}{k(1+\alpha\theta)(r+z)^2} e^{\frac{-grz}{k(1+\alpha\theta)(r+z)}} - \frac{cgr^2}{(r+z)^2}.$$

By multiplying by $2dz$, then integrating and denoting the constant arbitrary by C, it becomes($r$)

$$c\frac{dz^2}{dt^2} = \mathrm{C} - 2\Pi e^{\frac{-grz}{k(1+\alpha\theta)(r+z)}} + \frac{2cgr^2}{r+z}.$$

If the balloon ascends from the surface of the earth without any initial velocity, we shall have at the same time $z = 0$, and $\frac{dz}{dt} = 0$; consequently, we must have

$$\mathrm{C} = 2\Pi - 2cgr;$$

and there will result for the square of the velocity at any height whatever such as $z$ ($s$),

$$\frac{dz^2}{dt^2} = \frac{2\Pi}{c}\left[1 - e^{\frac{-grz}{k(1+\alpha\theta)(r+z)}}\right] - \frac{2grz}{r+z}.$$

By resolving this equation with respect to $dt$, the time which the balloon takes to attain a given height may be determined by the method of quadratures.

By making the value of $\frac{d^2z}{dt^2}$ equal to cipher, we can determine the height at which the moveable would remain in equilibrio, if it reached it without any acquired velocity; and in like manner, the equation $\frac{dz^2}{dt^2} = 0$, will make known the greatest height to which the balloon can ascend in the atmosphere, on the supposition that neither its mass or volume undergo any change. The first of these two heights will be expressed by means of a logarithm; the second will depend on a transcendental equation, and can only be calculated by approximation ($t$).

628. Let us now apply equation (4) to the measurement of vertical heights.

Let $h$ and $h'$ be the heights of the mercury in the barometer at the inferior and superior station; T and T′ the corresponding temperatures of mercury, $t$ and $t'$ those of the air, which will be different from T and T′, when the mercury in the barometer has not sufficient time, during the observations, to acquire the temperature of the surrounding air. If $m$ be the density of the mercury at the inferior station, $\left(1 + \frac{\text{T} - \text{T}'}{5550}\right) m$ will be its density at the superior station, because the density increases by $\frac{1}{5550}$ for each degree of diminution in the temperature. For greater simplicity, let the factor $1 + \frac{\text{T} - \text{T}'}{5550}$ be comprehended in the height $h'$, which will then be the observed height multiplied by this quantity, and we will afterwards suppose that $m$ is the density of the mercury at the two stations. In this way we shall have

$$\Pi = mgh, \quad p = mg'h' = \frac{mgr^2h'}{(r+z)^2};$$

by means of which, equation (4) will become($u$)

$$\log \frac{h}{h'} + 2\log \frac{r+z}{r} = \frac{grz}{k(1+a\theta)(r+z)}. \qquad (5)$$

Agreeably to what is stated above, $\frac{1}{2}(t+t')$ should be taken for $\theta$. The value of $a$ is 0,00375 for dry air as well as for that which contains a constant quantity of aqueous vapour, and in any proportion. But it should be observed, that when the temperature rises, the quantity of vapour in the atmosphere, in general, increases; and, as under the ordinary pressures of the atmosphere, namely, $0^m,76$, the density of vapour is to that of air in the ratio of 10 to 16, it follows, that the density of the free air, the temperature of which increases, must diminish in a greater ratio than that which corresponds to the coefficient, 0,00375. Therefore, in order to take into account, as far as possible, the quantity of vapour which exists in the atmosphere, the coefficient $a$ should be increased, and for the convenience of the calculation we will make it equal to 0,004; so that we shall have($v$)

$$a\theta = \frac{2(t+t')}{1000}.$$

As the logarithms which occur in the first member of the preceding equation are Naperian, in order to convert them into vulgar logarithms, they should be multiplied by M the modulus of these last, the value of which is

$$\mathrm{M} = 0,4342945;$$

$g$ the expression for gravity, which its second member contains, is that which belongs to the inferior station, and to the latitude of the place of observation. If this angle be denoted by $\psi$, we shall obtain by comparing the gravity $g$ with that which occurs in the values of $k$ of No. 625, and which corresponds to the latitude of Paris($x$),

$$g = \frac{(1-0,002588\cos 2\psi)\,\mathrm{G}}{1-0,002588\cos 2(48°50'14'')}$$

Moreover, as in the coefficients of G in these two values of $k$, the one belongs to the state of extreme dryness of the air, the other, to the air when saturated with humidity, we can take the semi-sum of these two quantities, or $7961^m,10$ for the coefficient which corresponds to the ordinary state of the atmosphere. We shall then have($y$)

$$\frac{k}{M}[1-0,002588 \cos 2(48°50'14'')] = (18337^m,46)G;$$

and by means of this value combined with that of $g$, we shall obtain from equation (5) ($z$),

$$\left.\begin{aligned} z = \frac{18337^m,46\left[1+\frac{2(t+t')}{1000}\right]}{1-0,002588 \cos 2\psi}\Big[\log\frac{h}{h'} \\ +2\log\left(1+\frac{z}{r}\right)\Big]\left(1+\frac{z}{r}\right); \end{aligned}\right\} \quad (6)$$

in which formula the logarithms are actually those belonging to the system, the base of which is the number 10.

In applying this equation, we in the *first* place neglect $\frac{z}{r}$, the fraction contained in its second member; this will make known the first approximate value of $z$, and by substituting it in this second member, a second value of $z$ will be obtained, more accurate than the first, beyond which the approximation need not be continued.

If an unknown coefficient $a$ be substituted for the number 18337,46 in this equation, and if $a$ be then determined by means of a great number of heights $z$, measured trigonometrically, we find

$$a = 18336^m;$$

which differs very little from the coefficient $18337^m,46$, that was directly calculated.

629. If it was proposed to employ formula (6) to determine the elevation of a place on the earth above the level of the sea; $T'$, $t'$, $h'$, should be supposed to refer to this place and

T, $t$, $h$, to the shore of the nearest sea; and, for greater accuracy, $\psi$ should be taken equal to the mean latitude between those of these two points. Agreeably to what has been remarked in No. 255, the action of the stratum of the earth, the height of which is $z$, should be also taken into account; on the supposition that its density is equal to half of the mean density of the earth, we have then

$$g' = \frac{gr^2}{(r+z)^2} + \frac{3gz}{4r};$$

and, as the fraction $\frac{z}{r}$ is very small, when this value of the gravity $g'$ is made use of, the quantity $1+\frac{z}{r}$, contained in formula (6), should be replaced by $1+\frac{5z}{8r}(a')$.

In order to give an example of this formula thus modified, let us take that which has been cited in the yearly register of the Bureau of Longitudes, in which the height of Guanaxuato above the level of the sea has been determined.

The data of this example are

$$h = 0^m,76315, \quad \text{T} = t = 25°,3,$$
$$h' = 0^m,60095, \quad \text{T}' = t' = 21°,3, \quad \psi = 21°.$$

The height $h'$, when corrected and multiplied by the factor $1+\frac{\text{T}-\text{T}'}{5550}$, which should be employed in formula (6), becomes

$$h' = 0^m,60138.$$

If the fraction $\frac{z}{r}$ be neglected in this formula, we find for the first approximate value of $z$

$$z = 2077^m,98;$$

and if this value be substituted in the same formula, by putting $1+\frac{5z}{8r}$ in place of $1+\frac{z}{r}$, as has been stated above, and ob-

serving that $r = 6366198^m$, we obtain $z = 2081^m,96$, for the required height, which is less than that given in the register, by about 2 metres and one-half very nearly.

630. When $z$ the height is not very considerable, the fraction $\frac{z}{r}$ may be neglected in formula (6), and the number $18337^m,46$ should at the same time be replaced by a coefficient somewhat greater. That which results from numerous observations made by Raymond in the south of France, is 18393 metres, and as the corresponding latitude is very nearly $\psi = 45°$, equation (6) is reduced to

$$z = 18393^m\left[1 + \frac{2(t+t')}{1000}\right]\log\frac{h}{h'};$$

which is the barometrical formula that is commonly made use of.

In the same vessel, and under the same atmospherical pressure, the ebullition of distilled water commences always at the same temperature, and this temperature is less according as the external pressure diminishes. If, therefore, a table is formed, by experiment, of the temperatures at which water begins to boil under pressures that decrease by very small differences, and if a vessel containing the water be carried to different heights above the earth, the temperatures at which this water commences to boil will make known, by means of the above-mentioned table, the corresponding barometrical heights, which should be employed in the preceding formula. It is in this manner that philosophers have proposed to determine the differences of height above the earth, by observing at what temperature water begins to boil, and without having recourse explicitly to measurements of the barometer.

631. We shall conclude this chapter with some remarks on the weight and elastic force of vapours, which force has also been termed the *tension* of these fluids.

If a closed vessel contains any liquid in sufficient quantity to furnish all the vapour which can be generated in it, that which rises from this liquid attains, more or less quickly, a

*maximum* which depends solely on the temperature, and which is the same when the vessel is void of air, and when it contains air or any gas whatever, in a condensed or dilated state. Likewise, whether this quantity of vapour has reached its *maximum*, or is still below this point, its tension is added to that of the elastic force of the dry gas, and the sum constitutes the elastic force of the mixture. At the same temperature, the *maximum* tension is different in different vapours; and the law which it follows for the same vapour, in a function of the temperature, is not yet known. The most extensive experiments which have been hitherto made on the vapour of water, are those which are detailed in the tenth and eleventh volumes of the Academy of Sciences, and in Taylor's Scientific Memoirs, Part 8. The maximum elastic force of the vapour of water formed in a vacuo at the temperature of $18°,75$, for example, is measured by an elevation of mercury in the barometer equal to $0^m,016$. When it is produced in perfectly dry air, at this temperature and under the ordinary pressure of $0^m,76$, its elastic force is added to this pressure, and it appears from experiment, that in fact the pressure of the mixture is $0^m,776$.

If it was possible, without liquefying unmixed vapour, to raise its tension from $0^m,016$ to $0^m,76$, its density would, by the determination of M. Gay-Lussac, be to that of dry air, under the same pressure and at the same temperature, in the ratio of 10 to 16. Therefore, in virtue of the law of Mariotte, the density of the vapour which has been taken for our example is $\frac{10}{16} \cdot \frac{0,016}{0,76}$; that of air at the temperature of 18,75, and under the pressure of $0^m,76$, being taken for unity ($b'$). Consequently, if P denote the weight of a litre of air, and P′ that of a litre of vapour of water, we shall have

$$P' = \frac{P}{76}.$$

At the temperature zero, P would be equal to 1000 grammes

divided by 769,4 (No. 61); as the density of the air varies inversely as the volume, in order to obtain its value at the temperature of 18°,75, this quotient should be divided by 1 + (18,75) (0,00375); hence it follows that

$$\text{P} = 1^{gr}, 21433,$$

and consequently($c'$)

$$\text{P}' = 0^{gr}, 01597.$$

The weight of a litre of vapour at the temperature of 18°,75, and at its *maximum* of density, must also be that of the greatest quantity of vapour which a litre of air can contain at this temperature, whatever its density may be. Now, Saussure found, by direct experiment, that the weight of the greatest quantity of vapour which can be formed in a cubic foot of air, under the ordinary pressure of the atmosphere, and at the given temperature, is 10 grains; hence it follows, that it is 0,01346 for the cube, one of whose sides is a deci-metre, which differs very little from the preceding result. Generally, if $\Delta$ denotes the density of the dry air, $\Delta'$ that of air charged with moisture, and $a$ the tension of the vapour which it contains, we shall have ($d'$)

$$\Delta' = \frac{\Delta}{h}\left(h - a + \tfrac{10}{16}\, a\right);$$

for any volume whatever of air charged with moisture such as A, will consist of an equal volume of dry air, the elastic force of which is reduced to $h - a$, and the density to $\frac{\Delta}{h}(h - a)$, added to an equal volume of vapour, the density of which will be equal to $\frac{10}{16}\frac{a}{h}\Delta$; consequently the sum of these two densities multiplied by A, will express the mass $\text{A}\Delta'$ of the mixture; and by suppressing the common factor A, the preceding equation will be obtained. This equation will enable us to determine the weight of a given volume of air charged with moisture, when the weight of this volume of dry air at the same temperature, and the tension of the vapour contained in the air charged with moisture are given. The value of $\delta$ of No. 625 may be deduced from it by making $h = 0^m, 76$, and assuming

that the temperature is zero, in which case, the maximum tension of the vapour of water is $0^m,00508$.

632. If the atmosphere which envelopes us did not exist, it would be replaced by another atmosphere formed of the aqueous vapour which rises from the sea. The law of the densities of its strata, and its total height, would depend on the law of the temperature which would have place in this aqueous atmosphere, and which cannot by any possibility be known; but, whatever it may be, the entire weight of a vertical cylindrical column of this vapour, whose base is the unit of surface, will be always equal to the elastic force which belongs to its lowest point (No. 626); and when its density at this point attains its maximum, this force will only depend on the corresponding temperature(*e*′). Indeed, we are also ignorant what this temperature should be; but there is reason to think that it would be much lower than that which has place now at the surface of the earth, since the fluid that would then be in contact with this surface should have a density considerably less than that of the ordinary air(*f*′). For greater clearness, let the temperature in question be still 18°,75, then the weight of the aqueous vapour, the base of which is the square of the decimetre, cannot exceed that of a prism of mercury which would have the same base, and $0^m,016$ for height, that is to say, the weight of sixteen centiemes of a litre of mercury, or very nearly 2300 grammes. The weight of all the aqueous vapour which may be contained in a column of air of our atmosphere, depends on the law of the decrement of temperature in a vertical direction, and cannot be calculated; but if the base of the column is the square of the decimetre, and the inferior temperature 18°,75, it is easy to be satisfied that this weight must exceed 2300 grammes, as is evident by considering that throughout all the part of this column, the temperature of which differs little from 18°,75, and unto a height where the pressure will not be reduced to $0^m,016$, each litre of air may contain about sixteen milligrammes of vapour.

Thus the atmosphere which presses on the surface of the earth does not, as was formerly supposed, prevent liquids from being reduced into vapour, and thus dissipated into space;
9 on the contrary, its presence permits vapours to exist above the earth, in greater abundance than if there was no atmosphere.

## CHAPTER VI.

### OF THE ELASTIC FORCE, AND HEAT OF GAS.

633. The law of Mariotte obtains, as has been already stated (No. 622), on the supposition that the rarified or condensed fluid has had time to revert to its primitive temperature. If this precaution is not taken, the temperature increases or diminishes with the density, and hence as the elastic force also varies in the combined ratio of the density and temperature, it is easy to conceive that it must vary, for the same fluid, in a greater ratio than its density. When the fluid is contained in a vessel, whose sides are impermeable to heat, it retains all its caloric, while it is being condensed or dilated, and consequently, its temperature is accordingly increased or diminished. The same is the case whenever the variations of its density are so rapid, that its proper heat has not time, in the case of condensation, to escape in a radiating form, or to communicate itself by contact to the surrounding bodies, and, in the case of dilatation, for these bodies to communicate to the fluid, by radiation or by contact, a sensible quantity of caloric. This is the supposition that is made for example, as will be explained in the sequel, relatively to the variations of density which have place in sonorous waves, the duration of which is only some thousandths of a second.

In this question, and many others, it is important to know the expression of the elastic force of a gas in a function of the density and corresponding elevation or depression of the temperatures, when the quantity of heat of the fluid mass remains invariable. But in the present state of our knowledge, we have not the data requisite for the complete solution of this pro-

blem; and in this chapter, it is proposed to explain what, up to the present time, we have been enabled to deduce from the calculus and from experiment on this subject.

634. Let $\rho$ be the density of a gas, $\theta$ its temperature in degrees of the centigrade thermometer, and $p$ the pressure which it exerts on each unit of the surface, or the measure of its elastic force, we shall have (No. 624)

$$p = k\rho(1 + \alpha\theta); \tag{1}$$

$\alpha$ and $k$ being two coefficients independent of $\rho$ and $\theta$, of which the first is the same for all gases, and equal to 0,00375, and the second must be given for each gas in particular.

The absolute or entire quantity of heat contained in a given weight of any body, as for example in a gramme, cannot be calculated; it is considered as inexhaustible, and as extremely great relatively to the quantities by which it varies, when the density or temperature of this body is changed; and it is these variations, that is to say, the quantities which are added or subtracted, that are to be compared together, and submitted to calculation. Thus if $q$ denotes the excess of the quantity of heat contained in the gramme of gas, which is considered, over that which this gramme contains, when the gas has any temperature and density whatever, as for example, the temperature zero, and the density corresponding to the ordinary pressure of $0^m,76$, this quantity $q$ will be a function of $p$, $\rho$, $\theta$, or simply of $p$ and $\rho$, since these three variables are connected together by the preceding equation. Therefore we shall have

$$q = f(p, \rho);$$

in which $f$ denotes a function, the form of which it will be our object to determine. The specific heat of this gramme of the fluid is the quantity of heat which should be communicated to it in order to raise its temperature one degree, or so to speak, the increment of $q$ with respect to $\theta$, the expression of which will therefore be $\frac{dq}{d\theta}$. Now it may be

considered under two different points of view, we may either suppose the pressure $p$ to be constant, and at the same time that the gas can dilate itself, or, we may assume that the volume is constant, and then suppose that the pressure $p$ varies with the temperature. In virtue of equation (1), we have

$$\frac{d\rho}{d\theta} = \frac{-a\rho}{1+a\theta};$$

when $p$ is considered as constant, and($a$)

$$\frac{dp}{d\theta} = \frac{ap}{1+a\theta}$$

when $\rho$ is supposed to be invariable. If therefore the specific heat of the gas under a constant pressure be denoted by $c$, and its specific heat when the volume is constant by $c'$, so that we may have

$$c = \frac{dq}{d\rho}\frac{d\rho}{d\theta}, \quad c' = \frac{dq}{dp}\frac{dp}{d\theta},$$

there will result

$$c = -\frac{dq}{d\rho}\frac{a\rho}{1+a\theta}, \quad c' = \frac{dq}{dp}\frac{ap}{1+a\theta}, \tag{2}$$

and, consequently,

$$\rho\frac{dq}{d\rho} + \gamma p\frac{dq}{dp} = 0, \tag{3}$$

in which $\gamma$ denotes the ratio of the two specific heats, that is to say($b$),

$$\gamma = \frac{c}{c'}.$$

It is evident, *a priori*, that this ratio $\gamma$ must surpass unity, for more heat is required to increase the temperature of a gas, and at the same time to dilate its volume, than to increase its temperature by the same quantity, without increasing the distance of the molecules from each other($c$). But the value of $\gamma$ for different gases, and the manner in which it depends on the pressure and density, can only be determined by experiment. It will be shown immediately, that this value may be deduced

from the increment of temperature, which takes place when a small condensation is made in the gas, without any loss of heat.

635. Let the temperature of the gas be denoted as before by $\theta$, and let $\theta + \omega$ be what it becomes when the density of the fluid has been increased, by a very sudden compression, in the ratio of $1 + \delta$ to unity; $\delta$ being a very small fraction. If the loss of heat during the continuance of this compression had been insensible, $\omega$ the increment of temperature corresponding to the very small condensation $\delta$, would be the quantity which it is proposed to determine by the following experiment.

For this purpose, let the gas be supposed to be the atmospherical air contained in a closed vessel, and let the pressure, the density, and temperature, be the same as without the vessel, where they are supposed to be represented by $p$, $\rho$, $\theta$, during the entire continuance of the observation. Let a small portion of the interior air be abstracted, and, when the remaining air has resumed its primitive temperature, let $p'$ and $\rho'$ denote its pressure and density; if then the communication with the exterior air be opened again, the pressure, density, and temperature will increase together; so that after a very short portion of time, the interior pressure becomes equal to that which has place outside; at this instant let the communication be again cut off, $\rho''$ and $\theta + \omega$ denoting the interior density and temperature; finally, after the lapse of some time, this increment of temperature $\omega$ is dissipated, and thus the interior pressure diminishes and becomes $p''$, without $\rho''$ undergoing any variation.

As the density of the interior air passes *very rapidly* from $\rho'$ to $\rho''$, if

$$\delta = \frac{\rho'' - \rho'}{\rho'},$$

and if the small quantity of heat which may be absorbed by the vessel, during the time of this passage, is not taken into account, the increment of temperature $\omega$ will be that which

corresponds to the condensation $\delta$, and is the quantity the value of which is required. The indication of a thermometer plunged in the interior air, would be too slow to make known this increase of temperature, as it only subsists for a very short time; but the value of $\omega$ may be obtained from knowing the three pressures $p$, $p'$, $p''$, or from the three corresponding barometrical heights, which there is sufficient time to observe.

In fact, there are two epochs in the experiment, which has been described, in which the same temperature $\theta$ corresponds to two different densities $\rho'$ and $\rho''$, and to two given pressures $p'$ and $p''$. Therefore, by the law of Mariotte, we have

$$\frac{\rho''}{\rho'} = \frac{p''}{p'},$$

and, consequently,

$$\delta = \frac{p'' - p'}{p'};$$

by means of which the condensation $\delta$ can be determined. Moreover, there are also two epochs in which the same density $\rho''$ has place for the temperatures $\theta + \omega$ and $\theta$, under the pressures $p$ and $p''$. Therefore by the law of elastic forces, when the densities are equal, we shall also have

$$\frac{p}{p''} = \frac{1 + a(\theta + \omega)}{1 + a\theta}; \qquad (4)$$

from which the value of $\omega$ corresponding to the condensation $\delta$ can be obtained($d$).

In an experiment made by MM. Desormes and Clement, in which the change of the density from $\rho'$ to $\rho''$ is effected in less than half a second, we have

$$p = 0^m,7665, \quad p' = 0^m,7527, \quad p'' = 0^m,7629;$$

which gives

$$\delta = 0,0133.$$

We have also $\theta = 12°,5$, and as $a = 0,00375$, we obtain from equation (4)

$$\omega = 1°,3173;$$

from which it follows that for a condensation of 0,0133, without any loss of heat, the temperature of the air will increase by 1°,3173, or very nearly a degree for a condensation(*e*)

$$\delta = \frac{0{,}01331}{1{,}3173} = 0{,}0101.$$

This increase of temperature may be also deduced from the velocity of sound; and in this way, the author found formerly that for a condensation of $\frac{1}{116}$, without any loss of heat, there was an increase of one degree; a result which does not differ much from that which has been just cited.

636. It now is easy to perceive that the expression of the ratio $\gamma$ of No. 634 is

$$\gamma = 1 + \frac{a\omega}{(1+a\theta)\delta}. \qquad (5)$$

In fact, if $p$ and $\theta$, denoting as above the elastic force and temperature of any gas, the condensation $\delta$ be supposed to be equivalent to that which the fluid experiences when its temperature is diminished by ever so small a quantity without changing the pressure, and if this small variation of temperature be denoted by $\varepsilon$, we shall have(*f*)

$$\delta = \frac{a\varepsilon}{1+a\theta}.$$

If the quantity of heat which should be communicated to a gramme of the gas that we are considering, in order to raise its temperature from $\theta - \varepsilon$ to $\theta$, without changing the pressure $p$, be denoted by $\Gamma$, we shall also have, the specific heat at a constant pressure being $c$,

$$\Gamma = c\varepsilon.$$

After this communication of heat, if this fluid be suddenly compressed, so as to be reduced to its primitive volume, it will then experience the condensation $\delta$; and if it has not lost any portion of heat, its temperature being increased by $\omega$, will become $\theta + \omega$. In this state, the pressure of the fluid

will be greater than $p$; but if, without changing the volume, the temperature is allowed to fall to $\theta - \varepsilon$, this pressure will also be diminished, and become again equal to $p$. During this depression, the gas will lose a quantity of heat proportional to the small diminution of temperature $\varepsilon + \omega$, and expressed by $c'(\varepsilon + \omega)$, because $c'$ is its specific heat at a constant volume. Since the volume, the temperature. and the pressure are the same after this loss of heat as they were before the quantity of heat $\Gamma$ had been communicated to the fluid, the loss of heat $c_{,}(\varepsilon + \omega)$ must be equal to $\Gamma$; therefore we have

$$c\varepsilon = c'(\varepsilon + \omega);$$

from which we obtain

$$\gamma = \frac{c}{c'} = 1 + \frac{\omega}{\varepsilon};$$

and it is evident, by substituting for $\varepsilon$ its preceding value, that this value of $\gamma$ coincides with that given in formula(5).

637. If, in this formula, $\frac{p'' - p'}{p'}$ be substituted in place of $\delta$, and for $\omega$ its value deduced from equation (4), we shall have

$$\gamma = 1 + \frac{(p - p'')\,p'}{(p'' - p')p''};$$

by means of which, the value of $\gamma$, which results from the experiment described above, will be known, when the pressures $p, p', p''$, or the corresponding barometrical heights, are given.

By substituting the numerical values of $p, p', p''$, cited above, we find

$$\gamma = 1{,}3482,$$

for the value of the ratio of the two specific heats $c$ and $c'$, in the case of atmospherical air.

MM. Gay-Lussac and Welter have obtained, by an analogous process, a value for this ratio somewhat different, namely,

$$\gamma = 1{,}3748;$$

and they have established, that this quantity is independent of the pressure and temperature of the air; so that in equation (3) applied to this fluid, $\gamma$ may be regarded as a constant quantity. M. Dulong has found by a different process, for air perfectly dry,

$$\gamma = 1{,}421,$$

and a value sensibly equal to this for both oxygen and hydrogen gas. But for other fluids, such as carbonic acid and the olefiant gas, this process, which will be pointed out in the sequel, furnishes very discordant values for $\gamma$, and always less than the preceding, so that this ratio depends, in general, on the nature of the gas to which it refers.

638. If $\gamma$ be considered as a constant quantity, the integral of equation (3) of partial differences will be

$$q = f\left(\frac{p^{\frac{1}{\gamma}}}{\rho}\right);$$

$f$ denoting an arbitrary function. We shall have conversely (*g*)

$$p = \rho^{\gamma}\phi q,$$

and, on account of equation (1),

$$\theta = \frac{1}{ak}\rho^{\gamma-1}\phi q - \frac{1}{a};$$

$\phi$ being the inverse function of $f$.

If while the quantity $q$ remains the same, $p, \rho, \theta$ become $p', \rho', \theta'$, we shall have in the same manner

$$p' = \rho'^{\gamma}\phi q, \quad \theta' = \frac{1}{ak}\rho'^{\gamma-1}\phi q - \frac{1}{a};$$

$\frac{1}{a} = 266{,}67$, and in order that $\theta$ and $\theta'$ may be degrees of the centigrade thermometer, this factor of their expressions should be expressed in similar degrees. If $\phi q$ be eliminated be-

tween these last and the preceding equations, there will result$(h)$

$$\left.\begin{array}{l} p' = p\left(\frac{\rho'}{\rho}\right)^{\gamma} \\ \theta' = (266°,67 + \theta)\left(\frac{\rho'}{\rho}\right)^{\gamma-1} - 266°,67. \end{array}\right\} \quad (6)$$

These equations (6) contain the laws of the elastic force and temperature of gases, which are either compressed or dilated without experiencing any variation in their quantity of heat; they depend on the sole hypothesis, that $\gamma$ the ratio of the two specific heats does not vary, in the same fluid, with the pressure and temperature, which has been verified in the case of atmospherical air, by the experiments that have been cited in the preceding number.

639. It is necessary to make a second supposition in order to determine the arbitrary function $f$ which occurs in the value of $q$. The simplest is to assume, that under a constant pressure, a gas is dilated uniformly, for equal additions of the quantity of heat; which implies that the specific heat $c$ is constant, when the increment of one degree of temperature, by which it is estimated (No. 634), is measured by an air thermometer. In this hypothesis, $q$ must be a linear function of $\theta$; now, if in the function $f$ the value of $\rho$ deduced from equation (1) be substituted, we obtain

$$q = f\left[akp^{\frac{1}{\gamma}-1}\left(\frac{1}{a}+\theta\right)\right];$$

consequently, we shall have

$$q = \text{A} + \text{B}\,(266°,67 + \theta)\,p^{\frac{1}{\gamma}-1}; \qquad (7)$$

A and B being quantities independent of $p$ and $\theta$, and relative to the nature of the gas which is considered$(i)$.

By equations (2), we shall have

$$c = \text{B}p^{\frac{1}{\gamma}-1}, \quad c' = \frac{\text{B}}{\gamma}p^{\frac{1}{\gamma}-1};$$

and, in order to know the specific heat under a constant pressure, or at a constant density under all pressures, it is sufficient if it be known under a determinate pressure. For example, according to MM. Laroche and Berard, we have $c = 0{,}2669$ for dry air under the pressure of $0^m{,}76$, the specific heat of water being taken for unity, therefore, if the pressure corresponding to this barometrical height be denoted by $\Pi$, we shall have

$$0{,}2669 = \text{B}\Pi^{\frac{1}{\gamma}-1};$$

and likewise, if $h$ denotes the height of the barometer, expressed in metres, which belongs to any pressure whatever such as $p$, so that we may have $\frac{p}{\Pi} = \frac{h}{0^m{,}76}$, there will result from this ($k$)

$$c = 0{,}2669 \,.\left(\frac{0{,}76}{h}\right)^{1-\frac{1}{\gamma}};$$

and if this expression be divided by the constant $\gamma$, the value of $c'$ will be obtained. As this constant is greater than unity, the exponent $1 - \frac{1}{\gamma}$ will be positive, consequently, the specific heat of a gramme of air will diminish, when its elastic force or the height $h$ is increased.

If the quantity of heat lost by a gramme of air when its temperature is depressed $n$ degrees, without the elastic force undergoing any change, be denoted by $m$, we shall have ($l$)

$$m = n(0{,}2669)\left(\frac{0{,}76}{h}\right)^{1-\frac{1}{\gamma}}.$$

The weight of this volume of air, having the same primitive temperature, but under a pressure measured by the barometrical height $h'$, will be increased in the ratio of $h'$ to $h$. Therefore, if $m'$ be the quantity of heat lost by this same volume under the pressure $h'$, when the depression of temperature is the same as before, namely, $n$ degrees, we shall have

$$m' = \frac{h'n}{h}(0,2669)\left(\frac{0,76}{h'}\right)^{1-\frac{1}{\gamma}};$$

from which there results,

$$\frac{m'}{m} = \left(\frac{h'}{h}\right)^{\frac{1}{\gamma}},$$

for the ratio of the quantities of heat lost by the same volume of air, under different pressures.

In a case in which

$$h' = 1^m,0058, \quad h = 0^m,7405,$$

MM. Laroche and Berard have found, by taking the mean of two observations,

$$\frac{m'}{m} = 1,2396;$$

the formula gives for these values of $h$ and $h'$, and by making $\gamma = 1,421$,

$$\frac{m'}{m} = 1,2405,$$

which does not differ sensibly from the result of experiment.

640. In order to be able to apply formula (7) to the vapour of water, it is necessary to suppose

1st. That when a gramme of vapour is formed, which remains unchanged, so that as none of it is precipitated, neither is it increased by additional vapour, the ratio denoted by $\gamma$, of its specific heat at a constant pressure, to its specific heat under a constant volume, does not change with the temperature and density.

2ndly. That the quantity of heat necessary to raise the temperature of this gramme of vapour, by any number whatever of degrees, either under a constant pressure, or under an invariable volume, is proportional to this number, the temperature being marked by an air thermometer.

This being established, if c represents the quantity of heat necessary to convert into vapour, under the barometrical pressure of $0^m,76$, and at a temperature of $100°$, a gramme of

water, the primitive temperature of which was zero: and if $q$ denotes the quantity of heat which is requisite to vaporize this same gramme of water, and to raise the vapour to the temperature $\theta$, under any pressure whatever such as $p$; and finally, if $c$ denotes the specific heat of the vapour of water under the constant pressure of $0^m,76$, and if in equation (7), the pressure $p$ be replaced by the barometrical height that measures it, and which we shall denote by $h$, this formula must give $q = \text{c}$, when $h = 0^m,76$, and $\theta = 100°$, and $\frac{dq}{d\theta} = c$, when $h = 0^m,76$. Now if by means of these conditions, A and B, the two arbitrary constants which it contains, be determined, this equation becomes($m$)

$$q = \text{c} + c\left[(266°,67 + \theta)\left(\frac{0^m,76}{h}\right)^{\frac{\gamma-1}{\gamma}} - 366°,67\right]. \quad (8)$$

It were much to be wished, that the degree of accuracy of which this formula is susceptible, was verified by experiment, and that the three constants c, $c$, $\gamma$, which it contains, were precisely determined. If the specific heat of a gramme of water at the temperature zero, be taken for the unit, we have

$$\text{c} = 550;$$

c being the mean of the values of this quantity, which have been obtained by different philosophers. At the same time, there results from an experiment, which it must be admitted is far from being conclusive, and which ought therefore be repeated,

$$c = 0,847.$$

With respect to the value of $\gamma$, as yet, it is altogether unknown.

641. Whether $\rho$ the density of the vapour of water corresponding to the pressure $p$ and the temperature $\theta$, has attained its *maximum*, or is below this point, equation (1) which is applicable both to vapours and permanent gases, will always make known the value of $\rho$, when those of $p$ and $\theta$ are

given. If D be the density at the temperature of 100°, and under the ordinary presure of $0^m, 76$, and if $h$, as before, be the barometrical height which is produced by $p$, we obtain from this equation (2) ($n$)

$$\rho = \frac{\text{D}h}{0^m,76} \cdot \frac{366°,67}{266,67+\theta}.$$

As the weight of a litre of dry air, at the temperature zero, and under the pressure of $0^m,76$, is $1^{gr},21433$ (No. 631), it will become $\frac{1^{gr},21433}{1,375}$, or $0^{gr},883$, at the temperature of 100° ($o$); and the weight of a litre of vapour of water at the same temperature and under the same pressure, will be $\frac{10}{16}$(0,883), or $0^{gr},55$; consequently, we shall have ($p$)

$$\frac{v\rho}{\text{D}}(0^{gr},55) = \frac{vh}{0^m,76}\,\frac{201^{gr},66}{266,67+\theta},$$

for the weight of a volume $v$ of vapour, at the temperature $\theta$, and under any pressure whatever such as $h$. Therefore, if v denotes the quantity of heat necessary to produce this weight of vapour, the water being originally at the temperature zero, v will be the product of this number of grammes, and of $q$ the quantity given by formula (8); so that we shall have

$$\text{v} = \frac{vhq}{0^m,76} \cdot \frac{201^{gr},66}{266,67+\theta}.$$

The unit to which the quantity v should be referred, is the quantity of heat required to raise the temperature of a gramme of water one degree from zero; and we know that this unit is equal to seventy-five times the quantity of heat which should be employed in order to melt a gramme of ice at the temperature zero, without raising this temperature.

Different observations have induced several philosphers to think, that when the vapour of water has attained to the *maximum* of density corresponding to its temperature, the quantity of heat denoted by $q$ varies no longer with this temperature.

This is the state in which this fluid is employed in steam engines; the ratio of V the quantity of heat produced to its tension $h$, will be therefore then, every thing else being the same, the reciprocal of $266°,67 + \theta$; consequently, the ratio of the consumption of heat to the effort made on the piston, the measure of which is $h$, will diminish when the temperature becomes greater, and this ratio will be less in engines of high pressure. But the economy of fuel which should result in their favour from these considerations, is very far from corresponding to that which seems to be pointed out by experiment; and these engines are certainly indebted to other circumstances for the superiority which they possess.

642. Let the part of the vertical cylinder ABCD (fig. 53) which is comprised between EF the surface of the water, and GH the base of the piston, be supposed to be filled with the vapour of water at its *maximum* of density, corresponding to $\theta$, which is the temperature of this vapour, of the water that is beneath it, as also of the cylinder and of the piston. In this state, if the elastic force of the vapour is in equilibrio with the weight of the piston, so that denoting this force by $p$, this weight by P (the external pressure which the piston sustains being comprised under P), and by $\lambda$ the area of its horizontal base, we shall have

$$P = \lambda p.$$

If the weight P be increased until it becomes $P + \Pi$, the piston will descend, and the space occupied by the vapour will diminish; but as it is supposed to have attained its *maximum* of density, a part of it will be liquefied; and if the temperature $\theta$ be invariable, the pressure $p$ will be so likewise($q$). Indeed, in the first instant of the compression, the temperature $\theta$ will increase, so that the liquefaction cannot take place immediately, and thus $p$ the pressure may increase. But if the motion of the piston is not very rapid, this increase of temperature will disappear before the displacement of this moveable

is appreciable, so that $\theta$ and $p$ may be considered as constant during the entire of its motion. It should be also observed, that the condensation of the fluid, by which it is reduced to water, and which is produced immediately at GH the upper part in contact with the piston, is transmitted as far as EF, in an extremely short portion of time, during which the displacement of the piston is insensible; hence it follows that during the descent of the piston, the density of the fluid is sensibly the same through the entire height. This being the case, the motive force of this body will be constant and equal to the excess of P + $\Pi$ over $\lambda p$ or to $\Pi$; consequently, if the friction against the sides of the cylinder, is not taken into account, its motion will be uniformly accelerated; in like manner, if the motion communicated to the vapour be not taken into account, that is to say, if its mass relatively to that of the weight $\Pi$ be neglected, the accelerating force of this motion will be the weight diminished in the ratio of $\Pi$ to P + $\Pi$. Therefore, if the height of GH above EF be denoted by $l$, the value of the living force, produced by the entire descent of the piston, will be $2\Pi l$.

If the piston being first stopped at GH, the temperature of the inferior water is then suddenly depressed, so as to become equal to $\theta'$, which is less than $\theta$, the stratum of vapour in contact with EF will be liquefied by the cold, and will be replaced by another stratum, which will in like manner be liquefied; and if the quantity of water is so considerable, that these successive strata of vapour do not cause any sensible variation in its temperature, the liquefaction will continue until the entire mass of vapour attains the elastic force $p'$, which corresponds to the temperature $\theta'$, and to its maximum of density, relative to this temperature. However, neither the temperature of the column of vapour nor the density will be the same throughout its entire height; and it would be an interesting problem to determine the laws of its temperature and density, when the temperatures at its two extremities are constantly $\theta$ and $\theta'$, the density of each stratum being consi-

dered to be such a function of its temperature, that the elastic force may be constant and equal to $p'$. This constancy of pressure throughout the entire height of the column of vapour is evidently the condition of the equilibrium of its successive strata; and when the equilibrium is established, it is equally evident that the value of the constant pressure cannot be greater than that which corresponds to the least of the two temperatures $\theta$ and $\theta'$, while it may be less than the elastic force which corresponds to the greater. In fine, it appears from experiment that the vapour attains, in an extremely short interval of time, to the state of equilibrium in question; so that if aqueous vapour, at its *maximum* of density and pressure, is contained in a closed vessel, whose sides have attained to its temperature; and if the temperature of a portion of these sides is then unequally depressed, one part of the vapour will be liquefied, and the remaining portion will acquire, with very great rapidity, the maximum elastic force which corresponds to the least temperature.

This being established, when the piston is no longer retained, it will descend, and it is evident, as in the preceding case, that its motion will be uniformly accelerated, its motive force being equal to $\mathrm{P} - \lambda p'$ or $\lambda(p - p')$, its accelerating force will be equal to the weight multiplied by the ratio $\frac{p-p'}{p}$, and, finally, the living force produced throughout the entire descent is $2\lambda(p-p')l$. When the piston has arrived at EF, if the temperature of the inferior liquid be raised, and if this liquid be converted into vapour, the constant pressure of which on the base of the piston is greater than the weight of this body, it will ascend with a motion uniformly accelerated; and the living force that is produced when it has traversed a length $l'$, is equal to twice $l'$ multiplied by the excess of this pressure over this weight, it being understood always that the friction is not taken into account.

It is by considerations such as these that the living force

due to the descent or ascent of the piston in steam engines is calculated. This force is then distributed through the system to which the engine is applied, and is partly destroyed by the frictions, and partly employed to produce an useful effect. The calculation would be different, if the density of the vapour contained in the body of the pump had not reached its *maximum*, this is, in fact, what happens during what is termed the *detente* of the vapour, that is to say, during the time the communication of this fluid with the boiler is suspended, when the vapour is dilated, without any new addition being made to it; the motion of the piston is then that which has been considered in No. 358; and it appears from the following number 359, that the living force produced, during the time it traverses any given length, is equal to $2pv \log \frac{p'}{p}$, $v$ denoting the primitive volume of the fluid, and $p$ and $p'$ its elastic forces at the beginning and end of the motion.

643. It only now remains for us to consider the elastic forces and quantities of heat of the mixtures of several gases, compared with those of these fluids.

If two different gases, whose volumes are $a$ and $a'$, have the same temperature $\theta$, and, being subjected to the same pressure $p$, are then superimposed in a closed vessel, the capacity of which is $a + a'$, it is evident that they may continue thus in equilibrio, since they have the same temperature, and exert, the one against the other, the same pressure; but it appears from experiment that this equilibrium is not stable, for these two fluids gradually penetrate one another's dimensions, until they are perfectly mixed together; experiment also shows that, in this operation, there is neither any variation of temperature, nor any loss or absorption of heat; so that after a certain time, which is different for different fluids, we have a homogeneous mixture, in which the proportion of the two gases is every where the same throughout, and of which the temperature and elastic force are always $\theta$ and $p$. From these facts, which are

established by observation, another result may be inferred, which is also verified by experiment.

If two gases mixed together occupy a volume $v$ at the temperature $\theta$, and if $p$ and $p'$ denote the pressures on the unit of surface, which these two gases separately sustain at the same temperature, and under this volume $v$, the elastic force of the mixture will be $p+p'$. In fact, if we suppose, first, that the two gases are separated, and that $p'>p$, then if the gas subjected to the pressure $p'$ is dilated, without any change being made in its temperature, so that its elastic force may be reduced to $p$, its volume will, by the law of Mariotte, be $\frac{vp'}{p}$; if then, the two gases be superimposed in a closed vessel, the capacity of which is $v+\frac{vp'}{p}$ or $\frac{v}{p}(p+p')$; these gases will, by what has been just stated, mix without undergoing any change of temperature; and there will result a homogeneous mixture at a temperature $\theta$, and under the pressure $p$. Now, as the law of Mariotte is applicable to mixtures as well as to simple gases, if this mixture be compressed, without changing the temperature, until its volume $\frac{v}{p}(p+p')$ is reduced to $v$; its elastic force $p$ will become $p+p'$; which it was proposed to demonstrate($r$). The same principle likewise obtains for three or any greater number of gases, and for a mixture of gas and vapours; the pressure of the mixture is always the sum of the pressures which these fluids would separately sustain, at the same temperature and under the same volume as the mixture.

644. Let $n$ and $n'$ be the actual numbers of grammes of the two gases mixed together, and filling the volume $v$ at the temperature $\theta$, and under the pressure $p$; and let $c$ and $c'$ denote the specific heats of a gramme of these gases under a constant pressure equal to $p$, and $c''$ the specific heat of a gramme of the mixture under the same pressure, we shall have

$$(n+n')c''=nc+n'c', \tag{9}$$

In fact, if the two gases, instead of being perfectly mixed together, were only superimposed, so that they might occupy $a$ and $a'$ separate portions of the volume $v$; by what has been just stated, the quantity of heat will be the same in the two gases when separate, and when mixed together; also this equality of heat will still subsist, if $\theta$, the common temperature of the two gases and of the mixture, be respectively increased by one degree. Now, it is necessary, in order to effect this increase, to communicate a quantity $(n + n')c''$ of heat to the mixture, and the quantities $nc$ and $n'c'$ to the two gases, $p$ the pressure being supposed to remain the same. Therefore the first quantity must be equal to the sum of the two others, this gives equation (9), which may be extended without difficulty to any number whatever of elastic fluids. By means of it, the specific heat of a mixture is known, when those of all the gases or vapours which compose it, and the proportions of these fluids, are given; conversely, it may be made use of to determine the specific heat of one of the components, when those of all the others and of the mixture are known; and it may be observed, that they do not imply that the specific heats of the mixed gases are independent of their common temperature.

We might, instead of considering the specific heats $c$, $c'$, $c''$ of the gases and of the mixture under a constant pressure, consider, in the same manner, their specific heats under a constant volume; and if they are represented by $c$, $c_{\prime}'$, $c_{\prime}''$, an equation similar to the preceding will be obtained, namely,

$$(n + n')c_{\prime}'', = nc_{\prime} + n'c_{\prime}'. \tag{10}$$

Now, if we make

$$\frac{c}{c_{\prime}} = \gamma, \quad \frac{c'}{c_{\prime}'} = \gamma', \quad \frac{c''}{c_{\prime}''} = \gamma'',$$

there will result from equations (9) and (10) ($s$)

$$\gamma'' = \frac{n\gamma c_{\prime} + n'\gamma' c_{\prime}'}{nc_{\prime} + n'c_{\prime}'}; \tag{11}$$

by means of which equation, $\gamma''$ the ratio relative to the mixture will be known, when the similar quantities $\gamma$ and $\gamma'$, and the values of $c_{,}$ and $c'_{,}$, are known, for the two mixed gases. Whether the value of $\gamma$ relative to dry air, be taken equal to 1,375, or 1,421 (No. 637), and whatever may be the unknown value of $\gamma'$ which corresponds to the vapour of water, the value of $\gamma''$ in ordinary air, will differ little from $\gamma$, because the proportion of vapour which this air contains is inconsiderable.

It appears from No. 639, that if the ratios $\gamma$ and $\gamma'$ are independent of the pressure $p$, but different for the two gases, the quantities $c_{,}$ and $c'_{,}$ will be expressed by unequal powers of $p$; hence it results, in virtue of equation (11), that $\gamma''$ the ratio of the mixture, cannot be also independent of the pressure. Consequently, the hypothesis of the invariability of the ratio of the specific heat of the same fluid under a constant pressure, to its specific heat under a constant volume, and the formulæ which have been deduced from them, cannot apply at the same time to simple gases, for which this ratio is not the same, and to their mixtures in any proportion whatever; and if, in the experiments made on air subjected to different pressures (No. 637), this ratio has appeared to be constant, the reason is, that it is sensibly the same for air and oxygen, and, consequently also, for the oxygen and azote, or nitrogen, of which the air is composed.

# BOOK THE SIXTH.

## HYDRODYNAMICS.

### CHAPTER I.

#### GENERAL EQUATIONS OF THE MOTION OF FLUIDS.

645. The equations of the equilibrium of fluids which were established in No. 582, are founded on the characteristic property which is common to liquids and aeriform fluids, of transmitting equally in every direction the pressures applied to their surface, and of exercising about each point of their mass, in virtue of the molecular action, equal pressures in every direction. This property arises, as has been already stated (No. 576), from the circumstance that the molecules of a fluid that has been compressed or dilated reverts very promptly to an arrangement similar to that which they previously had about any point whatever, so that after its compression or dilatation, a fluid is a system of material points similar to what it was before, but constituted on a greater or less scale. The time which it takes to return to such a similar state has no influence on the laws of equilibrium, which does not take place until this time is lapsed; but, however short this interval may be, it is easy to conceive that it can influence the laws of their motion, especially in the case in which the vibrations of the fluid molecules are performed with great rapidity; so that the principle of the equality of pressure in all directions, though applicable in the case of hydrostatics, or

the equilibrium of fluids, is not always so to hydrodynamics, that is to say, to the part of mechanics which treats of the motion of fluids.

A corresponding difference between the state of equilibrium and the state of motion, relatively to the law of Mariotte, was long since remarked by Laplace. According to this law, it is necessary that the temperature of the fluid should become the same after the pressure, as it was before; and the principle of the equality of pressure in all directions supposes also, that the molecules of a fluid have had time to revert to a relative arrangement similar to their original one. This law has not place, or it ought to be modified, in those extremely rapid vibrations of gas, in which the primitive temperature has not had time to reestablish itself; and, in like manner, the principle of the equality of pressure in all directions is not rigorously and always applicable to the motions of liquids and aeriform fluids. The influence of this modification of the law of Mariotte, has been observed in the velocity of the propagation of sound; and there are, doubtless, also phenomena of the motion of fluids, which, in general, depend on the circumstance that the pressure in all directions, resulting from the cause that has been adverted to, is not perfectly equal. In consequence of this circumstance, terms are introduced into the general equations of the motion of fluids, which cannot be deduced from their equations of equilibrium. The author took these into account in the memoir already cited, (No. 576), and he intended in another treatise to revert to the consideration of this important question. But in the present work he assumes, agreeably to the method which is commonly pursued, that the property of pressing equally in every direction, is applicable to the state of equilibrium and also to the state of motion. On this hypothesis, the equations of hydrostatics which are founded on this property, may be extended at once to hydrodynamics, by means of the principle of D'Alembert, which is applicable to all possible systems of material points.

646. Let the fluid mass ABCD (fig. 36), of which the equations of equilibrium have been already determined, be again considered; and let it now be supposed to be in motion, and that all the notations of No. 581 refer to the end of $t$ any time whatever, reckoned from the commencement of this motion. Thus let $x$, $y$, $z$, be the coordinates of $dm$ any element whatever, of a fluid mass, whether homogeneous or heterogeneous, liquid or aeriform, at the end of the time $t$; let $\rho$ denote the density of the fluid in this point and at this instant, and $\mathrm{X}dm$, $\mathrm{Y}dm$, $\mathrm{Z}dm$, the components of the motive force of $dm$, parallel to the axes of $x$, $y$, $z$, at this same instant. The quantities X, Y, Z will be given functions of $x$, $y$, $z$, when they arise from attractions or repulsions, which emanate from fixed centres; these given functions will contain the time explicitly, when the centres of these forces are in motion. When these points are those of the fluid, X, Y, Z will be functions of $x$, $y$, $z$, $t$, which depend on its figure at each instant, and on the law of the densities in its interior.

The coordinates $x$, $y$, $z$ will vary with the time; they will also vary from one point to another of the fluid; and if their initial values, that is to say, the coordinates of the point of space which the element $dm$ occupies at the commencement of the motion, be denoted by $x'$, $y'$, $z'$; then $x$, $y$, $z$, the coordinates of this same element at the end of the time $t$, will be unknown functions of $x'$, $y'$, $z'$, $t$; so that the complete solution of the problem will consist in determining these three functions of the four independent variables.

If the components parallel to the axes O$x$, O$y$, O$z$, of the velocity with which the element $dm$ is actuated at the end of the time $t$, be denoted by $u$, $v$, $w$, we shall have

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt}; \tag{1}$$

$u$, $v$, $w$ may be considered as unknown functions either of $t$, $x$, $y$, $z$, or of $t$, $x'$, $y'$, $z'$; it is under this second point of view

that the author proposes to consider these three quantities at present(*a*); and then in order to obtain their increments in the interval $dt$, they should be differenced with respect to $t$ and the coordinates $x$, $y$, $z$. Now if $q$ denotes any function whatever of $t$, $x$, $y$, $z$, and $q'dt$ its differential taken with respect to $t$, and the variables $x$, $y$, $z$, considered as functions of $t$, we shall have by the known rule for the differentiation of functions

$$q'=\frac{dq}{dt}+\frac{dq}{dx}\cdot\frac{dx}{dt}+\frac{dq}{dy}\cdot\frac{dy}{dt}+\frac{dq}{dz}\cdot\frac{dz}{dt},$$

or, by considering equations (1),

$$q'=\frac{dq}{dt}+u\frac{dq}{dx}+v\frac{dq}{dy}+w\frac{dq}{dz}. \tag{2}$$

Therefore, if the increments of $u$, $v$, $w$, be denoted by $u'dt$, $v'dt$, $w'dt$, we shall have

$$u'=\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz},$$

$$v'=\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz},$$

$$w'=\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz};$$

and, in this manner, it appears that the components of the velocity of the same element $dm$, in the two positions which it successively occupies, will be $u$, $v$, $w$, and $u+u'dt$, $v+v'dt$, $w+w'dt$.

If the fluid be homogeneous and incompressible, the density $\rho$ will be a given constant; in the case of a heterogeneous liquid, the density $\rho$ corresponding to a determinate element $dm$, will be a given function of its three initial coordinates, $x'$, $y'$, $z'$; and finally, if the fluid is compressible, this density $\rho$ will be an unknown function of $t$, $x'$, $y'$, $z'$, the initial value of which will be solely given. With the exception of the case in which $\rho$ is constant, this density relative to the position of $dm$ at the end of the time $t$, must be always considered as an

$\rho = f(x' y' z') = F(x y z t)$

unknown function of $x, y, z, t$. Hence, if its increment during the instant $dt$ be denoted by $\rho' dt$, we shall have by formula (2),

$$\rho' = \frac{d\rho}{dt} + u\frac{d\rho}{dx} + v\frac{d\rho}{dy} + w\frac{d\rho}{dz};$$

and in the case of an incompressible fluid, whether homogeneous or heterogeneous, this value of $\rho'$ must become zero.

647. The components of the force lost by the element $dm$ during the instant $dt$, will be

$$(\mathrm{X} - u')\, dm, \quad (\mathrm{Y} - v')\, dm, \quad (\mathrm{Z} - w')\, dm;$$

therefore, if $\mathrm{X} - u'$, $\mathrm{Y} - v'$, $\mathrm{Z} - w'$, be substituted in place of X, Y, Z, in equations (2) of No. 582, there will result the three following equations of its motion:

$$\frac{dp}{dx} = \rho\,(\mathrm{X} - u'), \quad \frac{dp}{dy} = \rho\,(\mathrm{Y} - v'), \quad \frac{dp}{dz} = \rho\,(\mathrm{Z} - w'),$$

$p$ being the pressure on the unit of surface, that has place at the end of the time $t$, at the point, whose coordinates are $x, y, z$, which pressure is supposed to be the same in all directions.

If this point appertains to a fixed side of the vessel, $p$ will express the normal pressure that this surface must sustain, and which must be destroyed by its resistance. If this point exists on the free surface of a liquid, we should have $p = 0$, or more generally, $dp = 0$, so that the differential equation of the free surface of the liquid in motion, will be (b)

$$(\mathrm{X} - u')\, dx + (\mathrm{Y} - v')\, dy + (\mathrm{Z} - w')\, dz = 0.$$

By what has been remarked in 585, the value of $p$ when it is determined, should be constantly positive in the interior of this liquid, unless the parts of the fluid mass separate during the motion, in which case it will be negative in one point of the side of separation, this can only have place in the case of a liquid, and as then this surface is no longer pressed from without inwards, the parts of the liquid will be separated.

By means of the values of $u'$, $v'$, $w'$, the preceding equations become

$$\left.\begin{aligned}
\frac{1}{\rho}\frac{dp}{dx} &= \mathrm{X} - \frac{du}{dt} - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz}, \\
\frac{1}{\rho}\frac{dp}{dy} &= \mathrm{Y} - \frac{dv}{dt} - u\frac{dv}{dx} - v\frac{dv}{dy} - w\frac{dv}{dz}, \\
\frac{1}{\rho}\frac{dp}{dz} &= \mathrm{Z} - \frac{dw}{dt} - u\frac{dw}{dx} - v\frac{dw}{dy} - w\frac{dw}{dz}.
\end{aligned}\right\} \quad (3)$$

As the quantity $p$ which it contains is, as well as each of the velocities $u, v, w,$ an unknown function of $x, y, z, t,$ a *fourth* equation is necessary, when the quantity $\rho$ is a *given constant*, and, in the general case, in which this quantity is also an unknown function of $x, y, z, t,$ *two* additional equations are necessary. These equations can be obtained in the following manner.

648. Each of the elements, such as $dm$, will change its form during the instant $dt$, and it will also change its volume, if the fluid is compressible; but as the mass must always remain the same, it follows, that the product of its volume at the end of the time $t + dt$, and of its density $\rho + \rho' dt$, which corresponds to the same instant, must be the same as at the end of the time $t$, consequently, the variation of this product in the instant $dt$, will be equal to zero; this will furnish a new general equation of motion.

In order to obtain it, let us consider the rectangular parallellopiped, whose volume is $dx . dy . dz$, at the end of the time $t$, and what will be the form of this element of the fluid at the end of the time $t + dt$, Let M (fig. 54) be the summit of this parallellopiped, which corresponds to the coordinates $x, y, z$; likewise let MA, MB, MC be the three sides adjacent to this summit, and respectively parallel to the axes $ox$, $oy$, $oz$, so that we may have

$$\mathrm{MA} = dx, \quad \mathrm{MB} = dy, \quad \mathrm{MC} = dz;$$

now if D, E, F, G be the four other summits, and if, during the

instant $dt$, the eight points M, A, B, C, D, E, F, G are transferred to M′, A′, B′, C′, D′, E′, F′, G′; then the polyhedron, of which these last points are the summits, will be an oblique angled parallelopiped; this can be shown to be the case, by determining and comparing together the lengths of its twelve sides M′A′, M′B′, &c.

As $x, y, z$, the coordinates of the point M become

$$x + udt, \quad y + vdt, \quad z + wdt,$$

at the end of the instant $dt$, these quantities are the coordinates of the point M′; those of any other summit may be deduced from them, by substituting the primitive coordinates of this summit for $x, y, z$; thus the coordinates of C′ will be obtained by retaining $x$ and $y$, and substituting for $z$, $z + dz$, since $x, y, z + dz$ are the coordinates of C. In this manner, the coordinates of C′ will be

$$x + udt + \frac{du}{dz} dz\, dt,$$

$$y + vdt + \frac{dv}{dz} dz\, dt,$$

$$z + dz + wdt + \frac{dw}{dz} dz\, dt;$$

and from a comparison of them with those of M′, we infer ($c$)

$$\text{M}'\text{C}' = \sqrt{\left(\frac{du}{dz}\right)^2 dz^2 dt^2 + \left(\frac{dv}{dz}\right)^2 dz^2 dt^2 + \left(dz + \frac{dw}{dz} dz\, dt\right)^2};$$

therefore, by extracting the square root, and neglecting infinitely small quantities of the third and higher orders, we obtain

$$\text{M}'\text{C}' = dz + \frac{dw}{dz} dz\, dt.$$

The coordinates of D′ may be obtained from those of M′, and the coordinates of G′ from those of C′, by substituting $x + dx$ and $y + dy$ in place of $x$ and $y$; consequently, the length

of the side D′G′ may be obtained in the same manner from that of the side M′C′; and it gives

$$D'G' = dz + \frac{dw}{dz}dz\,dt + \frac{d^2w}{dx\,dz}dx\,dz\,dt + \frac{d^2w}{dy\,dz}dy\,dz\,dt;$$

therefore, if the two last terms which are of the third order, be neglected, the value of D′G′ will be the same as that of M′C′. In the same manner it may be proved, that the sides A′E′ and B′F′ are equal to the side M′C′, when quantities of the third order are neglected, so that we shall have

$$M'C' = A'E' = B'F' = D'G'.$$

If $z$ be changed into $y$, and $w$ into $v$, in the value of M′C′, it will become that of M′B′, namely,

$$M'B' = dy + \frac{dv}{dy}dy\,dt;$$

In like manner, by changing $z$ into $x$ and $w$ into $u$, we shall have the value of M′A′, which will be

$$M'A' = dx + \frac{du}{dx}dx\,dt;$$

and we shall also find

$$M'B' = A'D' = C'F' = E'G',$$

$$M'A' = B'D' = C'E' = F'G'.$$

It appears, therefore, that the sides which are equal in the primitive parallellopiped, continue to be equal after its change of form; and the parallelism of the sides is a consequence of their equality; hence the element of volume, which has been considered, retains at the end of $dt$, the form of a parallellopiped, which, however, is not rectangular, as at the commencement of this instant.

The volume of this parallellopiped will be obtained by multiplying one of its faces, for example, the face M′A′D′B′ by C′P′, the perpendicular let fall from the point C′ on this face;

the area of the parallellogram M′A′D′B′ is equal to the product of its two sides M′A′ and M′B′, multiplied into the sine of the angle A′M′B′; and the perpendicular C′P′ is equal to the side C′M′ multiplied by the sine of the angle C′M′P′; consequently, the value of the volume of the new parallellopiped will be

$$\text{M}'\text{A}'.\,\text{M}'\text{B}'.\,\text{M}'\text{C}'.\,\sin \text{A}'\text{M}'\text{B}'.\,\sin \text{C}'\text{M}'\text{P}'.$$

But, as the angles A′M′B′, and C′M′P′, were right angles in the original parallellopiped, each of them will now differ from a right angle only by an infinitely small quantity, therefore, the sine of each of these angles will only differ from unity by an infinitely small quantity of the second order; consequently, if infinitely small quantities of the fifth order be neglected, we should make sin A′M′P′ $=1$, and sin C′M′P′ $=1$ (*d*), in the preceding product; by which means it is reduced to

$$\text{M}'\text{A}'.\,\text{M}'\text{B}'.\text{M}'\text{C}'.$$

Therefore, if for each of the factors, its preceding value be substituted, and then the multiplication be performed, this product will be, by neglecting infinitely small quantities of the fifth order(*e*),

$$\left(1+\frac{du}{dx}dt+\frac{dv}{dy}dt+\frac{dw}{dz}dt\right)dx\,dy\,dz.$$

This, therefore, is the value, at the end of the time $t+dt$, of the volume which was $dx\,dy\,dz$ at the end of the time $t$. The density $\rho$ becomes, at the same time, $\rho+\rho' dt$; therefore, if after this volume is multiplied by $\rho+\rho' dt$, the primitive mass $\rho\,dx\,dy\,dz$ be taken from the product, the remainder will be the variation of this mass during the instant $dt$, and as this variation should be cipher, there results the equation

$$\rho'+\rho\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0,$$

infinitely small quantities of the fifth order being neglected as before, and the factor $dt\,dx\,dy\,dz$, which is common to all the

terms, being suppressed. Consequently, if for $\rho'$ its value given in the preceding number be substituted, there will result by concinnating

$$\frac{d\rho}{dt} + \frac{d.\rho u}{dx} + \frac{d.\rho v}{dy} + \frac{d.\rho w}{dz} = 0, \tag{4}$$

which will be the fourth equation of the motion that it was proposed to form.

649. This equation is common to liquids and to aeriform fluids; but as the quantity $\rho'$ is cipher, in the case of incom-
1° pressible liquids, this equation naturally divides itself into the two following:

$$\left.\begin{aligned} \frac{d\rho}{dt} + u\frac{d\rho}{dx} + v\frac{d\rho}{dy} + w\frac{d\rho}{dz} &= 0, \\ \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} &= 0. \end{aligned}\right\} \tag{5}$$

By means of these and of the three equations (3), we shall have a number of equations equal to that of the five unknown quantities $\rho$, $p$, $u$, $v$, $w$, which they ought to determine in functions of $x$, $y$, $z$, $t$. When the liquid is homogeneous,
2° the density $\rho$ is a given constant; this reduces the unknown quantities to four, and at the same time causes the first equation (5) to disappear.

3° In the case of elastic fluids also, there are only four equations, namely, equations (3) and (4); but as then the density is connected with the pressure, the two unknown quantities $\rho$ and $p$ are reduced to one. If the temperature be supposed to be the same throughout the entire mass of the fluid in the state of rest, the dilatations or compressions of the elements of this fluid, which take place during its motion, will cause this temperature to vary, so that the pressure $p$ will be no longer proportional to the density $\rho$, in the state of motion, as it is in the state of equilibrium. It will be shewn in the sequel, how we should take this circumstance into account, when the motion is very rapid; at present we shall assume that

the motion is too slow to have any sensible influence, so that the expresssion of $p$ in a function of $\rho$ may be that which agrees to the state of equilibrium, namely (No. 624),

$$p = k\rho\,(1 + a\theta)\,; \tag{6}$$

in which $\theta$ denotes the temperature common to all the points of the fluid, $a$ the coefficient 0,00375, of the dilatation of gases, $k$ a constant which depends on the nature of the fluid in question.

When the values of $\rho, p, u, v, w$ shall have been determined, either by means of the five equations (3) and (5), or by the five equations (3), (4), (6), we can deduce from them the values of $x, y, z$, in functions of $t$, and of their initial values $x', y', z'$, by means of equations (1). The integrals of all these equations of partial differences will contain arbitrary functions, which must be determined by the initial state of the fluid, and by means of certain conditions relative to its surface, which will be considered farther on.

650. When the temperature is not the same, at the origin of the motion, throughout the entire fluid mass, it varies then from one point to another, and, for the same point, from one instant to another, so that if the temperature, which corresponds to the points of which $x, y, z$, are the coordinates, at the end of the time $t$, be denoted by $\theta$, this quantity $\theta$ is an unknown function of $t, x, y, z$, and in order to determine it, besides the preceding equations, an additional one is required. This equation will be different in the two cases of a liquid, and of an aeriform fluid, which we now proceed to consider successively.

1st. Let us suppose that the question is respecting a homogeneous liquid, water for example; then as the temperature $\theta$ varies from one point to another, the density $\rho$ will also vary, and will be a determinate function of $\theta$, which we shall denote by $f\theta$; the manner of determining the form of this function, is given in the *Traité de Physique de M. Biot*, tom. 1, chapter xi. The quantity $\rho'$ will be no longer cipher,

neither will equation (4) be decomposable into the two equations (5). The specific heat of the liquid and the measure of its conductibility will be also determinate functions of $\theta$; but if the communication of heat in the interior of the water be supposed to take place as in a solid body, by radiation to an insensible distance, the equation relative to the motion of heat in a heterogeneous body, which the author determined in the *Journal d'Ecole Polytechnique*, No. 19, page 87, will be applicable to the mass of water that is considered; for it makes known the instantaneous increment of temperature, which has place in any point whatever of a body, in which the specific heat and conductibility vary arbitrarily from one point to another; and from the manner in which it has been formed, it appears that the heat depends neither on the motion of the material point in question, nor on the motion of the surrounding points. Thus, if the increment of $\theta$ during the instant $dt$ be denoted by $\theta' dt$, we shall have($f$)

$$g\theta' = \frac{d.h\frac{d\theta}{dx}}{dx} + \frac{d.h\frac{d\theta}{dy}}{dy} + \frac{d.h\frac{d\theta}{dz}}{dz}; \qquad (7)$$

in which equation, we assume, as in formula (2), that

$$\theta' = \frac{d\theta}{dt} + u\frac{d\theta}{dx} + v\frac{d\theta}{dy} + w\frac{d\theta}{dz},$$

and in which $g$ and $h$ are functions of $\theta$, that denote respectively the specific heat relative to the unit of mass, and the measure of the conductibility. As each of these functions is supposed to be known, and also $f\theta$, the number of equations (3), (4), (7), will be the same as that of the unknown quantities $\theta, p, u, v, w$, which they contain. In the case of a heterogeneous liquid, the three quantities, $\rho$, $g$, $h$, relative to the point of its mass, the coordinates of which are $x$, $y$, $z$, will depend on the temperature $\theta$, and the matter of the fluid in this point, and they will be consequently given functions of $\theta$, and of $x'$, $y'$, $z'$, the initial coordinates of this same point.

2ndly. If the fluid in question is a mass of air, or any gas whatever, of which $\theta$ the temperature varies from one point to another, and if, in the state of motion, the temperature is always supposed to be proportional to the density, as in the preceding number, equation (6) will always have place; but equation (7) will no longer subsist, for it is founded on the supposition, that the communication of heat in the interior of the body is effected by a radiation to an insensible distance; while, on the contrary, radiating heat traverses aeriform fluids to very great depths, so that there is an interchange of heat between molecules very far removed from each other. This equation should therefore be replaced by another, which, together with equations (3), (4), (6), make up a number equal to that of the unknown quantities $\rho, p, \theta, u, v, w$. For example, in the problem of the trade winds, which are produced by the differences of temperature of the atmospherical strata, a sixth equation is formed in the following manner, which it will be sufficient for us now merely to point out.

The quantity of heat received during the instant $dt$, by $dm$ any element whatever of the fluid mass, and which may be supposed to be proportional to $dmdt$, is made up of the solar heat absorbed by $dm$ during this instant $dt$, and of the radiating heat that this element receives in this same instant, from a part of the surface of the earth, and from the part of the atmosphere, the communication of which with $dm$ is not interrupted by this surface, and also of the portion of heat which can be communicated to $dm$ by the surrounding elements, as in solid bodies. If from this sum be taken the quantity of heat emitted by the element $dm$, during the instant $dt$, either by communication, or by radiation to a great distance, the instantaneous increase of the heat of $dm$, which we shall represent by $\Delta dmdt$, will be obtained; $\Delta$ being a coefficient, of which we shall content ourselves merely to indicate the origin. On the other hand, this increase of heat is equal to $g\theta' dtdm$, $g$ and $\theta' dt$ denoting always the spe-

cific heat for the unit of mass, and the instantaneous increment of temperature; therefore we shall have $\Delta dmdt = g\theta' dmdt$ or $\Delta = g\theta'$ for the required equation, which should be substituted in equation (7).

651. Before we proceed any further, an important remark may be made relatively to equation (4).

From the manner in which it has been formed, it is evident that the mass of $dm$ the differential element of the fluid, does not vary during the instant $dt$; but it is solely for the sake of conciseness that the volume of this part of the fluid has been considered as infinitely small; and if the entire volume be divided into parts of a finite but insensible magnitude, each of which may, notwithstanding, contain a very great number of molecules, equation (4) expresses actually that each of these parts contains always the same molecules, and, consequently, that its mass is invariable. It is on this account that it is denominated the *equation of the continuity of the fluid.* Now there are motions in which this continuity is interrupted, and in which the equation that refers to it cannot be made use of. In the case, for example, of water contained in a vertical cylinder, which is open at its upper surface, if it be heated from above, the temperature will increase, and the density diminish from the bottom to the surface; the length of the fluid mass will increase, the horizontal strata will successively replace each other, and the equation of continuity will be applicable to this motion ($g$). But if the liquid is heated from below, the density will increase, and temperature diminish from below upwards; in strictness, the horizontal strata may still successively replace each other; but such a motion will not be stable; and it appears from observation, that the molecules of water rise from the bottom to the surface by traversing the superior strata. All the very small parts of the liquid do not then constantly consist of the *same* molecules; consequently, equation (4) does not obtain in this kind of motion; and it is even doubtful, whether equations (3), which

are founded on the principle of the equality of the pressure in all directions, can be applied to them; so that in the actual state of the science we have no means of determining the motion of a liquid, whose strata mutually traverse each other, the one by ascending, the other by descending. The same remarks are applicable to the vertical motions which may exist in each atmospherical column, the inferior strata of which, when heated by contact with the earth, and thus rendered lighter, rise by traversing the superior strata. The determination of these motions, which are of a different kind from those that have been hitherto considered, and their influence on the diurnal variations of the barometer, are questions to which it is of great consequence to direct the attention of philosophers.

652. In the motions of fluids which have been subjected to calculation, it is customary to suppose that the points which, at a determined epoch, exist on a fixed or moveable side, or which appertain to the free surface of a liquid, will remain on this side, or will appertain to this surface, during the entire continuance of the motion; so that those complicated motions, in which the points of a fluid, after having appertained to its surface, penetrate again into the interior of the mass, or conversely, are not taken into account; and in like manner, those cases are excluded in which the points of a liquid pass alternately from the free surface to the surface in contact with a fixed or moveable side. Those particular conditions to which the motions that are considered are subjected, may be expressed by the following equations:

Let $x$, $y$, $z$ be always the variable coordinates of a point of the fluid, and

$$f(t, x, y, z) = 0,$$

the equation of a fixed or moveable surface that passes through this point at the end of the time $t$, and which, for conciseness, we shall denote by s. Likewise, let $x'$, $y'$, $z'$ be the initial coordinates of this same point, so that $x$, $y$, $z$ may

be functions of $t$, $x'$, $y'$, $z'$. If their values be substituted in the given equation, it will be changed into

$$\text{F}(t, x', y', z') = 0;$$

and all the points of the fluid, the initial coordinates of which satisfy this equation will be those which, at the end of the time $t$, appertain to the surface S; consequently, in order that these points may be constantly the same, the function F should not contain the variable $t$. If therefore S is the equation of this free surface, or that of a fixed or moveable side, the function $f(t, x, y, z)$ must be independent of $t$; $x$, $y$, $z$ being considered as functions of the preceding variables; therefore its complete differential with respect to $t$ must be cipher; and by formula (2) we shall have, to express the condition stated above, the equation($g$)

$$\frac{df}{dt} + u\frac{df}{dx} + v\frac{df}{dy} + w\frac{df}{dz} = 0. \tag{8}$$

In the case of a *fixed* side, the function $f$ will not contain the time $t$ explicitly; if it be denoted by L, so that $\text{L} = 0$, may be the given equation of the side, equation (8) will become

$$u\frac{d\text{L}}{dx} + v\frac{d\text{L}}{dy} + w\frac{d\text{L}}{dz} = 0. \tag{9}$$

If the resultant of the velocities $u$, $v$, $w$ be denoted by $\zeta$, and the angles which it makes with the directions of $x$, $y$, $z$, by $\alpha$, $\beta$, $\gamma$, and also if $a$, $b$, $c$ be the angles which the normal to the side makes with the same directions, we shall have at the same time, and by making

$$\left(\frac{d\text{L}}{dx}\right)^2 + \left(\frac{d\text{L}}{dy}\right)^2 + \left(\frac{d\text{L}}{dz}\right)^2 = \lambda^2,$$

$$u = \zeta\cos\alpha, \quad v = \zeta\cos\beta, \quad w = \zeta\cos\gamma,$$

$$\frac{d\text{L}}{dx} = \lambda\cos a, \quad \frac{d\text{L}}{dy} = \lambda\cos b, \quad \frac{d\text{L}}{dz} = \lambda\cos c;$$

and, by substituting these values in equation (9), and then suppressing the common factor $\zeta\lambda$, it will become

$$\cos\alpha\cos a + \cos\beta\cos b + \cos\gamma\cos c = 0.$$

Therefore equation (9) indicates that the direction of the velocity of each point of the fluid adjacent to a fixed side, is normal to this surface; and, in fact, it is the condition which must be satisfied, and is sufficient to prevent this point from being detached from the side, so that it can only slide on its surface.

At the free surface of a liquid, the pressure $p$ is in general a constant quantity; but it may depend on, or be a function of $t$, and be only independent of $x$, $y$, $z$, if the external pressure, which is common to all the points of this surface, varies with the time; therefore denoting this function by $\varpi$, the equation of the free surface will be $p - \varpi$, and by putting $p - \varpi$ in place of $f$, in equation (8), we shall have

$$\frac{dp}{dt} + u\frac{dp}{dx} + v\frac{dp}{dy} + w\frac{dp}{dz} = \frac{d\varpi}{dt}, \tag{10}$$

which will have place at the same time as $p - \varpi = 0$, or simultaneously with the differential equation of the free surface, which has been given in No. 647.

It may be remarked that equations (8), (9), (10) will likewise still obtain without any sensible error, when the points of the fluid only deviate from its superficies by insensible quantities. Consequently, if, as in the preceding number, a portion of the fluid is considered, the dimensions of which, though insensible, are still of a finite magnitude, so that it may, notwithstanding, contain an immense number of molecules, and if a part of its surface be supposed to appertain to that of the fluid at a determinate epoch, these equations will express, in reality, that it will have place during the entire continuance of the motion. The extent of this part, which belongs in common to the two surfaces, may besides vary in any ratio whatever; and the

small portion of the fluid in question may, when it is displaced, at the surface of the fluid, enlarge or contract, without its volume undergoing any change in the case of a liquid, or its mass in the case of any fluid whatever($h$). Thus, for example, when a heavy liquid oscillates in a vessel which is open at its upper surface, the extent of its free surface, and that of its surface of contact with the sides of the vessel, vary during the motion, so that the number of material points of the liquid, which are situated on one or other of these two surfaces, is not constantly the same; but equations (9) and (10) may have place notwithstanding, if it is considered that they do not belong solely to detached points, but rather refer to small portions of the liquid which are of an insensible magnitude and variable form.

By means of these particular equations, which were introduced by Lagrange in the theory of fluids, combined in each case with the initial state of the system, the arbitrary functions contained in the equations of the motion can be determined.

653. There is a very extended case, in which the three equations (3) can be reduced to an equation of partial differences of the first order, and the three unknown, $u$, $v$, $w$, made to depend on one sole quantity. This case has place when the formula $udx + vdy + wdz$ is an exact differential of a function of $x$, $y$, $z$, regarded as independent variables, and the fluid in question is homogeneous and has every where the same temperature in the state of equilibrium. Let then

$$udx + vdy + wdz = d\phi;$$

$\phi$ denoting an unknown function of the four variables $t$, $x$, $y$, $z$, but in which, however, the differential $d\phi$ is taken solely with respect to $x$, $y$, $z$, so that we may have

$$u = \frac{d\phi}{dx}, \quad v = \frac{d\phi}{dy}, \quad w = \frac{d\phi}{dz}. \qquad \text{(a)}$$

By the nature of the forces X, Y, Z, which always are supposed to arise from attractions or repulsions, whose centres are

fixed or moveable points, or points of the fluid itself, we have likewise

$$\text{X}dx + \text{Y}dy + \text{Z}dz = d\text{V},$$

and, consequently,

$$\text{X} = \frac{d\text{V}}{dx}, \quad \text{Y} = \frac{d\text{V}}{dy}, \quad \text{Z} = \frac{d\text{V}}{dz};$$

V being a function of $t$, $x$, $y$, $z$, which is differenced solely with respect to $x$, $y$, $z$. In the case of an elastic fluid whose density is constant in a state of repose, the integral $\int \frac{dp}{\rho}$ will be expressed by a logarithm, provided the law of Mariotte be supposed still to obtain in a state of motion also($i$); if the variations of temperature which are produced by those of the density during the motion be taken into account, this integral will be a different function of $p$; and in the case of a homogeneous liquid, it will be reduced to $\frac{1}{\rho}p$, without taking into account the arbitrary constant. In order to comprise all these cases under one, let

$$\int \frac{dp}{\rho} = \text{P};$$

there results from this($k$)

$$\frac{1}{\rho}\frac{dp}{dx} = \frac{d\text{P}}{dx}, \quad \frac{1}{\rho}\frac{dp}{dy} = \frac{d\text{P}}{dy}, \quad \frac{1}{\rho}\frac{dp}{dz} = \frac{d\text{P}}{dz};$$

and by means of these and the preceding values, equations (3) will become

$$\frac{d\text{P}}{dx} = \frac{d\text{V}}{dx} - \frac{d^2\phi}{dxdt} - \frac{d\phi}{dx}\frac{d^2\phi}{dx^2} - \frac{d\phi}{dy}\frac{d^2\phi}{dxdy} - \frac{d\phi}{dz}\frac{d^2\phi}{dxdz},$$

$$\frac{d\text{P}}{dy} = \frac{d\text{V}}{dy} - \frac{d^2\phi}{dydt} - \frac{d\phi}{dy}\frac{d^2\phi}{dydx} - \frac{d\phi}{dy}\frac{d^2\phi}{dy^2} - \frac{d\phi}{dz}\frac{d^2\phi}{dydz},$$

$$\frac{d\text{P}}{dz} = \frac{d\text{V}}{dz} - \frac{d^2\phi}{dzdt} - \frac{d\phi}{dz}\frac{d^2\phi}{dzdx} - \frac{d\phi}{dy}\frac{d^2\phi}{dzdy} - \frac{d\phi}{dz}\frac{d^2\phi}{dz^2}.$$

If these equations be multiplied by $dx$, $dy$, $dz$, respectively, and then added together, there results ($l$)

$$dP = dV - d.\frac{d\phi}{dt} - \tfrac{1}{2}d.\left[\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right];$$

and as all the terms of this equation are exact differentials of the three variables $x$, $y$, $z$, we deduce immediately

$$V - P = \frac{d\phi}{dt} + \tfrac{1}{2}\left[\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right]. \qquad \text{(b)}$$

The arbitrary constant which ought to be added to this integral may be considered as contained in the unknown quantity $\phi$, and thus the integrals V and P may be regarded as quantities entirely determinate.

This equation, which replaces the three equations (3), will make known the value of $p$, when that of $\phi$ shall have been determined; likewise equations (a) will determine the three unknown qnantities $u$, $v$, $w$; and with respect to the value of $\phi$, it can be deduced from equation (4), which becomes

$$\frac{d\rho}{dt} + \frac{d.\rho\frac{d\phi}{dx}}{dx} + \frac{d.\rho\frac{d\phi}{dy}}{dy} + \frac{d.\rho\frac{d\phi}{dz}}{dz} = 0. \qquad \text{(c)}$$

In the case of an incompressible fluid, this equation will be reduced to

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0;$$

in the case of an aeriform fluid, we should substitute for $\rho$ its value in a function of $p$, and for $p$ its value deduced from equation (b).

654. In order that the formula $udx + vdy + wdz$ may be an exact differential during the entire continuance of the motion, it should be so at the commencement, and the initial values of $u$, $v$, $w$, which are given arbitrarily in functions of $x$, $y$, $z$, should satisfy the conditions of integrability. Conversely, though it be admitted that it is sufficient to prove this formula to be an exact differential relative to a determinate value of $t$, in order to be satisfied that it is one also for all values of this quantity; still this proposition is not so

general as has been supposed. It may be demonstrated in the following manner:

Let $t_1$ be a particular value of $t$, and, for this value, let

$$udx + vdy + wdz = d\phi_1;$$

$\phi_1$ being a function of $x, y, z$. If $\varepsilon$ denotes an infinitely small interval of time, then when the time $t$ becomes $t_1 + \varepsilon$, the quantities $u, v, w$ will likewise vary, and, on the supposition that their expressions in functions of $t$ are developable according to the powers of $\varepsilon$, we shall have($m$)

$$u = \frac{d\phi_1}{dx} + \varepsilon u_1, \quad v = \frac{d\phi_1}{dy} + \varepsilon v_1, \quad w = \frac{d\phi_1}{dz} + \varepsilon w_1,$$

and, consequently,

$$udx + vdy + wdz = d\phi_1 + \varepsilon(u_1dx + v_1dy + w_1dz),$$

in which $u_1, v_1, w_1$ denote functions of $x, y, z$. In order to obtain the values of the partial differences $\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}$, which occur in equations (3), these values of $u, v, w$ should be differenced with respect to $\varepsilon$; this gives

$$\frac{du}{dt} = u_1, \quad \frac{dv}{dt} = v_1, \quad \frac{dw}{dt} = w_1.$$

By substituting them with those of $u, v, w$, and of their partial differences relative to $x, y, z$, in these equations, and then suppressing the terms multiplied by $\varepsilon$, there results

$$\frac{1}{\rho}\frac{dp}{dx} = \mathrm{X} - u_1 - \frac{d\phi_1}{dx}\frac{d^2\phi_1}{dx^2} - \frac{d\phi_1}{dy}\frac{d^2\phi_1}{dxdy} - \frac{d\phi_1}{dz}\frac{d^2\phi_1}{dxdz},$$

$$\frac{1}{\rho}\frac{dp}{dy} = \mathrm{Y} - v_1 - \frac{d\phi_1}{dx}\frac{d^2\phi_1}{dydx} - \frac{d\phi_1}{dy}\frac{d^2\phi_1}{dy^2} - \frac{d\phi_1}{dz}\cdot\frac{d^2\phi_1}{dy.dz},$$

$$\frac{1}{\rho}\frac{dp}{dz} = \mathrm{Z} - w_1 - \frac{d\phi_1}{dx}\cdot\frac{d^2\phi_1}{dzdx} - \frac{d\phi_1}{dy}\frac{d^2\phi_1}{dzdy} - \frac{d\phi_1}{dz}\cdot\frac{d^2\phi_1}{dz^2};$$

hence by the preceding notations we deduce($n$)

$$u_1 dx + v_1 dy + w_1 dz$$
$$= d\text{V} - d\text{P} - \tfrac{1}{2} d.\left[\left(\frac{d\phi_1}{dx}\right)^2 + \left(\frac{d\phi_1}{dy}\right)^2 + \left(\frac{d\phi_1}{dz}\right)^2\right];$$

it follows from this that the quantity $(u_1 dx + v_1 dy + w_1 dz)\varepsilon$, by which the formula $udx + vdy + wdz$ is increased during the time $\varepsilon$, will be an exact differential. Consequently, this formula will be an exact differential at the end of the time $t_{\prime} + \varepsilon$, since it is supposed to be so at the end of the time $t_{\prime}$; it will be so at the end of the time $t_{\prime} + 2\varepsilon$, since it is so at the end of the time $t_{\prime} + \varepsilon$, and so on. And as $\varepsilon$ may be either positive or negative, it follows that this formula $udx + vdy + wdz$ is an exact differential for all values of $t$, if it is so for any value whatever of this variable.

But this demonstration supposes that the values of $u$, $v$, $w$, which correspond to $t + \varepsilon$, may be developed according to the powers of $\varepsilon$, or, what comes to the same thing, it supposes that the expressions of $u$, $v$, $w$, in functions of $t$, satisfy the equations of the problem and all those which may be deduced from it by differentiations relative to $t$. Now, this is not always the case, with respect to expressions of $u$, $v$, $w$ in series of exponentials and of sines and cosines, the exponents and arcs of which are proportional to $t$; and as the demonstration then fails, the proposition may be likewise at fault, and it is in point of fact faulty in certain cases, examples of which have been met with by the author. In each problem, the expressions of $u$, $v$, $w$, in question, satisfy the equations relative to the *mass* and *surface* of the fluid in motion; and by determining in a suitable manner the coefficients of the exponentials and of the sines and cosines, they represent the given initial state of all the points of the fluid; and if the series which result from them, are moreover convergent, this is sufficient, in order that they may contain the solution of the question, although one of their particular characters may not satisfy always the equations which may be deduced from those of the motion, by new differentiations.

655. The condition of integrability of the formula $udx + vdy + wdz$ has not place in the motion of a fluid which turns about a fixed axis without changing its form. In fact, the components of the velocity of any point whatever are then the same as in the case of a solid body; therefore if the fixed axis be assumed to be that of $z$, and if the angular velocity of rotation be denoted by $\omega$, we shall have (No. 387),

$$u = -y\omega, \quad v = x\omega, \quad w = 0;$$

from which there results

$$udx + vdy + wdz = \omega(xdy - ydx);$$

this quantity is not an exact differential, because the factor $\omega$ is independent of the coordinates $x$ and $y$.

Hence in order to determine the pressure $p$ in any point whatever, we must have recourse, in this example, to equations (3). Now, if the values of $u, v, w$ be substituted in this expression, there results, when $\omega$ is considered to be a constant quantity with respect to $t$, as well as with respect to $x, y, z$,

$$\frac{1}{\rho}\frac{dp}{dx} = \mathrm{X} + \omega^2 x, \quad \frac{1}{\rho}\frac{dp}{dy} = \mathrm{Y} + \omega^2 y, \quad \frac{1}{\rho}\frac{dp}{dz} = \mathrm{Z};$$

hence we obtain ($o$)

$$\frac{1}{\rho}dp = \mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz + \omega^2(xdx + ydy);$$

an equation which coincides with that which has been obtained in No. 589, from the consideration of the equilibrium of the given forces that act on all the points of the fluid, and of their centrifugal forces resulting from its motion of rotation.

## CHAPTER II.

### OF THE PROPAGATION OF SOUND.

586. As it does not fall in with the plan of this treatise, to detail the numerous results which have been obtained from the general equations of the motion of fluids that have been given in the preceding chapter, we shall merely point out those treatises in which they can be found. In the following chapter, the motion of a fluid which flows out of a vessel, is determined on a particular hypothesis, which, for the most part, gives results sufficiently accurate in practice; in the present one, we shall select for examples of the application of the general equations, the simplest cases of the theory of sound.

1st. In the second and third books of the *Méchanique Céleste*, the reader will find detailed all that is as yet known about the oscillations of the sea and of the atmosphere, produced by the attractions of the sun and moon.

2nd. In the second volume of the *Mechanique Analytique* there is given the determination, by means of convergent series, of the motion of a heavy liquid, both in a very narrow canal, and also in a very deep vessel.

3rd. Relatively to the oscillations of this liquid in a vessel of any depth whatever, the reader is referred to a memoir inserted by the author on this subject, in the nineteenth volume of the Journal of M. Gergonne.

4th. For the problem of the propagation of waves at the surface and in the interior of stagnant water, the reader is also referred to a memoir of the author inserted in the first volume of the Academy of Sciences.

5th. On the propagation of elastic fluids in vessels and

narrow tubes, the memoir of M. Navier, which is inserted in the ninth volume of this Academy, may be consulted.

Finally, for every thing which concerns the theory of sound, and generally the propagation of the motion in an elastic medium or in several superimposed media, the student may consult the memoirs written by the author on this subject, which are contained in the fourteenth Number of the Journal of the Polytechnic School, and in the eleventh and tenth volumes of the Academy of Sciences.

657. To give an application of the general equations, let an elastic homogeneous fluid be considered, whose density and temperature may be throughout the same in its state of equilibrium, and in which when it is made to deviate from this state ever so little, the velocities of its different points, and likewise the dilatations and condensations with which they are accompanied, in the motion which results, may be very small fractions, so that the squares and products of these quantities can consequently be neglected; by which means the equations of the motion are reduced to a linear form, the integrals of which may be obtained in a finite form. Moreover, as the density of the fluid, in a state of equilibrium, is by hypothesis constant, the forces X, Y, Z should be made equal to cipher.

Let this density be denoted by D; $\rho$ being that which has place in the state of motion, at the end of the time $t$, and for the point whose coordinates are $x$, $y$, $z$, we shall have

$$\rho = \mathrm{D}(1 + s),$$

in which equation, $s$ is a very small fraction, either positive or negative. Likewise, let $h$ and $mgh$ denote the height and barometrical pressure corresponding to the density D, $g$ the gravity, and $m$ the density of the mercury. In the state of motion, the pressure $p$ which corresponds to the density $\rho$, will by the law of Mariotte, be $gmh(1 + s)$, if the temperature of the fluid be invariable; but in consequence of the condensation or rarefaction denoted by $s$, the temperature increases or dimi-

nishes; and if the motion be so rapid that the fluid has not time to revert to its original temperature, the pressure will vary in a greater ratio than the density($a$). Therefore we shall suppose that in general

$$p = gmh\,(1 + s + \sigma);$$

in which $\sigma$ denotes a quantity having the same sign as $s$, and is a certain function of it. In consequence of the smallness of $s$, the quantity $\sigma$ may be supposed to be proportional to $s$, and such that

$$\sigma = \beta s;$$

$\beta$ being a positive coefficient independent of $s$.

By means of these values, we shall have

$$dp = gmh(1 + \beta)\,ds;$$

and by supposing that the integral vanishes with $s$, and making, for conciseness,

$$\frac{gmh(1+\beta)}{\text{D}} = a^2,$$

there will result

$$\int \frac{dp}{\rho} = a^2 . \log(1 + s).$$

If this integral be taken for the value of the quantity P comprised in equation (b) of No. 653, we shall obtain by neglecting the square of $s$,

$$\text{P} = a^2 s;$$

in like manner if the squares of the velocities $\frac{d\phi}{dz}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$, be also neglected, this equation will become, by suppressing the term V which arises from the forces X, Y, Z,

$$s = -\frac{1}{a^2}\frac{d\phi}{dt}; \qquad (1)$$

and by joining it with equations (a), namely,

$$u = \frac{d\phi}{dx}, \quad v = \frac{d\phi}{dy}, \quad w = \frac{d\phi}{dz}; \qquad (2)$$

these four equations will make known the condensation, the magnitude and direction of the velocity of the fluid, at the end of the time $t$, and for the point whose coordinates are $x, y, z$, when the function $\phi$ shall have been determined in a function of $x, y, z, t$.

If the displacements of the molecules of the fluid are likewise supposed to be very small, that is to say, if the molecules of the fluid make only very small oscillations, and have no common motion of translation or rotation, the variables $x, y, z$ will differ very little from $x', y', z'$, which are the initial coordinates of the points to which they belong, so that they may be regarded as equal to $x', y', z'$, when the values of $udt$, $vdt$, $wdt$ are integrated, in order to deduce from them at any instant whatever, the displacements of this point in the direction of the three axes of the coordinates; and then, we shall have

$$x - x' = \int udt, \quad y - y' = \int vdt, \quad z - z' = \int wdt;$$

the integrals being taken so that they may vanish when $t = 0$ (*b*).

With respect to the quantity $\phi$, in order to obtain the equation on which it depends, let D $(1 + s)$ be put in place of $\rho$ in equation (c) of the number cited above, then by neglecting the products of $s$ and $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$, it becomes

$$\frac{ds}{dt} + \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0,$$

or, what comes to the same thing by substituting for $s$ its preceding value (*c*),

$$\frac{d^2\phi}{dt^2} = a^2\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right) \qquad (3)$$

These equations (1), (2), (3) are those of the theory of sound in air whose temperature and density are constant. They suppose that the formula $udx + vdy + wdz$ is an exact diffe-

rential, and this is, in point of fact, the case in the two particular instances, to which we proceed to apply them.

658. Let us, in the first place, suppose that the air is contained in a cylindrical tube, and that its points move parallel to the axis which is assumed to be horizontal, in order that the gravity should not cause the density to vary. If the axis of $x$ coincides with this direction, $v$ and $w$ will be respectively equal to cipher, and the quantity $\phi$ will only be a function of $x$ and $t$, so that equation (3) will be reduced to

$$\frac{d^2\phi}{dt^2} = a^2 \frac{d^2\phi}{dx^2}.$$

The same consequences may be deduced from this as were obtained from equation (1) of No. 494, with respect to the longitudinal vibrations of an elastic rod. When the tube extends indefinitely, $a$ will be the velocity of the propagation of sound in the direction of its length; when it is of a finite length equal to $l$, the number of vibrations of the fluid in the 1 unit of time, corresponding to the gravest sound, will be in the inverse ratio of $l$; when the tone is raised, this number will 2 increase in the same ratio as that of the nodes of vibrations; and if the distance between two consecutive nodes be denoted by $\lambda$, and the corresponding number of vibrations by $n$, we shall have

$$n = \frac{a}{2\lambda}.$$

In those points, the velocity of the molecules of the air is cipher, but the condensation does not vanish; there are, on the contrary, other points, where this condensation is cipher, and where still the fluid is in motion. The distances between these other points are the same as for the first, as is evident from the formulæ of No. 495. They possess a property which appertains exclusively to them, by means of which they can be determined by experiment. If an opening is made in the side of the tube at one of these points where the condensation is

cipher, and if a communication is thus established with the external air, the motion of the interior fluid is not in any way affected, nor the tone which is produced. If $\lambda$ be the distance between two of these consecutive points, and $n$ the number corresponding to the observed tone, the preceding equation will make known the value of $a$, and, consequently, that of $\beta$ the quantity that occurs in the expression of this velocity. It is preferable, for this object, to make use of the elevated tone which corresponds to an *aliquot* part of $l$, rather than the *fundamental* tone, which may be influenced by the mode of blowing into the tube, and by the circumstances relative to the mouth-piece. It is in this manner that M. Dulong has determined for air and different gases, the values of the quantity $\gamma$ of No. 637; which quantity is equal to $1+\beta$, as we shall see immediately.

659. For a second example, let the mass of air be supposed to extend indefinitely on every side, and that it is agitated in like manner, in all directions, about a fixed point which is assumed for the origin of the coordinates. If $r$ be the radius vector of the point, whose coordinates are $x, y, z$, at the end of the time $t$, and $\zeta$ its velocity, it will be directed along this radius, and its magnitude will be a function of $r$ and $t$, as well as the condensation $s$; for it is evident that every thing should be symmetrical about the origin of the coordinates, during the entire continuance of the motion. We shall have

$$u=\zeta\frac{x}{r},\quad v=\zeta\frac{y}{r},\quad w=\zeta\frac{z}{r};$$

and because

$$x^2+y^2+z^2=r^2,\quad xdx+ydy+zdz=rdr;$$

there will result,

$$udx+vdy+wdz=\zeta dr;$$

so that this formula will be an exact differential of a function

of $r$ and $t(d)$. As this function is the quantity $\phi$, which has been determined by equation (3), we shall have

$$\zeta = \frac{d\phi}{dr},$$

or the resultant of the velocities $u, v, w$.

By differentiating it with respect to $x, y, z$, we shall also have

$$\frac{d\phi}{dx} = \frac{d\phi}{dr}\cdot\frac{x}{r}, \quad \frac{d\phi}{dy} = \frac{d\phi}{dr}\cdot\frac{y}{r}, \quad \frac{d\phi}{dz} = \frac{d\phi}{dr}\cdot\frac{z}{r};$$

by differentiating a second time, we obtain

$$\frac{d^2\phi}{dx^2} = \frac{d^2\phi}{dr^2}\cdot\frac{x^2}{r^2} + \frac{d\phi}{dr}\cdot\frac{y^2+z^2}{r^3},$$

$$\frac{d^2\phi}{dy^2} = \frac{d^2\phi}{dr^2}\cdot\frac{y^2}{r^2} + \frac{d\phi}{dr}\cdot\frac{z^2+x^2}{r^3},$$

$$\frac{d^2\phi}{dz^2} = \frac{d^2\phi}{dr^2}\cdot\frac{z^2}{r^2} + \frac{d\phi}{dr}\cdot\frac{x^2+y^2}{r^3};$$

and, by substituting these values of $\frac{d^2\phi}{dx^2}, \frac{d^2\phi}{dy^2}, \frac{d^2\phi}{dz^2}$, in equation (3), it becomes $(e)$

$$\frac{d^2\phi}{dt^2} = a^2\left(\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr}\right),$$

or, what comes to the same thing $(f)$,

$$\frac{d^2.r\phi}{dt^2} = a^2.\frac{d^2.r\phi}{dr^2}. \qquad (4)$$

The complete integral of this equation is (No. 484),

$$r\phi = f(r+at) + \text{F}(r-at);$$

in which $f$ and F denote two arbitrary functions. If therefore, for any variable whatever such as $z$, we make

$$\frac{dfz}{dz} = f'z, \quad \frac{d\text{F}z}{dz} = \text{F}'z,$$

we can deduce from this integral,

$$\left.\begin{aligned} \zeta &= \frac{1}{r}\left[f'(r+at) + \mathrm{F}'(r-at)\right] \\ &\quad - \frac{1}{r^2}\left[f(r+at) + \mathrm{F}(r-at)\right], \\ s &= \frac{1}{ar}\left[\mathrm{F}'(r-at) - f'(r+at)\right]; \end{aligned}\right\} \quad (5)$$

and by means of these formulæ, when the functions $f$ and $f'$ are determined for all values of $r + at$, which is a positive variable, and the functions $\mathrm{F}$ and $\mathrm{F}'$ for all the positive or negative values of $r - at$, the velocity and condensation in any point and at any instant whatever will be known($g$).

660. As by hypothesis, every thing is alike about the origin of the coordinates, the centre of the agitation of the fluid must continue immoveable during the entire continuance of the motion; the first formula (5) must, therefore, vanish at the same time as $r$; this implies, that when this radius is infinitely small, we should have

$$f(r+at) + \mathrm{F}(r-at) = \mathrm{T}r,$$
$$f'(r+at) + \mathrm{F}'(r-at) = \mathrm{T};$$

$\mathrm{T}$ denoting an unknown function of $t$. Therefore, if the radius $r$ be made altogether equal to cipher in the first of these equations, and in its differential with respect to $at$, namely,

$$f'(r+at) - \mathrm{F}'(r-at) = \frac{r}{a}\frac{d\mathrm{T}}{dt},$$

we shall obtain, by substituting $z$ in place of $at$($h$),

$$fz + \mathrm{F}(-z) = 0, \quad f'z - \mathrm{F}'(-z) = 0. \qquad (6)$$

but solely for the positive values of $z$. These equations will make known the values of $\mathrm{F}(-z)$ and $\mathrm{F}'(-z)$ by means of those of $fz$ and $f'z$, so that it only remains to determine the values of $fz, f'z$, $\mathrm{F}z$, $\mathrm{F}'z$, for all positive values of $z$.

For this purpose, let $\psi r$ and $\frac{1}{a}\Psi r$ be the *initial* values of

$\zeta$ and $s$, so that $\psi r$ and $\Psi r$ may denote given functions from $r = 0$ to $r = \infty$, the first of which should be cipher for $r = 0$, and both the one and the other denote always certain velocities$(i)$. By making $t = 0$ in equations (5), we shall have$(k)$

$$\psi r = \frac{d.\frac{1}{r}fr}{dr} + \frac{d.\frac{1}{r}\text{F}r}{dr},$$

$$r\Psi r = \frac{d\text{F}r}{dr} - \frac{d.fr}{dr};$$

hence we obtain, by making

$$\textstyle\int \psi r dr = \psi_1 r, \quad \int r\Psi r dr = \Psi_1 r;$$

$$\left.\begin{aligned} \frac{1}{r}fr + \frac{1}{r}\text{F}r &= \psi_1 r + b, \\ \text{F}r - fr &= \Psi_1 r + c; \end{aligned}\right\} \quad (7)$$

$b$ and $c$ denoting two arbitrary constants introduced by the integration; as we may suppose that the two integrals $\psi_1 r$, $\Psi r_1$, vanish for any value we please of $r$, we will presently assume this value to be $r = \infty$.

If we have solely regard to the constants $b$ and $c$, the preceding equations will give

$$fr = \tfrac{1}{2}br - \tfrac{1}{2}c, \quad f'r = \tfrac{1}{2}b,$$
$$\text{F}r = \tfrac{1}{2}br + \tfrac{1}{2}c, \quad \text{F}'r = \tfrac{1}{2}b;$$

hence there results,

$$f(r + at) = \tfrac{1}{2}b(r + at) - \tfrac{1}{2}c,$$
$$f'(r + at) = \tfrac{1}{2}b;$$

likewise for $r > at$, we shall have

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$$\text{F}(r - at) = \tfrac{1}{2}b(r - at) + \tfrac{1}{2}c,$$
$$\text{F}'(r - at) = \tfrac{1}{2}b;$$

for $r \angle at$, we shall have

$$f(at - r) = \tfrac{1}{2}b(at - r) - \tfrac{1}{2}c,$$
$$f'(at - r) = \tfrac{1}{2}b;$$

and in virtue of equations (6), there will result from this

$$\text{F}(r-at)=\tfrac{1}{2}b(r-at)+\tfrac{1}{2}c,$$
$$\text{F}'(r-at)=\tfrac{1}{2}b;$$

as in the case of $r>at$. Now, it will be found, that if these different values be substituted in formulæ (5), they will be reduced to cipher; so that the two arbitrary constants $b$ and $c$ must disappear from the expressions of $\zeta$ and $s(l)$.

Therefore, if they are not taken into account, there results, by substituting $z$ in place of $r$ in equations (7) and in their differentials,

$$\left.\begin{aligned} fz &= \tfrac{1}{2}z\psi_1 z - \tfrac{1}{2}\Psi_1 z, \\ f'z &= \tfrac{1}{2}\psi_1 z + \tfrac{1}{2}z(\psi z - \Psi z), \\ \text{F}z &= \tfrac{1}{2}z\psi_1 z + \tfrac{1}{2}\Psi_1 z. \\ \text{F}'z &= \tfrac{1}{2}\psi_1 z + \tfrac{1}{2}z(\psi z + \Psi z), \end{aligned}\right\} \quad (8)$$

for the values which it was required to find$(m)$.

As formulæ (5) will not contain any unknown quantity, they give the complete solution of the problem. It may be observed here with reference to $f(-z)$, that there is nothing in the question to enable us to determine its value, but it is evident, that a knowledge of this function is not required in formulæ (5).

661. The following consequences relative to the theory of sound may be deduced from these formulæ.

Let $\varepsilon$ be the radius of the primitive agitation, so that the given values of $\psi r$ and $\Psi r$ may be of an arbitrary magnitude from $r=0$ to $r=\varepsilon$, and cipher from $r=\varepsilon$ to $r=\infty$. The integrals $\psi_1 r$ and $\Psi_1 r$ will be constant quantities for all values of $z$ which surpass $\varepsilon$; and, as they are by supposition cipher for $r=\infty$, they will be so likewise from $r=\varepsilon$ to $r=\infty$. This being established, if first a point of the fluid comprised within the extent of the primitive agitation be considered, we shall have $r \angle \varepsilon$; as long as $t$ will be less than $\frac{\varepsilon-r}{a}$, the values of $f(r+at)$ and $f'(r+at)$ will not be cipher, and they

may be deduced from equations (8); the same will be the case with respect to the values of $\mathrm{F}(r-at)$ and $\mathrm{F}'(r-at)$ as long as $t \angle \frac{r}{a}$; when $t$ is greater than $\frac{r}{a}$, these can be deduced from those of $f(at-r)$ and $f'(at-r)$ by means of equations (6); finally, when the time $t$ becomes greater than $\frac{r+\varepsilon}{a}$, all the terms of formulæ (5) will be cipher, and all the points contained in the extent of the primitive agitation will have reverted to a state of repose. Thus, for all points contained within the sphere, whose radius is $\varepsilon$, the duration of the motion will decrease from the centre to the surface, between the limits $\frac{\varepsilon}{a}$ and $\frac{2\varepsilon}{a}$ (*n*).

Beyond the primitive agitation, we shall have $r+at>\varepsilon$, this will cause $f(r+at)$ and $f'(r+at)$ to disappear from formulæ (5), and reduce them to

$$\zeta = \frac{1}{r}\mathrm{F}'(r-at) - \frac{1}{r^2}\mathrm{F}(r-at),$$

$$s = \frac{1}{ar}\mathrm{F}'(r-at),$$

when $r>at$, or, what comes to the same thing in virtue of equations (6), to

$$\zeta = \frac{1}{r}f'(at-r) + \frac{1}{r^2}f(at-r),$$

$$s = \frac{1}{ar}f'(at-r),$$

when $at>r$. The values of the quantities comprised in these expressions of $\zeta$ and $s$ will be given by formulæ (8), they will be cipher when $r>at+\varepsilon$, and will become so again when $r \angle at-\varepsilon$; hence it follows, that sound is propagated in the open air, with the same velocity as in the interior of a cylindrical tube; that the motion of each molecule of air will subsist during an interval of time equal to $\frac{2\varepsilon}{a}$, and that the breadth of the sonorous wave will be equal to $2\varepsilon$ the diameter of the primitive agitation (*o*).

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At a great distance from the centre of this agitation, the second terms of the values of $\zeta$, which are divided by $r^2$ may be neglected, relatively to the first, the divisor of which is $r$; we shall then have

$$s = \frac{\zeta}{a},$$

during the entire continuance of the motion, as in No. 497, where $s$ represents the dilatation instead of the condensation. The velocity of each molecule of air will then decrease in the inverse ratio of $r$. The intensity of sound is supposed to be proportional to the square of this velocity; so that at a great distance from the point of the primitive agitation, it will decrease in the inverse ratio of the square of this distance; which is conformable to experiment. These results likewise have place when the agitation is not the same in all directions. At a considerable distance with respect to its diameter, the velocity of sound is uniform and equal to the constant $a$, the form of the waves is nearly the spherical, and the intensity of sound in the direction of each radius varies in the inverse ratio of the square of the distance, whatever may be, in other respects, its variation in passing from one radius to another. This intensity also decreases with the density of the medium in which the sound is produced; so that, for example, it diminishes according as we approach to the summit of a high mountain. In considering the propagation of sound in air, composed of strata of different densities, it is found that at equal distances, its intensity depends solely on the density at the place of the primitive agitation; it follows from this, that a person in a balloon ought to hear the noise made at the surface of the earth, just as if it was at this surface; and, on the other hand, the noise made at the balloon would be heard in precisely the same manner by an individual at the surface, as if the same stratum of the atmosphere, in which the aëronaut floated, extended from the balloon to the earth$(p)$.

If the intensity of sound depends on the magnitude of the velocities of the molecules of air which strike the organs of hearing,

and if the elevation of the tone is regulated by the number of strokes in the same time, that is to say, by the more or less frequent repetition of the vibrations of the air, it may be demanded what it is that makes the difference between one syllable and another, when sung with the same force and on the same tone. According to Euler, this difference ought to be ascribed to the form of the function which expresses the law of the successive velocities of air during each vibration; so that the organ of the voice has the faculty of giving the suitable form to this function, and the organ of hearing, the faculty of appreciating the different forms.

662. The origin of the coordinates may be transferred to other points of the fluid, without the form of equation (4) undergoing any change. Hence, if $r_{\prime}, r_{\prime\prime}, r_{\prime\prime\prime}$, &c.; denote the radii vectores of the same point, reckoned from these different origins, and if $\phi$ be supposed to be successively a function of $t$ and of each of these radii, equation (4) may be satisfied by means of the value of $\phi$ of No. 659, and of the values which may be deduced from it, by substituting $r_{\prime}, r_{\prime\prime}, r_{\prime\prime\prime}$, &c., in place of $r$, and changing each time, the arbitrary functions. On account of the linear form of this equation, it may therefore be likewise satisfied, by taking for $\phi$, the sum of all these particular values, this gives

$$\left.\begin{aligned}\phi = {} & \frac{1}{r}\left[f(r+at) + \mathrm{F}(r-at)\right] \\ & + \frac{1}{r_{\prime}}\left[f_{\prime}(r_{\prime}+at) + \mathrm{F}_{\prime}(r_{\prime}-at)\right] \\ & + \frac{1}{r_{\prime\prime}}\left[f_{\prime\prime}(r_{\prime\prime}+at) + \mathrm{F}_{\prime\prime}(r_{\prime\prime}-at)\right] \\ & + \&c.\end{aligned}\right\} \qquad (9)$$

Now, it follows from this formula, that if the air is simultaneously agitated about each of the origins of $r$, $r_{\prime}$, $r_{\prime\prime}$, &c., $\frac{d\phi}{dt}$ the condensation at any point and instant whatever, which is always given by equation (1), will have for its value, the

sum of the condensations which would have place in virtue of each of these separate agitations. Moreover, it appears from equations (2), that the components of the velocity at the end of the time $t$, and in a point M, the coordinates of which are $x, y, z$, referred to the origin of $r$, will have for expressions

$$u = \frac{d\phi}{dx} = \frac{d\phi}{dr}\frac{dr}{dx} + \frac{d\phi}{dr_1}\frac{dr_,}{dx} + \frac{d\phi}{dr_{,,}}\frac{dr_{,,}}{dx} + \&c.,$$

$$v = \frac{d\phi}{dy} = \frac{d\phi}{dr}\frac{dr}{dy} + \frac{d\phi}{dr_,}\frac{dr_,}{dy} + \frac{d\phi}{dr_{,,}}\frac{dr_{,,}}{dy} + \&c.,$$

$$w = \frac{d\phi}{dz} = \frac{d\phi}{dr}\frac{dr}{dz} + \frac{d\phi}{dr_,}\frac{dr_,}{dz} + \frac{d\phi}{dr_{,,}}\frac{dr_{,,}}{dz} + \&c.;$$

in which the partial differences $\frac{d\phi}{dr}, \frac{d\phi}{dr_,}, \frac{d\phi}{dr_{,,}}$, &c., are so taken, that $r, r_,, r_{,,}$, are considered as independent variables. But if $x_1, y_1, z_1$, be the coordinates of M referred to the origin of $r_,$, and to axes parallel to those of $x, y, z$, these coordinates $x_, y_, z_,$ will only differ from $x, y, z$, by a constant quantity; so that we shall have

$$\frac{dr_,}{dx} = \frac{dr_,}{dx_,} = \frac{x_,}{r_,}, \quad \frac{dr_,}{dy} = \frac{dr_,}{dy_,} = \frac{y_,}{r_,}, \quad \frac{dr_,}{dz} = \frac{dr_,}{dz_,} = \frac{z_,}{r_,};$$

we shall have likewise

$$\frac{dr_{,,}}{dx} = \frac{x_{,,}}{r_{,,}}, \quad \frac{dr_{,,}}{dy} = \frac{y_{,,}}{r_{,,}}, \quad \frac{dr_{,,}}{dz} = \frac{z_{,,}}{r_{,,}},$$

$x_{,,}, y_{,,}, z_{,,}$, being the coordinates of the same point, the origin of which is the same as that of $r_{,,}$, and so on. Therefore, the preceding formulæ will become

$$u = \frac{d\phi}{dr}\frac{x}{r} + \frac{d\phi}{dr}\frac{x_,}{r_,} + \frac{d\phi}{dr_{,,}}\frac{x_{,,}}{r_{,,}} + \&c.,$$

$$v = \frac{d\phi}{dr}\frac{y}{r} + \frac{d\phi}{dr}\frac{y_,}{r_,} + \frac{d\phi_,}{dr_{,,}}\frac{y_{,,}}{r_{,,}} + \&c.,$$

$$w = \frac{d\phi}{dr}\frac{z}{r} + \frac{d\phi}{dr}\frac{z_,}{r_,} + \frac{d\phi}{dr}\frac{z_{,,}}{r_{,,}} + \&c.;$$

hence it follows, that the resultant of $u, v, w$, will be the same as that of the velocities $\frac{d\phi}{dr}, \frac{d\phi}{dr_{\prime}}, \frac{d\phi}{dr_{\prime\prime}}$, &c., which act in the directions of the radii vectores $r, r_{\prime}, r_{\prime\prime}$, &c., and, consequently, in virtue of formula (9), the same, in magnitude and direction, as if all the agitations about the centres of these radii obtained separately; this agrees with the principle of the superimposition of small motions $(q)$.

663. By means of this formula (9), the reflexion of sound on a fixed plane can be determined.

For this purpose, let us suppose that the mass of air is terminated by a fixed plane AB (fig. 55), and that the primitive agitation has place about the point C, the origin of the radius vector $r$, and that it does not extend to the plane AB. From this point, let a perpendicular CD be let fall on this plane, and prolonged to $C_{\prime}$, so that $DC_{\prime}$ may be equal to CD, let $C_{\prime}$ be the origin of $r_{\prime}$, and let the line $CDC_{\prime}$ be taken for the axis of the ordinates $x$ and $x_1$. If the length of CD be denoted by $h$, we shall have $x = h$ and $x_1 = -h$, for all the points of the plane AB; therefore, it is necessary that for these values of $x$ and $x_1$, the velocity $u$ perpendicular to this plane, should be constantly cipher (No. 652). Now this condition, and the initial state of the fluid, can be both satisfied, by making $\phi$ equal to formula (9) reduced to its two first terms, namely,

$$\phi = \frac{1}{r}[f(r+at) + F(r-at)] + \frac{1}{r_1}[f_1(r_1 + at) + F_1(r_1 - at)];$$

and determining in a suitable manner, the arbitrary functions $f, F, f_1, F_1$.

In fact, the two first may be determined, as before, by means of the initial state of the fluid about the point C; and as the points which correspond to $r_{\prime} \angle h$, do not appertain to the fluid, we can assign any value we please to each of the functions $fr_{\prime}$ and $Fr_{\prime}$, without changing this initial state; therefore, we can assume for the functions indicated by $f_{\prime}$ and $F_{\prime}$,

the same functions as were found for those of which the indices are $f$ and F; the preceding formula will then become

$$\phi=\frac{1}{r}[f(r+at)+\mathrm{F}(r-at)]+\frac{1}{r_1}[f(r_1+at)+\mathrm{F}(r_1-at)], \quad (10)$$

and will no longer contain any unknown quantity. Moreover, as for all the points of the plane AB, we have $r_{,}=r$; we shall have, therefore, $\frac{d\phi}{dr}=\frac{d\phi}{dr_{,}}$; and since we have likewise for these same points $x=h$, and $x_{,}=-h$, there results from this $u=0$; so that formula (10) will represent the initial state of the fluid, and satisfy the condition relative to the points adjacent to the plane AB; which it was proposed to obtain($r$).

Let M be the point of the fluid whose radii vectores CM and C,M are $r$ and $r_{,}$; in virtue of the two parts of which formula (10) consists, this point will be first agitated at the end of a portion of time equal to $\frac{r-\varepsilon}{a}$, and then at the end of a portion of time equal to $\frac{r_{,}-\varepsilon}{a}$, in which $\varepsilon$ denotes, as before, the radius of the primitive agitation. The first motion will produce the direct sound, and the second the reflected sound. This last will be the same as if the plane AB had no existence, and a second agitation, identical with that which has place about the point C, had place simultaneously about the point C,. It will be propagated with the same velocity as the direct sound, namely $a$, and will have an intensity corresponding to the distance C,M, or to the line CEM, the parts of which are CE and EM, E being supposed to be the point where the radius C,M cuts the plane AB. Finally, as EF is the normal to this plane, CE and ME the two parts of the sonorous ray which is reflected at the point E, will make CEF the angle of incidence equal to MEF the angle of reflexion. Thus it results from formula (10) that the laws of the reflexion of sound from a fixed plane, are precisely the same as those of light.

664. Let us now compare $a$ the velocity as given by

theory, with that which has been determined by experiment; and for this purpose let us first consider what the quantity $\beta$, which occurs in this expression, denotes.

It appears from No. 657, that

$$p = \frac{gmh\rho(1+s+\beta s)}{\mathrm{D}(1+s)},$$

or more simply, by neglecting the square of $s$, $(s)$

$$p = \frac{gmh}{\mathrm{D}}\rho(1+\beta s). \tag{a}$$

Let $\eta$ be the increase of temperature which corresponds to this condensation $s$; so that the temperature, which was $\theta$ in the state of equilibrium, becomes $\theta+\eta$ at the end of the time $t$, in the state of motion. At this instant, the pressure $p$, the density $\rho$, and the temperature $\theta+\eta$ will have place simultaneously; therefore we shall have by equation (1) of No. 644,

$$p = k\rho[1+a(\theta+\eta)],$$

in which $k$ denotes a coefficient independent of the density and temperature, and $a$ the coefficient 0,00375, that expresses the dilatation of gases. In the state of equilibrium, we have

$$p = gmh, \quad \rho = \mathrm{D}, \quad \eta = 0;$$

therefore the preceding equation when applied to this state, will be

$$gmh = k\mathrm{D}(1+a\theta);$$

consequently, we shall have in the state of motion $(t)$,

$$p = \frac{gmh}{\mathrm{D}}\rho\left(1+\frac{a\eta}{1+a\theta}\right);$$

and, by comparing this value of $p$ with formula (a), there will result

$$\beta = \frac{a\eta}{(1+a\theta)s}.$$

Now, if the oscillations of the air are supposed to be so rapid, that the condensation $s$ has place without any loss of heat, $s$ and $\eta$ may be substituted in place of $\delta$ and $\omega$ in equation (5) of No. 636; this gives

$$1+\beta=\gamma;$$

$\gamma$ expressing the ratio of the specific heat of air under a constant pressure, to its specific heat under a constant volume.

By this means, the value of $a^2$ of No. 657, will become

$$a^2=\frac{gmh\gamma}{\text{D}}.$$

If $\Delta$ be the density of the air under the pressure $gmh$ and at the temperature zero, we shall have (No. 624)

$$\text{D}=\frac{\Delta}{1+\alpha\theta},$$

and, consequently,

$$a=\sqrt{\frac{gmh\gamma}{\Delta}(1+\alpha\theta)}.$$

Since by hypothesis the quantity $\gamma$ is independent of the pressure and temperature (No. 637), it appears—1st, that the velocity $a$ will increase with the temperature $\theta$, in the ratio of $\sqrt{1+\alpha\theta}$ to unity; 2ndly, that it will not vary with the heights of the barometer, since $h$ and $\Delta$ increase simultaneously in the same ratio. The French Academicians who were sent to Peru to measure the arch of the meridian, found, in fact, that the velocity of sound at Quito, where the pressure of the barometer was only $0^m,55$, was very nearly the same as at Paris, where this pressure amounted to $0^m,76$. The hygrometrical state of the air has some little influence on the value of $a$; for since the density diminishes, every thing else being the same, according as the air contains a greater quantity of vapour, $a$ the velocity will increase with the degree of humidity; but from the data of No. 631, it appears that the density of dry air at

the temperature of 18°, 75, for example, hardly exceeds by $\frac{1}{125}$, that of air loaded with the greatest quantity of vapour that it can contain: and this only causes a variation of $\frac{1}{250}$ in the velocity of sound in these two extreme states of the hygrometer($u$).

It has been found in the latest experiments made by persons, who were deputed by the Bureau of Longitude, that

$$a = 340^m, 89,$$

the second being taken for the unit of time, and the temperature of the air being 15°, 9 of the centigrade thermometer. Now, if in formula (b) we make

$$g = 9^m, 80896, \quad h = 0^m, 76, \quad \frac{m}{\Delta} = 10,462,$$

$$\alpha = 0,00375, \quad \theta = 15°, 9, \quad \gamma = 1,3748,$$

we obtain

$$a = 337^m, 07;$$

which differs very little from the result of observation. By assuming (No. 637)

$$\gamma = 1,421,$$

and retaining all the other data, we find,

$$a = 342^m, 69,$$

which differs from the value given by observation in an opposite way from the preceding, but the difference is, as before, very small. If the observed velocity is made use of to determine the value of $\gamma$ by means of the formula

$$\gamma = \frac{a^2 \Delta}{gmh(1 + \alpha\theta)},$$

we obtain, by means of the preceding data,

$$\gamma = 1,4061.$$

665. When this last value of $\gamma$ is compared with the preceding, we should recollect, that in each of them the dilatation or condensation of the air is supposed to be so rapid, that the

quantity of heat of the fluid has not had time to vary in a sensible degree. Now in the propagation of sound in the open air, from which the value of $\gamma = 1,4061$ has been obtained, it is possible that the heat may escape or return with greater facility in a radiating form, than in the case of sound produced by air contained in a tube, the consideration of which has furnished the other value, namely $\gamma = 1,421$, and in which the quantity of heat of each stratum of air can only vary by contact with the sides of the tube. This remark enables us to account for the difference between the two results, and also induces us to think that the greatest value of $\gamma$ is the most exact.

If this quantity is not taken into account, the velocity of sound reduced to $\sqrt{\frac{gmh}{\text{D}}}$, is that given by Newton. It is too small by about a sixth. In order that it might agree with experiment, Lagrange remarked that the pressure must be supposed to vary in a greater ratio than the density, and to be very nearly proportional to the $\frac{4}{3}$ power; and in fact, if the square of $s$ be neglected, the value of $p$ which is made use of, is $(v)$

$$p = gmh(1 + s)^{1+\beta}$$

for the density $\text{D}(1 + s)$. But he did not assign any cause of this more rapid variation of the elastic force of the air; and it was Laplace who first attributed it to the variation of temperature with which the alternate condensations and dilatations of the air are accompanied in the phenomenon of sound.

It is to this same cause that the propagation of sound in vapour produced from water at its *maximum* of density is to be ascribed. If a sonorous body is made to vibrate in a closed vessel which contains this vapour, without any mixture of air, experiment shows that sound is produced in this vapour, and is heard outside it. Now, if the temperature of the stratum of vapour adjacent to this sonorous body, was not increased when it is condensed by the vibrations of this body, it would be re-

duced to water, and precipitated on this body, since it is supposed to be at its *maximum* of density relative to the temperature of the space in which it exists; but as its temperature is increased by the compression, the stratum adjacent to the sonorous body may maintain itself in a state of vapour; it then condenses the following contiguous stratum, this condenses the stratum which is next to it, and so on; so that the sound is propagated as in a medium of permanent gas to the inner side of the vessel. The dilatations of the strata of vapour, which succeed their condensations, are accompanied with a diminution of temperature, by which, however, they are not reduced to water, since their density diminishes at the same time, and falls below the *maximum* relative to the temperature of the space in which the phenomenon takes place.

666. If water be considered as a fluid a little compressible, and perfectly elastic, sound will be propagated in it according to the same laws as in a mass of air. When the sound has reached to the surface of the water, it will be partly transmitted into the external air, and partly reflected back into the water; in this distribution, the direction of the sonorous waves, both transmitted and reflected, will be determined according to the laws of the reflection and refraction of light. The *velocity* of reflected sound will be the same as that of the direct sound, and the ratios of the *intensities* of transmitted and reflected sound to each other, and to the intensity of direct sound, will depend on the ratio of the velocities of the propagation of sound in the two superimposed media, that is to say, in air and water. These points are detailed at length in the memoirs cited at the commencement of this chapter; so that we shall here restrict ourselves to the determination of the numerical value of the velocity of sound in a mass of water.

It appears from what has been observed in the case of an elastic fluid, that this velocity will be the same as if the water was contained in a very narrow tube, the diameter of which was the same throughout; and in this case, this velocity is

also the same as that of the propagation of the motion, along the length of an elastic rod of the same material as water. Now let us suppose that a column of water contained in a vertical cylinder is pressed at its upper surface by the weight $\Delta$, and let $l$ be its natural length, and $l - \delta l$ what it becomes by the effect of this pressure, so that $\delta$ may be a very small fraction which expresses the condensation of the liquid; likewise let $p$ be its weight, and $g$ the gravity; if, as in No. 494, we make

$$\frac{\Delta}{\delta} = q, \quad \frac{glq}{p} = a^2,$$

$a$ will be the required velocity, as has been observed in No. 497.

Let $b$ denote the horizontal section of the column of water; then if the pressure $\Delta$ is supposed to be equal to the weight of a column of mercury whose base is $b$, and height is equal to $h$, we shall have

$$\Delta = gmhb, \quad p = g\rho lb,$$

$m$ denoting the density of the mercury, and $\rho$ that of water; and there will result from this

$$a^2 = \frac{gmh}{\rho\delta};$$

so that in order to calculate the value of $a$, it is sufficient to know the fraction $\delta$ relative to a given height $h$.

The English philosopher Canton found

$$\delta = 0,000046,$$

at the temperature of 10 degrees of the centigrade thermometer, and under a pressure equivalent to the ordinary pressure of the atmosphere. This result has been confirmed by experiments recently made, under more considerable pressures, as has been already observed in No. 575, and these show that the condensation is proportional to the pressure, and equal to the preceding value of $\delta$, for each atmospherical pressure. More-

over these experiments, however great the pressure may have been, do not indicate any sensible increase of temperature, so that there is no reason to think that the propagation of sound in water is accompanied, as in the air, with a variation of temperature which can influence its velocity. This being the case, if this value of $\delta$ be substituted in the preceding formula, and if we make

$$g = 9^m,80896, \quad h = 0^m,76, \quad \frac{m}{\rho} = 13,5975,$$

we obtain from it

$$a = 1484^m;$$

so that the velocity of sound in water is more than the quadruple of its velocity in air($x$).

# CHAPTER III.

## OF THE MOTION OF FLUIDS IN A PARTICULAR HYPOTHESIS.

667. THE supposition which is made in this chapter is known under the denomination of the *hypothesis of the parallelism of the slices.* It consists in supposing that when a heavy fluid, water for example, flows out of a vessel, and issues through a horizontal orifice made in the bottom of the vessel, the infinitely slender horizontal slices continue parallel, while they successively replace each other. This implies, that the differences of the vertical velocities of the points which belong to the same horizontal slice are neglected, so that each slice may be regarded as composed of the same points of the fluid during the entire continuance of the motion. Likewise, the horizontal velocities which are by hypothesis very small with respect to the vertical velocities, and which have but a slight influence on the vertical velocity common to all the points of the same slice, are neglected. These suppositions always agree better with observation, as the horizontal dimensions of the vessel vary less, and as their differences, from one slice to another, are smaller, with respect to the height of the liquid above the orifice. When these conditions are satisfied, it is observed, in fact, that particles of any light powder thrown into the liquid, and carried along in its motion, move, very nearly vertically, with a velocity which is almost the same for all the particles situated in the same horizontal slice. They retain these directions as long as they do not come very near to the orifice; when they are at an inconsiderable distance from it, and the area of the orifice differs considerably from that of the lower sections of the vessel, they

assume oblique directions, which shows that then the parallelism of the slices ceases to be admissible, for there is every reason to suppose that these light particles are attached to the liquid, and exactly assume the motion of the points to which they belong.

Therefore in the hypothesis of the parallelism of the slices, such as it has been now explained, there are only two unknown quantities to be determined in functions of two variables; namely, the velocity of any slice whatever, and the pressure to which it is subjected, in functions of the distance from a horizontal plane and of the time. The question will be thus reduced to its greatest possible simplicity, and will be susceptible, as we now proceed to show, of a complete solution, in the case of a homogeneous incompressible fluid.

668. Let ABCD be the vessel (fig. 56), AB the horizontal orifice, EF the level of the liquid, O$x$ a vertical axis, on which the distances of the horizontal sections from a fixed point O, or from the horizontal plane drawn through this point, are reckoned. Likewise, let MNM′N′ be any slice whatever, comprised between MN and M′N′ two horizontal sections of the vessel, whose distance from the point O at the end of any time whatever, such as $t$, is $x$, and breadth $dx$. Let $v$ denote its velocity at this same instant, and $p$ the pressure relative to the unit of surface, which is made on the upper surface MN, and is transmitted by the fluid on the lower section M′N′, and on MM′ and NN′ the sides of the vessel. Let $y$ represent MN the area of the section MN of the vessel, which, in each example, will be given in a function of $x$. Finally, let $g$ be the gravity, and $\rho$ the constant density of the fluid; the question will consist, as has been stated, in determining the values of $v$ and $p$ in functions of $t$ and $x$.

The mass of the slice which is considered will be the product of the density $\rho$ and of its volume $ydx$, and therefore equal to $\rho ydx$. If it was free, the increment of its velocity

would be $gdt$ in the instant $dt$, it actually increases by $dv$, consequently the velocity lost is $gdt - dv$; and we have

$$\left(g - \frac{dv}{dt}\right)\rho y dx$$

for the force that is lost, that is to say, for the part of the weight $g\rho y dx$ which is destroyed by the pressure of the other slices. Therefore by the principle of D'Alembert, there should be an equilibrium in the fluid, if all its slices were solicited by similar forces; in this state, the pressure $py$ which acts on MN the upper surface of the slice $\rho y dx$, will be transmitted on the inferior base M′N′, and will consequently, as the pressures are in the proportion of the surfaces (No. 577), become $py'$, $y'$ denoting the area of M′N′; hence, if to this transmitted pressure, be added the preceding motive force, the entire pressure exerted on M′N′ will be obtained; and if this pressure on the unit of surface be denoted by $p'$, we shall have

$$p'y' = py' + \left(g - \frac{dv}{dt}\right)\rho y dx.$$

Now, as the quantities $p'$ and $y'$ are what $p$ and $y$ become, when $x + dx$ is substituted for $x$, there will result, by neglecting infinitely small quantities of the second order,

$$p' = p + \frac{dp}{dx}dx, \quad y' = y + \frac{dy}{dx}dx,$$

and, consequently ($a$),

$$(p' - p)y' = \frac{dp}{dx} y dx;$$

this reduces the preceding equation to the following

$$\frac{dp}{dx} = \rho\left(g - \frac{dv}{dt}\right); \tag{1}$$

this might also be obtained by substituting $g - \dfrac{dv}{dt}$ in place of x in the first equation of equilibrium of No. 582.

669. The second equation which is necessary to determine the two unknown quantities, will be furnished by the consideration of the incompressibility of the fluid. It follows from it, that the volume of the liquid which passes, during the instant $dt$, through each horizontal section of the vessel, must be the same for all sections; consequently, the velocities of the fluid, which correspond, at the same time, to two different sections of the vessel, must be reciprocally proportional to the areas of these sections. If therefore $u$ denotes the velocity at the end of the time $t$, at the horizontal orifice AB, and $a$ the area of this orifice, this velocity $u$ will be to $v$ the velocity at MN any section whatever, as $y$ to $a$; hence we obtain

$$v = \frac{au}{y}. \tag{2}$$

In this value of $v$, $u$ is a function of $t$, and $y$ a function of $x$; the differential may therefore be taken with respect to one or other of these two variables: the differential relative to $x$, expresses the difference between the velocities of two consecutive slices which have place at the same instant; by differentiating with respect to $t$, the difference between the velocities of two slices of the fluid, which successively correspond to the same section of the base, will be obtained; but, in order to obtain the difference between the successive velocities of the same slice, which is displaced in the instant $dt$, the value of $v$ should be differentiated, at the same time, with respect to the variables $x$ and $t$; this gives

$$\frac{dv}{dt} = \frac{a}{y}\frac{du}{dt} - \frac{au}{y^2}\frac{dy}{dx}\frac{dx}{dt}.$$

Moreover, we have $\frac{dx}{dt} = v$; and by taking into account equation (2), there results from it

$$\frac{dv}{dt} = \frac{a}{y}\frac{du}{dt} - \frac{a^2u^2}{y^3}\frac{dy}{dx}.$$

It is this value of $\frac{dv}{dt}$ that should be employed in equation (1), which becomes, in consequence,

$$\frac{dp}{dx}=g\rho-\frac{\alpha\rho}{y}\frac{du}{dt}+\rho\frac{\alpha^2u^2}{y^3}\frac{dy}{dx}.$$

If these two members be multiplied by $dx$, and then integrated with respect to $x$, there results, by observing that the quantities $u$ and $\frac{du}{dt}$ must then be considered as constant($b$),

$$p=\varepsilon+g\rho x-\alpha\rho\frac{du}{dt}\int\frac{dx}{y}-\frac{\alpha^2\rho u^2}{2y^2};$$

$\varepsilon$ being an arbitrary constant, which may be a function of $t$. In order to determine it, let $\Pi$ represent the atmospheric pressure, which we suppose to be that which has place at EF the upper surface of the liquid. Previously to the commencement of the motion, this surface is horizontal, and as each horizontal slice is assumed to be constantly composed of the same points of the fluid, it follows that the surface EF will remain horizontal during the entire continuance of the motion. At the end of the time $t$, let $\theta$ denote the distance of EF from the point O, and $\omega$ the area of this variable section of the vessel, so that $\omega$ may be the same function of $\theta$, as $y$ is of $x$; we shall have, at the same time,

$$p=\Pi,\quad x=\theta,\quad y=\omega;$$

and if the integral $\int\frac{dx}{y}$ be supposed to commence when $x=\theta$, the preceding equation will give

$$\varepsilon=\Pi+\frac{\alpha^2\rho u^2}{2\omega^2}-g\rho\theta;$$

in consequence of which, this equation will become

$$p=\Pi+g\rho\,(x-\theta)-\alpha\rho\,\frac{du}{dt}\int\frac{dx}{y}-\frac{\rho u^2}{2}\left(\frac{\alpha^2}{y^2}-\frac{\alpha^2}{\omega^2}\right). \quad (3)$$

By means of equations (2) and (3), the values of the two unknown quantities $v$ and $p$ will be given, when the value of $u$ shall have been determined.

670. For this purpose it is to be observed, that the pressure which has place at the orifice AB will be given; for if the liquid flows into the open air, it will be the same as the atmospherical pressure, which presses at its level EF; and if it flows into a vacuum it will be cipher; for greater generality, we shall suppose that it flows into air whose elastic force is equal to the pressure $\Pi$ diminished by $g\rho c$, the pressure corresponding to $c$, a given height of the liquid; so that if $l$ denotes the distance of the orifice AB from the point O, we shall have constantly

$$p = \Pi - g\rho c,$$

for $x = l$. Likewise, let $h$ denote the height of EF, the level of the liquid, above this orifice, or the difference $l - \theta$, and $\frac{1}{\lambda}$ the value of the integral $\int \frac{dx}{y}$ extended to the entire volume of the liquid; so that $\lambda$ is a line, the length of which is a function of $h$, depending on the figure of the vessel, and given in each example. Therefore, at the orifice, we shall have at the same time, the preceding value of $p$, and

$$y = a, \quad x = l = \theta + h, \quad \int_{\theta}^{\theta+h} \frac{dx}{y} = \frac{1}{\lambda};$$

consequently, equation (3), applied to this section of the vessel, will become($c$)

$$g\,(h + c) - \frac{a}{\lambda}\frac{du}{dt} - \tfrac{1}{2}\beta^2 u^2 = 0, \tag{4}$$

in which we make, for conciseness,

$$1 - \frac{a^2}{\omega^2} = \beta^2.$$

We may remark, that this numerical quantity $\beta^2$ will be always positive and less than unity; for, in order that it

should become negative, the area of the least section of the vessel should surpass that of the orifice, and the liquid should be detached from the vessel at the place of this least section, which will then become the true orifice through which the flowing takes place.

When the level of the liquid always remains at a constant height above the orifice, the three quantities $h, \theta, l$, will be given constants, and equation (4) will suffice to determine the value of $u$ in a function of $t$. When the level EF is depressed, during the flowing of the liquid, $h$ will be a variable, which must be also determined in a function of $t$. Now, at this level, $y = \omega$ and $v = \frac{d\theta}{dt}$, and because the sum $\theta + h$ is equal to $l$ a constant quantity, we have also $\frac{d\theta}{dt} = -\frac{dh}{dt}$; therefore, in virtue of equation (2) we shall have

$$\frac{dh}{dt} + \frac{au}{\omega} = 0; \qquad (5)$$

and thus the values of $u$ and $h$ will depend on the two differential equations (4) and (5), which are of the first order. The two arbitrary constants which their integrals will contain, can be determined, by means of the initial height of the liquid, and by observing that $u = 0$, at the commencement of the motion.

Whether the level is depressed or does not vary, if $q$ denotes the volume of the liquid which has issued from the vessel at the end of the time $t$, its differential will be equal to $audt$, the volume of the slice which traverses AB the orifice during the instant $dt$; therefore, we shall have

$$dq = audt, \quad q = a\textstyle\int udt,$$

the integral being taken so that it may vanish when $t = 0$.

We now proceed to apply these different formulæ successively to the two cases of a constant and variable level.

671. In the first case, equation (4) gives(*d*)

$$\lambda dt = \frac{2\,adu}{2g(h+c)-\beta^2u^2};$$

hence we deduce, by integrating and substituting $h$ for $h+c$(*e*),

$$\lambda t = \frac{a}{\beta\sqrt{2gh}}\log\frac{\sqrt{2gh}+\beta u}{\sqrt{2gh}-\beta u};$$

it is not necessary to add any arbitrary constant, for we must have $u=0$ when $t=0$. We are at liberty, without changing this formula, to consider $\beta$ and $\sqrt{2gh}$, as either positive or negative; we shall suppose them to be positive. There results from the preceding expression,

$$\sqrt{2gh}-\beta u = (\sqrt{2gh}+\beta u)\,e^{-\frac{\beta\lambda t\sqrt{2gh}}{a}}, \qquad (6)$$

$e$ denoting, as usual, the base of the Naperian system of logarithms. According as $t$ increases, the second member of this equation will diminish; so that after the lapse of a certain time, it will be sensibly cipher; and, reckoning from this time, the velocity $u$ will be very nearly constant and equal

$$u = \frac{1}{\beta}\sqrt{2gh}.$$

In each point of the vessel, the pressure $p$ and the velocity $v$ will vary with the velocity $u$, and become sensibly constant at the same time as $u$. If in formula (3), $\frac{du}{dt}$ be made equal to cipher, there will result, by substituting its preceding value in place of $u$,

$$p = \Pi + g\rho(x-\theta) + \frac{g\rho h}{\beta^2}\left(\frac{a^2}{\omega^2}-\frac{a^2}{y^2}\right),$$

which will be the final value of $p$ relative to M any point whatever.

In the state of equilibrium, the pressure on this point

would be $\Pi + g\rho\,(x - \theta)$, therefore, it will be increased or diminished by the motion of the liquid, according as the last term of this formula is positive or negative; that is to say, according as the horizontal section MN or $y$ is greater or less than the section EF or $\omega$.

Equation (6) gives ($f$)

$$u = \frac{\sqrt{2gh}}{\beta}\left(\frac{e^{\frac{\beta\lambda t\sqrt{2gh}}{2a}} - e^{-\frac{\beta\lambda t\sqrt{2gh}}{2a}}}{e^{\frac{\beta\lambda t\sqrt{2gh}}{2a}} + e^{-\frac{\beta\lambda t\sqrt{2gh}}{2a}}}\right),$$

and as $q = a\int udt$, and $q = 0$ when $t = 0$, we shall therefore have($g$)

$$q = \frac{2\,a^2}{\beta^2\lambda}\log \tfrac{1}{2}\left(e^{\frac{\beta\lambda t\sqrt{2gh}}{2a}} + e^{-\frac{\beta\lambda t\sqrt{2gh}}{2a}}\right),$$

for the volume of the liquid that has issued from the vessel during the time $t$. After the lapse of a certain time, the second exponential may be neglected relatively to the first, and we shall have

$$q = \frac{ta\sqrt{2gh}}{\beta} - \frac{2a^2}{\beta^2\lambda}\log 2.$$

The first term is the volume corresponding to $\frac{1}{\beta}\sqrt{2gh}$, the constant velocity with which the liquid flows out; the total volume is less, since at the commencement, the variable value of $u$ is less than this final velocity.

672. In the case of a variable level, $u$ should be considered as a function of $h$, and by eliminating $dt$ between equations (4) and (5); there results($h$)

$$gh + \frac{a^2udu}{\lambda\omega dh} - \tfrac{1}{2}\beta^2u^2 = 0,$$

the constant $c$ being always supposed to be comprised in $h$.

If $z$ denotes the height due to the velocity $u$, so that

$$u^2 = 2gz, \quad udu = gdz,$$

the preceding equation will be changed into the following linear equation,

$$\frac{dz}{dh} - \frac{\beta^2 \lambda \omega}{a^2} z + \frac{\lambda \omega h}{a^2} = 0, \tag{7}$$

the integral of which may be obtained, as is well known, under a finite form($i$).

When $z$ and $u$ are known in functions of $h$, equation (5) will give $t$ in a function of $h$ by an immediate integration; so that the time which has lapsed, when the level of the liquid is at a certain height $h$ above the orifice, will be known, and, conversely, $h$ the height of the level EF, at the end of $t$ any time whatever. The entire time which all the liquid takes to flow out will be obtained, by integrating the value of $dt$ from the initial value of $h$ to $h = 0$. With respect to $q$ the volume of the fluid that has flown out, it will be equal, at each instant, to the portion of the vessel contained between the variable and initial level.

673. Let us suppose, for example, that the vessel is a vertical cylinder terminated by a segment of surface, the sagitta of which is very small, and in which the horizontal orifice AB is pierced. Let $a$ be the constant area of the horizontal section of the cylinder, and $n$ the ratio of $a$ to $\alpha$, so that we may have($k$)

$$\alpha = \frac{a}{n}, \quad \beta^2 = \frac{n^2 - 1}{n^2}.$$

If the inferior segment of the vessel, which is by hypothesis very small, is not taken into account, we may assume

$$\omega = a, \quad y = a, \quad \frac{1}{\lambda} = \frac{h}{a},$$

in equation (7), which will then become($l$)

$$\frac{dz}{dh} - \frac{(n^2-1)}{h} z + n^2 = 0.$$

Its complete integral is $(m)$

$$z = \mathrm{c}h^{n^2-1} - \frac{n^2 h}{2-n^2},$$

in which c denotes the arbitrary constant. If the initial value of $h$ be denoted by $\mathrm{H}$, it is necessary that $z$ should vanish for $h = \mathrm{H}$; this requires that

$$\mathrm{c} = \frac{n^2}{2-n^2}\mathrm{H}^{2-n^2},$$

hence there will result, at any instant whatever,

$$z = \frac{n^2}{2-n^2}(\mathrm{H}^{2-n^2} h^{n^2-1} - h).$$

We shall have, at the same time,

$$u = n\sqrt{2gh}\sqrt{\frac{\mathrm{H}^{2-n^2} h^{n^2-2} - 1}{2-n^2}}, \tag{8}$$

and, in virtue of equation (5) $(n)$,

$$dt = -\frac{dh}{\sqrt{2gh}}\sqrt{\frac{2-n^2}{\mathrm{H}^{2-n^2} h^{n^2-2} - 1}}. \tag{9}$$

It is this formula which should be integrated in order to obtain $t$ in a function of $h$. In the case of $n = 1$, we shall have

$$dt = -\frac{dh}{\sqrt{2g}\sqrt{\mathrm{H}-h}};$$

hence we obtain

$$t = \sqrt{\frac{2}{g}}\sqrt{\mathrm{H}-h},$$

and, consequently,

$$\mathrm{H} - h = \tfrac{1}{2}gt^2,$$

as we know it ought to be, since the orifice being then equal to the base of the cylinder, the motion of the liquid ought to be the same as that of a heavy solid body that descends in a

vacuum. Formula (9) may also be integrated in a finite form, when $n^2 = 3$, and it can only be effected in this case and for $n = 1$. But its definite integral taken from $h = \text{H}$ to $h = 0$, which expresses the time that the entire liquid takes to flow out, may be always reduced to the transcendentals, that M. Legendre has denominated *definite* Eulerian integrals of the second species, and of which he has given numerical tables. This reduction has been also effected by the author in the third volume of the correspondence of the Polytechnic school; here however he restricts himself to apply formula (9) to the case of $n^2 = 2$, in which it occurs under the form $\frac{0}{0}$.

Its true value, as furnished by the common rule, is($o$)

$$dt = -\frac{dh}{\sqrt{2gh}}\left(\log\frac{\text{H}}{h}\right)^{-\frac{1}{2}}.$$

Now if we make

$$h = \text{H}e^{-2x^2}, \quad dh = -4\text{H}e^{-2x^2}xdx,$$

there will result from it

$$dt = 2\sqrt{\frac{2\text{H}}{g}}\,e^{-x^2}dx.$$

The limits relative to $x$, which correspond to $h = \text{H}$, and $h = 0$, will be $x = 0$ and $x = \infty$. If therefore the time of the entire flowing out be denoted by T, we shall have

$$\text{T} = 2\sqrt{\frac{2\text{H}}{g}}\int_0 e^{-x^2}dx = \sqrt{\frac{2\pi\text{H}}{g}},$$

since the integral $\int_0^\infty e^{-x^2}dx$ is half of $\int_{-\infty}^{\infty} e^{-x^2}dx$, the value of which is $\sqrt{\pi}$, as was observed in No. 512. It follows therefore that the time T is that in which the small oscillations of a simple pendulum, the length of which is $\frac{2\text{H}}{\pi}$, are performed.

674. When AB the orifice is very small relatively to the horizontal sections of the vessel, the term multiplied by $a$ in

equation (4) may be neglected, unless the factor $\frac{du}{dt}$ is not very great; this is, in fact, what has place at the commencement of the motion, when the velocity $u$ varies with great rapidity. We may also substitute unity in place of $\beta$; and then, whether the level falls, or remains stationary, equation (4) is reduced to

$$g(h+c) - \tfrac{1}{2}u^2 = 0;$$

from which we obtain

$$u = \sqrt{2g(h+c)}.$$

It follows from this theorem, that the velocity of a liquid which issues from a vessel through a very small orifice, is equal to that which a heavy body would acquire in falling in a vacuum through a height equal to that of the level of the liquid above this orifice, (when the superior and inferior pressures are equal), or more generally, of the height of the level, increased by the constant $c$, when these two pressures are unequal.

In the case of a constant level, this theorem results from the final value of $u$ found in No. 671, by making in it $\beta = 1$. It results also from formula (8) applied to the case in which $n$ is a very great number, in order that the orifice $\alpha$ may be a very small part of $a$, the horizontal section of the cylinder. We can then substitute $n^2$ in place of $n^2 - 2$; this at once changes formula (8) into

$$u = \sqrt{2gh}\sqrt{1-\left(\frac{h}{\mathrm{H}}\right)^{n^2}}.$$

Now as $h$ is evidently less than H, the $n^2$ power of $\frac{h}{\mathrm{H}}$ will be a very small fraction, and this value of $u$ will be reduced very nearly to $u = \sqrt{2gh}$.

As the orifice AB is very small, if the section MN is not very near to this opening, the ratio $\frac{\alpha^2}{y^2}$, which occurs in for-

mula (3), will be very small; the ratio $\frac{\alpha^2}{\omega^2}$ is so likewise; the last term of this formula may therefore be suppressed; and if the term multiplied by $\alpha$ be also neglected, it will be reduced to

$$p = \Pi + g\rho(x - \theta);$$

hence it follows that in the case of a very small orifice, the pressure in all points of the vessel at a distance from this aperture, is sensibly the same during the motion as in the state of equilibrium.

675. The hypothesis of the parallelism of the slices requires, in general, that the orifice should be horizontal; but when the orifice is very small, this may be also assumed even when the liquid flows through a lateral opening, the plane of which may have any inclination whatever, and may even be vertical. It appears from observation that the liquid situated a short distance below this small opening remains stagnant, and that the horizontal slices, situated at an equal distance above this same opening, descend parallel to themselves, so that, as in the case of a small horizontal orifice, the parallelism of the slices is not disturbed, except for the part of the liquid which is very near to the orifice. $\sqrt{2g.(h+c)}$ may therefore be assumed to be the velocity with which the liquid flows through a very small opening, whatever the inclination of this orifice may be; $h$ being the constant or variable height of the level of the liquid above the centre of the orifice, and $c$ the constant arising from the difference of the exterior pressures which correspond to this level and this opening. If the vessel is situated in a vacuum, in which case the two pressures and this constant are cipher, the molecules of the liquid will be actuated as they issue from the vessel, by the velocity $\sqrt{2gh}$ which is due to the height $h$, which is also the velocity with which a body should move, in order that when projected in a vacuum upwards, it may ascend to this height (No. 130). Consequently, if a vertical direction is given to

the fluid by means of a pipe, it will reascend in the tube, to the height of its interior level; which is in fact conformable to experiment. In general, the molecules of the fluid will describe, in a vacuum, after a very short interval of time, parabolas, the tangents to which at their point of issuing from the orifice, will depend on the direction of the jet, and the parameter, on the height $h$ or $h + c$, if the constant $c$ is not cipher.

The pressure $p$ will be sensibly the same as in a state of equilibrium, except at the orifice, where it will be equal to $\Pi - g\rho c$ instead of $\Pi + g\rho h$. Now if it was also equal to $\Pi + g\rho h$ on this part of the vessel, the horizontal pressures will destroy each other, and the vertical pressures will be reduced to the weight of the liquid increased by $\Pi\omega$, the pressure which has place on the surface of the level; hence it follows, that in the state of motion, the entire load which the pressure will have to sustain, will be made up of the vertical pressure which would have place in the state of equilibrium, and of a force normal to the plane of the orifice, acting from without the vessel inwards, and equal to the excess of the pressure $(\Pi + g\rho h)\alpha$ over the pressure $(\Pi - g\rho c)\alpha$, or to $g\rho(h + c)\alpha$, in which $\alpha$ denotes, as before, the very small area of this orifice.

676. In the case of a constant level and of a very small orifice, either horizontal or inclined, the *consumption* in the time $t$, that is to say, the volume of liquid which issues from the vessel with the velocity $\sqrt{2gh}$, will be

$$q = \alpha t\sqrt{2gh};$$

this also results from the final value of $q$, which was found in No. 671, by neglecting the square of $\alpha$. But it should be kept in mind that the hypothesis of the parallelism of the slices, on which this value of $q$ is founded, is only an approximation, the accuracy of which cannot be estimated *a priori*, and of which therefore the results should not be employed without restriction; for it is well known that this theoretical value of the consumption of the liquid does not always accord with experiment.

If the side of the vessel is not very slender, and if the opening wrought in it widens internally; so that the fluid which flows out of the vessel may be a vertical cylinder, or a curved cylinder, the vertical sections of which are constant, and equal to $a$, the area of the orifice, measured on the exterior surface of the vessel; in this case it may be admitted that the observed consumption agrees with the preceding value of $q$. But if the side be very thin, the observed consumption is always proportional to the area of the orifice and to the square root of the elevation of the level, as in the theoretical formula; but though it is in this proportion, it differs from this formula in its absolute value, by a factor which is nearly constant, and less than unity. It appears from the most accurate experiments, that this factor is 0,62; so that the value of $q$ which is made use of in practice is

$$q=(0{,}62)at\sqrt{2gh},$$

when the orifice is a very small one wrought in a thin side, and the height of the level considerable with respect to the dimensions of this opening, whether horizontal or inclined.

This difference is ascribed to the inclined directions which the molecules of the liquid assume, as they approach the orifice, which for greater clearness we suppose to be horizontal, and which they retain, after having traversed the thin side of the vessel. It appears from this, that the exterior fluid vein contracts to a small distance from the vessel, where it attains its *minimum* breadth, which it afterwards retains. This phenomenon is termed the *contraction of the fluid vein*. The fluid flows out in precisely the same manner as if the area of the orifice was the same as that of the section of the vein at the place of its greatest contraction, so that if the area of this section be denoted by $a'$, and its constant distance from the level of the liquid by $h'$, the quantity of the liquid which will have flowed out will be denoted by $a't\sqrt{2gh'}$, or very nearly by $a't\sqrt{2gh}$, because $h$ and $h'$ differ very little from each other. Now it is found, in fact, by direct measurements of

the section $a'$, compared with the orifice $a$, that these two quantities are to each other in a ratio, which is very nearly independent of $h$, and that we have constantly $a' = (0,62)\, a$.

It appears from experiment, that if to an orifice in a thin side, a cylindrical adjutage be fixed without the vessel, perpendicular to the plane of the orifice, the quantity which flows out through it will be increased, and may amount to four-fifths of the result of theory. On the contrary, if this adjutage is fixed in the interior of the vessel, the quantity which flows out is diminished to one-half of the quantity which is given by theory, so that in the preceding value of $q$, the factor 0,62 should be replaced by 0,80 in the first case, and by 0,50 in the second.

These results of observation have been here only briefly pointed out, as they have not as yet been reduced to any precise theory.

677. The hypothesis of the parallelism of the slices is also assumed in the motion of an elastic fluid which issues from a vessel through any orifice whatever; and, when the sections of the vessel parallel to the plane of the orifice, do not differ much from each other, and the length of the vessel is considerable with respect to their dimensions, this hypothesis is not far from the truth.

In this case, the weight of the molecules of the fluid is not taken into account, so that the motion is solely due to the greater or less elastic force of the fluid, in the interior of the vessel than outside it. Therefore, the term depending on $g$ in equation (1), which is applicable both to liquids and elastic fluids, must be suppressed. Moreover, as the differential of $v$ which it contains, must be taken with respect to $t$ and the variable $x$, considered as a function of $t$, we shall have

$$dv = \frac{dv}{dt}\,dt + \frac{dv}{dx}\,\frac{dx}{dt}\,dt\,;$$

and as we have also $\dfrac{dx}{dt} = v$, this equation will become($p$)

$$\frac{dp}{dx} + \rho\frac{dv}{dt} + \rho v\frac{dv}{dx} = 0. \qquad \text{(a)}$$

As the fluid is compressible, the same volume will no longer pass, at each instant, through all the sections of the vessel, and equation (2) will not have place. The mass of fluid which passes in the instant $dt$, through the section MN, will be equal to $\rho yvdt$, the same mass will pass the following instant through the section M′N′, its volume being changed; and, during the entire continuance of the motion, its magnitude will not vary. Therefore the differential of the product $\rho yv$, taken with respect to $t$, and the variable $x$ considered as a function of $t$, will be cipher; and, because $y$ is solely a function of $x$, and that $\frac{dx}{dt} = v$, we obtain from it($q$)

$$\rho v^2\frac{dy}{dx} + y\frac{d.\rho v}{dt} + yv\frac{d.\rho v}{dx} = 0. \qquad \text{(b)}$$

Finally, if the temperature remains constant during the motion, in the entire mass of the fluid, we shall have

$$p = \rho k;$$

$k$ being a given constant coefficient.

This being established, if $\frac{p}{k}$ be substituted in place of $\rho$ in equations (a) and (b), we shall obtain two equations of partial differences of the first order, by means of which $v$ and $p$, the two unknown quantities of the problem, can be determined in functions of $t$ and $x$. As they are not integrable in a finite form, the values of $p$ and $v$ can only be obtained by approximation. These values will contain two arbitrary functions, which can be determined by two particular conditions; for this purpose, we shall suppose that the elastic fluid issues into the open air, so that if $\Pi$ denotes the atmospheric pressure, on the unit of surface, we may have constantly $p = \Pi$, at the orifice AB. We shall also suppose that the vessel communicates with a gasometer of *great* capacity($r$), by means of which

EF a section of the fluid parallel to AB, and fixed in position, sustains a constant given pressure, so that if this pressure on the unit of surface be denoted by $\Pi'$, we shall also have $p = \Pi'$ in this part of the vessel, during the entire continuance of the motion. If therefore the distance $x$ be reckoned from the plane EF, and if $l$ denotes the distance comprised between AB and EF, we shall have, whatever $t$ may be, $p = \Pi'$ for $x = 0$, and $p = \Pi$ for $x = l$; this will enable us to determine the two arbitrary functions, and thus completely solve the problem. But this solution is so complicated that it cannot be reduced to any useful result; and in practice it is sufficient to know the constant velocity with which the fluid flows through the orifice AB, when the pressure $p$ and the velocity $v$ become constant in each point of the vessel; this, in general, takes place after a very short interval of time.

678. If therefore we make $\frac{dv}{dt} = 0$, and $\frac{dp}{dt} = 0$, in equations (a) and (b), they will be reduced to two differential equations, namely,

$$\frac{k}{p}\frac{dp}{dx} + v\frac{dv}{dx} = 0, \quad pv\frac{dy}{dx} + y\frac{d.pv}{dx} = 0, \tag{c}$$

because $p = k\rho$.

The integral of the second of these equations is($s$)

$$ypv = c;$$

$c$ being an arbitrary constant. If the orifice AB be always denoted by $a$, and the velocity of the fluid at this orifice by $u$, so that we may have at the same time, $y = a$, $v = u$, $p = \Pi$, and, consequently, $c = a\Pi u$, there will result from this, at any point whatever of the vessel,

$$v = \frac{a\Pi u}{yp}. \tag{d}$$

By substituting this value of $v$ in the first equation (c), it becomes

$$\frac{k}{p}\frac{dp}{dx} + \frac{a^2\Pi^2u^2}{py}\frac{d.\frac{1}{py}}{dx} = 0;$$

hence, by integrating, and denoting the arbitrary constant by $c'$, we obtain($t$)

$$k\log p + \frac{a^2\Pi^2u^2}{2\Pi'^2y^2} = c'.$$

Therefore, if the area of the section EF be denoted by $a$, so that we may have at the same time $y = a$ and $p = \Pi'$, we shall obtain

$$k\log\Pi' + \frac{a^2\Pi^2u^2}{2\Pi'^2a^2} = c',$$

and, by subtracting this equation from the preceding,

$$k\log\frac{p}{\Pi'} + \tfrac{1}{2}a^2\Pi^2u^2\left(\frac{1}{p^2y^2} - \frac{1}{\Pi'^2a^2}\right) = 0. \qquad \text{(e)}$$

By means of equations (d) and (e), the velocity and pressure in any point whatever of the vessel will be known, when $u$ the velocity relative to the orifice is known. Now by making $p = \Pi$, and $y = a$, in equation (e), we obtain

$$u^2\left(1 - \frac{a^2\Pi^2}{a^2\Pi'^2}\right) = 2k\log\frac{\Pi'}{\Pi}; \qquad \text{(f)}$$

from which the value of $u$ can be deduced. In this formula we suppose

$$k = grh,$$

$g$ denoting the gravity, and $r$ the ratio of the density of the mercury to that of the interior fluid under a barometrical pressure, the height of which is $h$, hence the value of the volume of the fluid which issues from the vessel in the time $t$, will be $aut$.

It may be remarked, as in No. 675, that when the orifice is very small, it is no longer necessary that it should be parallel to the section EF, that is to say, it may be made in the lateral part of the vessel, and have any inclination whatever on the

plane of this section. We may then neglect the term depending on $\frac{a^2}{a^2}$ in the first member of equation (f), which will become

$$u^2 = 2grh.\log\frac{\Pi'}{\Pi};$$

consequently, the velocity with which a fluid issues through a very small orifice, will be that which is due to a height $rh$, multiplied by the Naperian logarithm of the ratio $\frac{\Pi'}{\Pi}$. The supposition that the temperature is invariable during the entire continuance of the motion, implies that the velocity $u$ should not be very considerable, otherwise the temperature would vary, as in the propagation of sound.

# ADDITION.

## RELATIVE TO THE APPLICATION OF THE PRINCIPLE OF LIVING FORCES IN THE CALCULATION OF MACHINES IN MOTION.

679. THE conditions of the equilibrium of forces applied to machines, are furnished immediately by the principle of virtual velocities; the theory of their motion is given by that of living forces, which enables us to calculate, in the most direct manner, the effects of the forces that are applied to them. This application of the principle of living forces constitutes, so to speak, the point at which rational and practical mechanics coincide. It is on this account that the author thought it necessary to give, in this addition, a brief sketch of the most general principles relative to this matter.

680. *Machines* may be defined to be instruments or systems of solid bodies, which are made use of to transfer the action of forces from one part to another of these bodies.

Therefore when a machine is in motion, certain points of it are subjected to the action of given forces, and other parts press on exterior bodies, or are reciprocally pressed by those bodies which it is proposed, by means of the machine, either to displace or to separate. The first description of forces are termed *moving forces*, and their points of application move along their directions, or, more generally, the directions of the motions of these points make *acute* angles with those of these forces(*a*). On the contrary, the pressures exerted by extraneous bodies are denominated *resisting forces*, and the directions of the motions of their points of application are opposite to those of these forces, or at least, they make with them obtuse angles.

The connexion of the parts of a machine is such, that it can, in general, only assume two motions, directly opposite the one to the other; it follows therefore, that, when the direction of the motion which it actually assumes is known, one equation is sufficient to determine this motion in a complete manner. This equation is that which is obtained by integrating the two members of equation (a) of No. 564, namely,

$$\tfrac{1}{2} d \,.\, \Sigma m v^2 = \Sigma m(\text{x}dx + \text{y}dy + \text{z}dz). \qquad \text{(a)}$$

After the lapse of $t$, any time whatever, reckoned from the commencement of the motion, $v$ denotes the velocity of the point, and $x$, $y$, $z$ its three coordinates referred to fixed rectangular axes; $m$ is the mass of this point; $dx$, $dy$, $dz$ are the projections, on these axes, of the space which it describes during the instant $dt$; $m\text{x}$, $m\text{y}$, $m\text{z}$ denote the components of its entire force parallel to these same axes, and the sums $\Sigma$ are supposed to extend to all points, such as $m$, of the system.

681. Before we proceed farther, it will be useful to distinguish, in the second member of equation (a), between the terms which arise from the moving forces and those which result from the resisting forces, and to assign another form to them.

For this purpose, let P be one of the moving forces, and $\alpha$, $\beta$, $\gamma$ the angles which its direction makes with lines parallel to the axes of $x$, $y$, $z$; we shall have, relatively to this force,

$$m\text{x} = \text{P}\cos\alpha, \quad m\text{y} = \text{P}\cos\beta, \quad m\text{z} = \text{P}\cos\gamma.$$

Likewise, if $ds$ be the space described by its point of application during the instant $dt$, and $\lambda$, $\mu$, $\nu$ the angles which the direction of $ds$ makes with its projections $dx$, $dy$, $dz$, we shall also have

$$dx = ds\cos\lambda, \quad dy = ds\cos\mu, \quad dz = ds\cos\nu.$$

Finally, if $dp$ denotes the projection of $ds$ on the direction of the force P, and $\sigma$ the angle contained between $dp$ and $ds$; we shall have

$$dp = ds\cos\sigma,\ \cos\sigma = \cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu;$$

and, from these different equations, we deduce(*b*)

$$m(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz) = \mathrm{P}dp,$$

for that term of the second member of equation (a), which corresponds to the force P.

If one of the resisting forces be denoted by Q, and the projection of the space described during the instant $dt$, by its point of application on the production of its direction, by $dq$, it may be shown, in the same manner, that $-\mathrm{Q}dq$ is the term of this second member, which arises from the force Q. In this manner, equation (a) will assume the form

$$\tfrac{1}{2}d.\Sigma mv^2 = \Sigma\mathrm{P}dp - \Sigma\mathrm{Q}dq; \qquad \text{(b)}$$

in which one of the sums $\Sigma$ contained in its second member is supposed to extend to all the moving forces of the machine, and the other to all the resisting forces. According to the hypotheses that have been made respecting the directions of these two descriptions of forces, relatively to the motions of the points where they act, the quantities $dp$ and $dq$, and also P and Q are positive, consequently the sums $\Sigma$ consist only of positive terms.

682. If $k$ denotes the initial velocity of $m$ any point whatever, or the value of $v$ corresponding to $t = 0$, then we shall obtain, by integrating the two members of equation (b),

$$\tfrac{1}{2}\Sigma mv^2 - \tfrac{1}{2}\Sigma mk^2 = \int\Sigma\mathrm{P}dp - \int\Sigma\mathrm{Q}dq; \qquad \text{(c)}$$

in which the integrals are taken in such a manner that they may vanish at the commencement of the motion. It is under this form that the equation of living forces should be employed in order to calculate the effects of machines in motion; it coincides with equation (b) of No. 564, when the integrations indicated in its second member can be effected.

The products $\mathrm{P}dp$ and $\mathrm{Q}dq$, the sums of which are subjected to these integrations, have received different denominations; they have been termed *the quantities of action, the*

*moments of activity, the dynamical effects* of the forces P and Q. It would be useful if they were always designated by the same term; the denomination *of quantities of elementary work,* which was proposed to be given to them by a French author of the name of Coriolis, is the one that will be here adopted. Consequently, the sums $\Sigma Pdp$ and $\Sigma Qdq$ will be the quantities of elementary work performed during the same instant by all the moving forces and all the resisting forces; and their integrals $\int \Sigma Pdp$ and $\int \Sigma Qdq$ will express the *entire motive work* and the *entire resisting work* of the machine, from the commencement of the motion to the instant in question.

Hence, equation (c) indicates that, in any machine in motion, the increment, during any time whatever, of the semi-sum of the living forces of all its parts, is always equal to the excess of the motive work over the resisting work, during the same interval.

683. If the moving force P, or the resisting force Q, is a weight $\Pi$, which descends in the first case, through a vertical height $h$, or which ascends, in the second case, to the same height, the product $\Pi h$ will be then the value of the motive or resisting work, whatever be the route traversed by this weight, $h$ being always the vertical projection of the right line or curve described by its centre of gravity. If there are points of contrary flexure in this line, or if it returns into itself, the motive work and the resisting work will alternately succeed one another; and, as $h$ is the difference of level of the points of departure and arrival, $\Pi h$ will be the excess of the first work over the second. In the case in which there are no such successions, the quantity of work corresponding to a weight $\Pi$ raised to a height $h$, is equivalent to the quantity of work which belongs to another weight $\Pi'$, when elevated to a height $h'$, which is such that $h' = \frac{\Pi h}{\Pi'}$.

Whatever may be the force P or Q, the integral $\int Pdp$ or

$\int Qdq$ is always equivalent to the product of a weight P and a height $h$; and, in order to compare together, and express in numbers, works of different descriptions, we may in this manner assimilate them to weights raised to given heights. In this case there is assumed for the unit of work, which is usually termed the *dynamical unit*, the work corresponding to a weight of 1000 kilogrammes, which is supposed either to be raised to the height of a metre, or to descend vertically through a space equal to a metre. This being agreed on, if the numerical values of the integrals $\int Pdp$ and $\int Qdq$ be computed, 1000 kilogrammes being assumed as the unit of force, and the metre being the linear unit, the numbers obtained in this manner will express, in dynamical units, the quantities of work represented by these integrals. Any sum whatever of living forces, such as $\frac{1}{2}\Sigma mv^2$, for example, may be also expressed in dynamical units; for if $l$ be the height due to the velocity $v$, $g$ the gravity, and $p$ the weight of $m$, we shall have

$$v^2 = 2gl, \quad p = gm, \quad \tfrac{1}{2}\Sigma mv^2 = \Sigma pl;$$

and this sum is of the same nature as the integrals $\int Pdp$ and $\int Qdq$, or as the product $\Pi h$.

684. When the machine sets out from a state of rest, equation (c) is reduced to

$$\tfrac{1}{2}\Sigma mv^2 = \int \Sigma Pdp - \int \Sigma Qdq. \qquad \text{(d)}$$

As its first member is always positive, it is necessary that, in the first instants, the motive work should exceed the resisting work. But as the velocities of the points of the machine cannot increase indefinitely, this first member attains its *maximum* after the lapse of a certain time, which is for the most part inconsiderable. By a means which will be pointed out farther on, the semi-sum $\frac{1}{2}\Sigma mv^2$ of the living forces is made to remain constant from the instant the maximum is attained, or at least it then only experiences very small variations, so that the machine is said to have attained to its permanent state.

In this constant state, we obtain, by differentiating the preceding equation,

$$\Sigma \text{P} dp = \Sigma \text{Q} dq ;$$

so that the effect of the machine is then to change, at each instant, the elementary motive work into an equal quantity of resisting work. But it is important to observe, that $\int \Sigma \text{Q} dq$ the quantity of resisting work into which the motive force $\int \Sigma \text{P} dp$ is changed, during any time whatever, does not express solely the work proposed to be done by means of this instrument, for the integral $\int \Sigma \text{Q} dq$ also contains the resisting work that arises from the friction of the parts of the machine, either against one another, or against extraneous bodies, and also that which is produced by the friction of the medium in which the machine moves ($d$).

For example, in order to take into account the frictions, it is necessary, agreeably to what has been observed in No. 568, to add to $\int \Sigma \text{Q} dq$, the integral arising from the resisting work properly so called, another integral $\int \Sigma f \text{N} ds$, in which $f$ is the coefficient of the friction, N the mutual pressure of the parts which rub against each other, and $ds$ the element of the curve described by their point of contact. In consequence of this addition, equation (d) will be changed into

$$\tfrac{1}{2} \Sigma m v^2 = \int \Sigma \text{P} dp - \int \Sigma \text{Q} dq - \int \Sigma f \text{N} ds. \qquad \text{(e)}$$

Hence it follows, that when a machine has attained to a permanent state, $\int \Sigma \text{P} dp$ the quantity of work performed during a given time by the moving forces, is not altogether represented by $\int \Sigma \text{Q} dq$ the effective part of the resisting work, for this part is always less than the motive work $\int \Sigma \text{P} dp$, by the entire quantity of work that corresponds to the frictions and the other resistances. A machine is more perfect the more the effective work $\int \Sigma \text{Q} dq$ approaches to an equality with the motive work $\int \Sigma \text{P} dp$; but the first integral can never, however the parts of the machine are combined, be equal to the second, much less surpass it. As an example of imperfect

machinery, in which the effective work is only a very small fraction of the motive work, or rather in which the greatest part of this is absorbed by the frictions, we may cite the old machine set up in Marly; in this the motive work consisted in the fall of a considerable body of the water of the Seine, and the effective part was the elevation of a quantity of water to a height, which was far from being an equivalent for the smallness of its quantity.

685. The *essential* parts of a machine are, the part to which the moving force is applied, that which is in contact with the body which it is proposed either to move or separate, and the intermediate part which transmits the action of the moving forces. It is of consequence, in order to save expense in the construction of machines, and also to diminish the frictions, that the solid mass should be as small as possible, consistently with the solidity of its parts; but there is another circumstance to be considered, in consequence of which it is necessary to increase this mass, and to add to the three essential parts of which it consists, a piece called a *fly*, and which in general consists of a solid body revolving about a fixed horizontal axis.

As the motions of the three first parts of a machine are either such as are alternately those of the motive and resisting kind, or such as produce a revolution, $\frac{1}{2}\Sigma mv^2$, the semi-sum of living forces relative to them, becomes a periodic quantity, after it attains its *maximum*; consequently, this will be the case with respect to the second member of equation (e); so that if the machine was reduced to its three essential parts, the motive work and the resisting work, in which last the effects of frictions are supposed to be included, would alternately predominate, the one over the other; and if the alternate variations of the motive work $\int\Sigma \text{P}dp$ and of the part $\int\Sigma f\text{N}ds$ of the resisting work, were not exactly regulated according to the periods of the machine, $\int\Sigma \text{Q}dq$ the quantity of effective work would continually vary. Now, in order to secure the good performance of the working of the machine, it is indispensably neces-

sary that, for the most part, the effective work should approach as near as possible to uniformity; and the chief use of the fly-wheel is to accomplish this object. In fact, if $dm$ is an element of the mass of the fly, $r$ its distance from the axis of rotation, $\omega$ the angular velocity about this axis, common to all the elements $dm$, and which may vary from one instant to another; $r\omega$ will be the absolute velocity of $dm$; consequently, $\int r^2\omega^2 dm$ will be the value of the sum of the living forces of the entire mass of the fly, or, what comes to the same thing, the product $\mu\omega^2$, in which $\mu$ denotes the moment of inertia of the fly with respect to its axis, that is to say, the integral $\int r^2 dm$ extended to the entire mass. If therefore $\frac{1}{2}\mu\omega^2$ be added to the first member of equation (e), and if the semi-sum $\frac{1}{2}\Sigma mv^2$ be supposed to refer to the three other parts of the machine, we shall have

$$\tfrac{1}{2}\mu\omega^2 + \tfrac{1}{2}\Sigma mv^2 = \int\Sigma \mathrm{P}dp - \int\Sigma \mathrm{Q}dq - \int\Sigma f\mathrm{N}ds;$$

from which we obtain

$$\int\Sigma \mathrm{Q}dq = \mathrm{R} - \tfrac{1}{2}\mu\omega^2,$$

in which we make, for conciseness,

$$\mathrm{R} = \int\Sigma \mathrm{P}dp - \int\Sigma f\mathrm{N}ds - \tfrac{1}{2}\Sigma mv^2.$$

Now, we may conceive that the variations of $\omega$ can be so regulated by those of this quantity R, that the entire variation of $\mathrm{R} - \frac{1}{2}\mu\omega^2$ may be reduced to very small amplitudes, and that consequently, the resisting work may be very nearly invariable in the permanent state of the machine; we may likewise conceive that, every thing else being the same, the variations of $\omega$ the velocity of the fly, will be so much less, as $\mu$ its moment of inertia is greater.

686. The quantity of motive force necessary to put the machine in motion, and to increase the total living force, until it reaches its *maximum*, is found to be augmented by the addition of the fly; but after the machine has attained to its permanent state, the masses of its different parts no longer

influence its work, provided we do not take into account the effect of their weights on the frictions.

If, during the motion of the machine, its parts experience a shock either between themselves or against extraneous bodies, and if after the shock the points of contact are actuated by a common velocity in a direction perpendicular to the surfaces, there will be a diminution of living force in the system; if the parts which experience the shock, are afterwards separated in virtue of their elasticity, there will be also a loss of living force when these parts are not perfectly elastic; and when they are perfectly elastic, there will be a loss of living force in the first part of the shock, and then an increase exactly equal to this loss in the second part (No. 572). Consequently, in order to reduce the machine to its permanent state, without any diminution being sustained in the quantity of resisting work, there must be a fresh consumption of motive work made by the moving force, similar to that which has place at the commencement of the motion, and equal to half of the living force that is lost during the shock($e$). This is the reason why, independently of the damage which these shocks produce in machines, it is also necessary to avoid them, in order to economize the moving forces.

687. In general, the resisting work which arises from frictions and the action of the medium in which the machine moves, is a continually increasing quantity; so that in order there may be some effectual work, or, at least, that the motion of the machine may be kept up, it is necessary that the quantity of motive work should also increase with the time, and in a ratio at least equal to that of the increment of the resisting work. If this is not the case, the resisting work will eventually become equal to the motive work, at this instant, the semi-sum of the living forces of all the points of the system will be cipher; the velocities of all these points will be zero, and the machine will stop, and be reduced to a state of rest.

There should be also added to the frictions and resistances

which produce this gradual exhaustion of living force, the communication of a part of the motion of the machine to its supports, which part is then transmitted and lost in the ground where the machine is placed. This communication does not arise solely from the defect of solidity in the supports; it is also produced by their elasticity, in consequence of which the motion is propagated in the same manner as sound; and there may result from this propagation a diminution of the velocity of the parts similar to that which is produced by the resistance of a medium. An example of this remarkable effect is furnished in the motion of a pendulum suspended at the extremity of an elastic horizontal rod of an indefinite length. The details of this discussion are given in the additions *a la Connaissance des temps pour l'annee* 1833, page 26.

When the action of the moving forces is suppressed, and the effective work of the machine has likewise ceased, the equation of living forces becomes

$$\tfrac{1}{2}\,\Sigma mv^2 = \tfrac{1}{2}\,\Sigma mk^2 - \int \Sigma f\text{N}ds\,;$$

$\Sigma mk^2$ being the sum of the living forces of all the points of the system at this instant, $\Sigma mv^2$ this sum at a subsequent epoch, and $\int \Sigma f\text{N}ds$ is supposed to comprise both what arises from frictions, the resistance of the medium, and also the loss of motion by the supports. Now this last term very soon becomes equal to $\frac{1}{2}\Sigma mk^2$; so that the living force of the machine will be completely exhausted, and it will cease to move, as has been already stated in No. 568.

688. When a man carries his own weight, or himself, which we shall denote by U, to a place the vertical height of which above the point from whence he set out is $h$, the quantity of work performed is, by the rule of No. 683, expressed by U$h$; but this quantity will give us a very imperfect notion of the muscular efforts that have been made by him, and of the entire muscular force which he has developed. Indeed it

would be difficult to obtain an exact measure of it; we can only show that it exceeds, for the most part considerably, the preceding quantity, which would be cipher, if the height $h$ was zero, although there can be no doubt, but that a man walking on a horizontal plane, exerts a quantity of mechanical work.

If as he walks the man has first the left foot before the right, his centre of gravity is then depressed below its natural position by a quantity which we shall denote by $\varepsilon$. Then by leaning on his left foot, the man, by means of the pressure of this foot against the ground, brings up his right foot to a level with the left, afterwards the right foot is advanced before the left, and placed on the ground; in this operation he makes an entire step, which is thus made up of two parts. Now, in the first part, the man raises his centre of gravity by the height $\varepsilon$, and thus performs a quantity of work equal to $\mathrm{U}\varepsilon$; he impresses at this same instant on this point a horizontal velocity, which we shall denote by $a$, at the end of the first half-step; this corresponds to another quantity of work equivalent to the semi-living force $\frac{1}{2}\frac{\mathrm{U}a^2}{g}$, in which $g$ denotes the gravity. There should be also added to $\frac{1}{2}\frac{\mathrm{U}a^2}{g}$, the part of the semi-sum of the living forces arising from the relative velocities of all the other points of the body (No. 569); but it is not necessary to take these into account in this estimation, which can be only a mere rough sketch. We shall likewise assume that the second half-step has place in virtue of the velocity acquired at the end of the first, and of the weight of the body which falls back on the ground, so that during the second half-step, the man exerts no effort whatever, and thus the vertical and horizontal velocities with which his centre of gravity are still actuated at the end of the entire step must be destroyed by the impact and friction of his foot against the ground. In this hypothesis, the quantity of work exerted by the man during the entire

step, will be the sum $U\varepsilon + \frac{1}{2}\frac{Ua^2}{g}$ or $U(\varepsilon + \alpha)$, in which $\alpha$ denotes the height due to the velocity $a$, so that $a^2 = 2g\alpha$.

It follows from this, that in a number such as $n$ of equal and similar steps, the value of the quantity of work performed by a man or animal carrying a load, and advancing on a horizontal route, will be $nK(\varepsilon + \alpha)$, in which K denotes his weight U increased by that of the burden. If the entire weight was raised vertically to a height $h$ about the point of departure, $Kh$ should be added to the quantity $nK(\varepsilon + \alpha)$; and if the load is drawn along a route on which it experiences a resistance denoted by F, which is a certain fraction of its weight, there will result another addition of work to be done equal to $Fl$, in which $l$ denotes the length of the route(*f*).

689. In calculating the effects of machines in motion, it is frequently useful to distinguish the velocities which are common to all its points, and also those that are relative to their different points. For this purpose, let $x, y, z$ be always the coordinates at the end of the time $t$, of any point whatever whose mass is $m$; $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ will be the components of its absolute velocity at this instant, and the coordinates of this same point will become $x + dx$, $y + dy$, $z + dz$, at the end of the time $t + dt$. Now, the motion of the system during the instant $dt$, may be decomposed into a motion of translation and rotation common to all its points, in which their distances are invariable, and into particular motions, in which these distances undergo suitable variations. Let $d'x$, $d'y$, $d'z$ denote the increments of $x$, $y$, $z$, which arise from the common motion, and $d_ix$, $d_iy$, $d_iz$, those which result from the relative motion of $m$, then we must have

$$dx = d'x + d_ix, \quad dy = d'y + d_iy, \quad dz = dz' + d_iz.$$

Likewise, relatively to this same point $m$, let $u'$, $v'$, $w'$ be the three components of the common velocity, they will be given functions of $t$, $x$, $y$, $z$, and will be respectively

$$u' = \frac{d'x}{dt}, \quad v' = \frac{d'y}{dt}, \quad w' = \frac{d'z}{dt};$$

those of its relative velocity will be likewise

$$\frac{d_{,}x}{dt}, \quad \frac{d_{,}y}{dt}, \quad \frac{d_{,}z}{dt},$$

so that the components of its absolute velocity will be

$$\frac{dx}{dt} = u' + \frac{d_{,}x}{dt}, \quad \frac{dy}{dt} = v' + \frac{d_{,}y}{dt}, \quad \frac{dz}{dt} = w' + \frac{d_{,}z}{dt};$$

and by differentiating, there will result

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \frac{du'}{dt} + \frac{dd_{,}x}{dt^2}, \\ \frac{d^2y}{dt} &= \frac{dv'}{dt} + \frac{dd_{,}y}{dt^2}, \\ \frac{d^2z}{dt^2} &= \frac{dw'}{dt} + \frac{dd_{,}z}{dt^2}, \end{aligned}\right\} \qquad \text{(f)}$$

which will be the accelerating forces of this point $m$ in the direction of the axes of the coordinates; the differentials relative to $t$ being taken with respect to this variable and to the coordinates $x$, $y$, $z$, considered as functions of $t$.

If this point $m$ is constrained to move on a surface, which may be either fixed or moveable, but whose form is invariable, it should remain constantly on this surface, in virtue of the motion common to all the points of the system. If $L = 0$ be the equation of this surface, $L$ will be a given function of the coordinates of $m$ referred to the moveable axes, which participate in the common motion, and this quantity may be changed into a function of the time and of the coordinates of $m$ referred to fixed axes, that is to say, into a function of $t$, $x$, $y$, $z$. Equation $L = 0$ ought to subsist when these four variables are replaced, either by $t + dt$, $x + dx$, $y + dy$, $z + dz$, in the absolute motion of $m$, or by $t + dt$, $x + d'x$, $y + d'y$, $z + d'z$, in the common motion of the system. Therefore, if infinitely

small quantities of the second order are neglected, we shall have simultaneously $(g)$

$$\frac{d\text{L}}{dt}+\frac{d\text{L}}{dx}(d'x+d_{,}x)+\frac{d\text{L}}{dy}(d'y+d_{,}y)+\frac{d\text{L}}{dz}(d'z+d_{,}z)=0,$$

$$\frac{d\text{L}}{dt}+\frac{d\text{L}}{dx}d'x+\frac{d\text{L}}{dy}d'y+\frac{d\text{L}}{dz}d'z=0,$$

and, consequently,

$$\frac{d\text{L}}{dx}d_{,}x+\frac{d\text{L}}{dy}d_{,}y+\frac{d\text{L}}{dz}d_{,}z=0. \qquad \text{(g)}$$

This being established, we now proceed to investigate the sum of the living forces due to the relative velocities of all the points of the system, and to compare it with the sum of living forces which result from their absolute velocities.

690. For this purpose, let us resume the general formula of No. 531, from which equation (a) of No. 680 has been deduced, and let us successively arrange the terms of this formula with respect to the different descriptions of forces which may act on the system that is considered. In the first place, let P be one of the given exterior forces; as $\delta p$ is the projection of the displacement of its point of application, on its direction, which projection is considered to be positive or negative, according as it falls on the direction itself of $m$ or on its production, we shall have, as in No. 681,

$$m(\text{X}\delta x+\text{Y}\delta y+\text{Z}\delta z)=\text{P}\delta p,$$

for the part of the above cited formula, that results from this force P.

Likewise, let R be the mutual action of two points of the system, whose masses are $m$ and $m'$, $r$ the distance $mm'$, of which R is a certain function; then as $x$, $y$, $z$, $x'$, $y'$, $z'$ are the coordinates of $m$ and $m'$, we shall have

$$r^2=(x-x')^2+(y-y')^2+(z-z')^2;$$

and if $\delta r$ be the variation of $r$ which results from their increments $\delta x$, $\delta y$, $\delta z$, $\delta x'$, $\delta y'$, $dz'$, we shall have likewise

$$r\delta r = (x - x')(\delta x - \delta x') + (y - y')(\delta y - \delta y') + (z - z')(\delta z - \delta z').$$

The components of the force R applied to the point $m$, will be

$$m\text{X} = \pm \left(\frac{x - x'}{r}\right)\text{R},$$

$$m\text{Y} = \pm \left(\frac{y - y'}{r}\right)\text{R},$$

$$m\text{Z} = \pm \left(\frac{z - z'}{r}\right)\text{R},$$

and those of the same force applied to the point $m'$,

$$m'\text{X}' = \pm \left(\frac{x' - x}{r}\right)\text{R},$$

$$m'\text{Y}' = \pm \left(\frac{y' - y}{r}\right)\text{R},$$

$$m'\text{Z}' = \pm \left(\frac{z' - z}{r}\right)\text{R};$$

hence there results

$$m(\text{X}\delta x + \text{Y}\delta y + \text{Z}\delta z) + m'(\text{X}'\delta x' + \text{Y}'\delta y' + \text{Z}'\delta z') = \pm\, \text{R}\delta r,$$

for the part of the general formula, which arises from the force R; in this the superior or inferior sign has place, according as this force is repulsive or attractive.

If the point $m$ is constrained to remain on the surface, whose equation was denoted by $\text{L} = 0$ in the preceding number, and if $\omega$ be the element of this surface corresponding to the point $m$ at the end of the time $t$, and $\omega\text{U}$ the resistance which it experiences, this force will be normal to the actual position of $\omega$; therefore its components will be

$$m\text{X} = \omega\text{UV}\frac{d\text{L}}{dx}, \quad m\text{Y} = \omega\text{UV}\frac{d\text{L}}{dy}, \quad m\text{Z} = \omega\text{UV}\frac{d\text{L}}{dz},$$

in which we make, for conciseness,

$$\mathrm{V} = \left[\left(\frac{d\mathrm{L}}{dx}\right)^2 + \left(\frac{d\mathrm{L}}{dy}\right)^2 + \left(\frac{d\mathrm{L}}{dz}\right)^2\right]^{-\frac{1}{2}};$$

and if we also assume

$$\mathrm{V}\left(\frac{d\mathrm{L}}{dx}\delta x + \frac{d\mathrm{L}}{dy}\delta y + \frac{d\mathrm{L}}{dz}\delta z\right) = \delta u,$$

there will result $\omega \mathrm{U} \delta u$, for the term of the general formula that arises from the resistance $\omega \mathrm{U}$. The factor $\mathrm{U}$ will express this resistance on the unit of surface; and $\delta u$ will be the projection of the displacement of $m$ on the normal to the element $\omega$; the sign will be doubtful in consequence of the radical $\mathrm{V}$, and $\delta u$ is to be regarded as positive or negative according as the projection of the displacement of $m$ falls on the direction itself of the resistance $\omega \mathrm{U}$, or on its production.

Besides the normal resistance of the surface on which the point $m$ is constrained to move, it likewise experiences a tangential resistance, that arises from the friction against this surface; if this force be denoted by $\omega \mathrm{F}$, and the projection of the displacement of the material point $m$ on its trajectory by $\delta s$, $\omega \mathrm{F} \delta s$ will be the corresponding term of the formula of No. 531.

In consequence of this, the formula may be written as follows:

$$\left.\begin{aligned} &\Sigma m\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right) \\ &= \Sigma \mathrm{P}\delta p \pm \Sigma \mathrm{R}\delta r + \Sigma \omega \mathrm{U}\delta u - \Sigma \omega \mathrm{F}\delta s; \end{aligned}\right\} \qquad \text{(h)}$$

in which the sum $\Sigma$ of the first member belongs to all the points of the system, and the sums $\Sigma$ of the second member extend, the first to the points which are subjected to the action of the extraneous motive forces, the second to the mutual actions of all the points of the system taken two by two, the third and fourth to all the elements of the surfaces that in moving resist, and produce friction. This being established, if the sole condition that restrains the motions of the entire system, be, that a part of the points of the system are constrained to

exist on these surfaces, all these points may now be regarded as entirely free, and $\delta x$, $\delta y$, $\delta z$, the variations of the coordinates of any point whatever, may be subjected to any condition we please.

691. In the first place we will suppose that

$$\delta x = dx, \quad \delta y = dy, \quad \delta z = dz;$$

in which case the displacement of $m$ any point whatever is that which has actually place during the instant $dt$. We shall have at the same time $\delta r = dr$, and $\delta p$, $\delta u$, $\delta s$ will be replaced by $dp$, $du$, $ds$, which denote the projections of the real displacement of the point $m$ on the directions of the forces P, $\omega$U, $\omega$F. If the velocity of any point whatever such as $m$ be denoted by $v$, then since

$$\frac{d^2x}{dt^2}dx + \frac{d^2y}{dt^2}dy + \frac{d^2z}{dt^2}dz = \tfrac{1}{2}\,d.v^2,$$

equation (h) will become

$$\tfrac{1}{2}\,d.\Sigma mv^2 = \Sigma \text{P}dp \pm \Sigma \text{R}dr + \Sigma\omega \text{U}du - \Sigma\omega \text{F}ds.$$

Hence we shall obtain, by integrating,

$$\tfrac{1}{2}\Sigma mv^2 - \tfrac{1}{2}\Sigma mk^2 = \textstyle\int\Sigma \text{P}dp \pm \int\Sigma \text{R}dr + \int\Sigma\omega \text{U}du - \int\Sigma\omega \text{F}ds; \quad \text{(i)}$$

in which the integrals are supposed to be reckoned from the commencement of the motion, $k$ being the initial value of $v$.

The term $\int\Sigma \text{P}dp$ comprises the part of the motive work that belongs to the weight of the system; and if this weight be denoted by $\Pi$, and the vertical height through which the centre of gravity falls in the time $t$ by $\zeta$, this part will be equal to $\Pi\zeta$.

When the distances of the points of the system that is considered continue invariable during the motion, $dr$ will be $= 0$, and the term $\int\Sigma rdr$ will disappear from equation (i). In the case of a fluid, this term comprises the mutual attractions and repulsions of its points, which extend to considerable distances; it also comprises those mutual actions that are pro-

perly denominated *molecular forces* (No. 588), which extend only to insensible distances, and which produce those interior pressures of which no account was taken in forming equation (i). The value of this integral $\int \Sigma \text{R} dr$ depends on the change of form and on the condensations or dilatations of the fluid during its motion; and for the very small variations of density which have place in the liquid, it may vary very considerably, on account of the molecular forces or interior pressures that result from it (No. 576).

The sums $\Sigma \omega \text{U} du$ and $\Sigma \omega \text{F} ds$ that occur in the two last integrals, are themselves double integrals, which extend to all elements such as $\omega$, of the resisting and rubbing surfaces. If the part of the system which produces friction by moving against one of these surfaces is a solid body, the force $\omega \text{F}$ will be independent of the velocity of this body, and proportional, for each element such as $\omega$, to the corresponding pressure, which is equal and contrary to the resistance $\omega \text{U}$. If this part of the system which produces friction is a fluid, the force $\omega \text{F}$ will depend on its relative velocity, and will be independent of the pressure (No. 456). When the surface of which $\text{L} = 0$ is the equation, is immoveable, the projection of the displacement of $m$, on the normal to this surface, will be cipher, since the point $m$ is constrained to exist on this surface; therefore we shall have $du = 0$; in consequence of which the integral $\int \Sigma \omega \text{U} du$ will disappear; and if besides, we do not take into account the friction, equation (i) will be reduced to the ordinary equation of living forces.

692. Let us now assume

$$\delta x = d_{\prime}x, \quad \delta y = d_{\prime}y, \quad \delta z = d_{\prime}z;$$

in which case the displacements of the points of the system implied in equation (h), are their relative displacements. Let $d_{\prime}p$, $d_{\prime}u$, $d_{\prime}s$ be the projections of the relative displacements of the points to which the forces $\text{P}$, $\text{U}$, $\text{F}$ are applied on their directions; $d_{\prime}p$, $d_{\prime}u$, $d_{\prime}s$, will be the values of $\delta p$, $\delta u$, $\delta s$, which correspond to those of $\delta x$, $\delta y$, $\delta z$ that are employed. More-

over, as the other parts $d'x$, $d'y$, $d'z$ of the total differentials $dx$, $dy$, $dz$, are supposed not to influence the mutual distances of the points of the system, $\delta r$ will be changed into the differential $dr$, as in the preceding number. Consequently, equation ($h$) will become

$$\left.\begin{aligned}&\Sigma m\left(\frac{d^2x}{dt^2}\,d_,x+\frac{d^2y}{dt^2}d_,y+\frac{d^2z}{dt^2}\,d_,z\right)\\&=\Sigma \text{P}d_,p \pm \Sigma \text{R}dr + \Sigma\omega\text{U}d_,u - \Sigma\omega\text{F}d_,s.\end{aligned}\right\} \qquad \text{(k)}$$

If the relative velocity of the point $m$, the components of which are $\frac{d_,x}{dt}, \frac{d_,y}{dt}, \frac{d_,z}{dt}$, be denoted by $v_,$, we shall have

$$v^2_,=\left(\frac{d_,x}{dt}\right)^2+\left(\frac{d_,y}{dt}\right)^2+\left(\frac{d_,z}{dt}\right)^2,$$

and, by differentiating,

$$\tfrac{1}{2}d.v_,^2=\frac{dd_,x}{dt^2}d_,x+\frac{dd_,y}{dt^2}d_,y+\frac{dd_,z}{dt^2}.$$

Hence we shall have in virtue of equations (f)

$$\frac{d^2x}{dt^2}d_,x+\frac{d^2y}{dt^2}d_,y+\frac{d^2z}{dt^2}d_,z$$

$$=\tfrac{1}{2}d.v^2+\frac{du'}{dt}d_,x+\frac{dv'}{dt}d_,y+\frac{dw'}{dt}d_,z.$$

Moreover, if we substitute $d_,x$, $d_,y$, $d_,z$ in the expression for $\delta u$ of No. 690, in order to obtain that of $d_,u$, we shall have

$$d_,u=\text{V}\left(\frac{d\text{L}}{dx}d_,x+\frac{d\text{L}}{dy}d_,y+\frac{d\text{L}}{dz}d_,z\right);$$

which will be cipher in virtue of equation (g). Equation (k) will assume, by means of these values, the form

$$\tfrac{1}{2}d.\Sigma mv_1^2+\Sigma m\left(\frac{du'}{dt}d_,x+\frac{dv'}{dt}d_,y+\frac{dw'}{dt}d_,z\right)$$

$$=\Sigma\text{P}d_,p\pm\Sigma\text{R}dr-\Sigma w\text{F}d_,s;$$

and by integrating, there will result

$$\left.\begin{aligned}\Sigma m v_{,}^{2} - \tfrac{1}{2}\Sigma m k_{,}^{2} = \textstyle\int \Sigma \mathrm{P} d_{,}p \pm \int \Sigma \mathrm{R} dr - \int \Sigma \omega \mathrm{F} d_{,}s \\ - \textstyle\int \Sigma m\left(\frac{du'}{dt} d_{,}x + \frac{dv'}{dt} d_{,}y + \frac{dw'}{dt} d_{,}z,\right)\end{aligned}\right\} \quad (1)$$

in which the integrals are reckoned from the origin of the motion, $k_1$ being the initial value of $v_{,}$.

693. This equation (1) will make known the increment, during the time $t$, of the semi-sum of living forces due to the relative velocities of all the points of the system.

If in the general formula of No. 531, we make the hypothesis which has conducted us to equation (e), that is to say, if we substitute $d_{,}x$, $d_{,}y$, $d_{,}z$, instead of $\delta x$, $\delta y$, $\delta z$, we shall have

$$\Sigma m\left(\frac{d^2x}{dt^2} d_{,}x + \frac{d^2y}{dt^2} d_{,}y + \frac{d^2z}{dt^2} d_{,}z\right) = \Sigma m\,(\mathrm{X} d_{,}x + \mathrm{Y} d_{,}y + \mathrm{Z} d_{,}z),$$

or, in consequence of what precedes($h$),

$$\tfrac{1}{2} d.mv_{,}^{2} = \Sigma m\left[\left(\mathrm{X} - \frac{du'}{dt}\right) d_{,}x + \left(\mathrm{Y} - \frac{dv'}{dt}\right) d_{,}y + \left(\mathrm{Z} - \frac{dw'}{dt}\right) d_{,}z\right]:$$

in which $\Sigma$, the sum that occurs in the second member, is supposed to extend to all the forces that act on the points of the system, with the exception of those forces, that arise from the normal resistances of the fixed or moveable surfaces, and which are made to disappear, by means of the values assumed for $\delta x$, $\delta y$, $\delta z$. Now if H denotes the force whose three components are

$$m\left(\mathrm{X} - \frac{du'}{dt}\right),\ \ m\left(\mathrm{Y} - \frac{dv'}{dt}\right),\ \ m\left(\mathrm{Z} - \frac{dw'}{dt}\right),$$

and $d_1 h$ the projection of the relative displacement of its point of application on its direction, we shall have

$$m\left(\mathrm{X} - \frac{du'}{dt}\right) d_{,}x + \left(m\mathrm{Y} - \frac{dv'}{dt}\right) d_{,}y + m\left(\mathrm{Z} - \frac{dw'}{dt}\right) d_{,}z = \pm \mathrm{H} d_{,}h,$$

the upper or lower sign has place, according as this projection $d_{,}h$ falls on the direction itself of the force H, or on its produc-

tion. Therefore, if in the first case H and $d_{,}h$ be retained, and if L and $d_{,}l$ be employed in the second, we shall obtain

$$\tfrac{1}{2}\Sigma mv_{,}^{2}-\tfrac{1}{2}\Sigma mk_{,}^{2}=\int\Sigma \text{H}d_{,}h-\int\Sigma \text{L}d_{,}l;$$

in which the sum $\Sigma \text{H}d_{,}h$ is supposed to extend to all the moving forces of the system, and the sum $\Sigma \text{L}d_{,}l$ to all the resisting forces.

This last equation is in fact equation (l), presented under a different form. It appears from a comparison of it with equation (d), that the principle of living forces likewise obtains with respect to the relative velocities of the points of the system, such as they have been defined in No. 689, provided that the given forces P and Q are replaced by other forces, H and L, which depend on the first and on the motion common to all the points of the system. We are indebted to M. Coriolis for this theorem. It may be usefully employed in many questions which do not fall under the head of rational mechanics, and for which the reader is referred to a memoir of his in the twenty-first number of the Journal of the Polytechnic School, on the principle of living forces in the relative motions of machines.

694. $\int\Sigma \text{R}dr$ the term that results from the action of molecular forces, is the same in the two equations (i) and (l); for the most part the term that arises from the friction is also the same in the absolute and relative motion of the system, and, consequently, it does not undergo any change when we pass from one equation to the other; in this case, then, if the second of these equations be taken from the first, we shall have

$$\left.\begin{aligned}&\tfrac{1}{2}\Sigma m(v^{2}-v_{,}^{2})-\Sigma m(k^{2}-k_{,}^{2})=\int\Sigma \text{P}(dp-d_{,}p)\\&+\tfrac{1}{2}\int\Sigma m\left(\frac{du'}{dt}d_{,}x+\frac{dv'}{dt}d_{,}y+\frac{dw'}{dt}d_{,}z\right)+\int\Sigma\omega \text{U}du.\end{aligned}\right\}\quad(\text{m})$$

If the forces P are reducible to the weights of the different parts of the system; and if $\Pi$ denotes the entire weight, and $\zeta'$ the vertical height described by its centre of gravity during

the time $t$, in the motion common to all its points, it is easy to perceive that we shall likewise have($i$)

$$\int \Sigma \mathrm{P}(dp - d_{,}p) = \Pi \zeta'.$$

Moreover, if there is only one resisting surface, and if it be, for example, a plane that moves parallel to itself, the given motion of this plane may be assumed for that of the common motion of the system, for it satisfies the two conditions of No. 689; it will not produce any change in the mutual distances of the points of the system; nor will it prevent the points that are in contact with this moveable plane, from remaining on its surface. Besides, it is evident that as this motion is perpendicular to the moveable plane, it will not have any influence whatever on the relative velocities of the points which slide on this plane, nor on the trajectories that they describe; hence it follows, that the resisting work arising from the friction against this plane, will be the same in the absolute and relative motion, as is implied in equation ($m$). In order to simplify still more this equation, the motion of the resisting plane is supposed to be uniform; so that all its points describe perpendiculars to its initial position with a common velocity, which should be rendered invariable and independent of the action of the system on this plane. Its components $u'$, $v'$, $w'$ will be constant, and we shall have

$$\int \Sigma m\left(\frac{du'}{dt}d_{,}x + \frac{dv'}{dt}d_{,}y + \frac{dw'}{dt}d_{,}z\right) = 0.$$

If this velocity be denoted by $a$, and the angle that its direction makes with that of gravity by $\alpha$, we shall have also($k$)

$$\zeta' = at\cos\alpha.$$

Moreover, let Q denote the pressure, at the end of the time $t$, on the entire surface of the given plane, and acting in the direction of the velocity $a$. Then the resisting work corresponding to this force, taken in a direction the opposite to that

in which it acts, that is to say, to the resistance of the plane, will be $-\int \mathrm{Q}a\,dt$, during the continuance of the time $t$, the integral being supposed to vanish with this variable. Therefore if the factor $a$ be taken from under the sign $\int$, we shall have

$$\int \Sigma \omega \mathrm{U} du = -a\int \mathrm{Q}dt,$$

for the value of the last term of equation (m), which will become

$$\tfrac{1}{2}\Sigma m(k^2 - k_{,}^2) - \tfrac{1}{2}\Sigma m(v^2 - v_{,}^2) + \Pi a t \cos\alpha = a\int \mathrm{Q}dt. \qquad \text{(n)}$$

The velocity $v_{,}$ is the resultant of $v$ and of the velocity $a$ estimated in a direction contrary to that in which it acts; if therefore, $\varepsilon$ denotes the angle that the direction of the velocity $v$ makes with that of $a$, we shall have

$$v_{,}^2 = v^2 - 2av\cos\varepsilon + a^2,$$

and if $\delta$ denotes the initial value of $\varepsilon$, we shall have also

$$k_{,}^2 = k^2 - 2ak\cos\delta + a^2;$$

hence there results

$$\tfrac{1}{2}\Sigma m(k^2 - k_{,}^2) = a\Sigma mk\cos\delta - \tfrac{1}{2}a^2\Sigma m,$$
$$\tfrac{1}{2}\Sigma m(v^2 - v_{,}^2) = a\Sigma mv\cos\varepsilon - \tfrac{1}{2}a^2\Sigma m;$$

this changes equation (n) into the following, namely,

$$\Sigma mk\cos\delta - \Sigma mv\cos\varepsilon + \Pi t\cos\alpha = \int \mathrm{Q}dt, \qquad \text{(o)}$$

when $a$ the factor common to all its terms is suppressed.

The sums $\Sigma mk\cos\delta$ and $\Sigma mv\cos\varepsilon$ express, at the commencement and end of the time $t$, the quantities of motion of all the points of the system, estimated in a direction perpendicular to the given plane; the product $\Pi t\cos\alpha$ is the quantity of motion produced in the same direction by $\Pi$ the weight of the system, during the continuance of the time $t$; and the integral $\int \mathrm{Q}dt$ is the quantity of motion destroyed during this time by the resistance of the given plane; now it is evident that this last quantity ought to be equal to the excess of the

first sum over the second, increased by the quantity $\Pi t \cos \alpha$; so that the preceding equation, which expresses this equality, may be regarded as a verification of our analysis.

695. When $k$ the velocity of each point of the system is suddenly changed into the velocity $v$, the action of the system on the given plane will become a percussion; during an extremely short time, $\Pi t \cos \alpha$ the effect of the gravity may be neglected; so that the quantity of motion destroyed by the plane will be the excess of $\Sigma mk \cos \delta$ over $\Sigma mv \cos \varepsilon$.

If the system is a solid body situated above the plane, and which continues in juxta-position with its surface after the shock, $v \cos \varepsilon$ the component of the velocity $v$ will be the same at this instant, for all the points of the body, and equal to the constant $a$; therefore, if equation (o) be differenced with respect to $t$, we shall have

$$\Pi \cos \alpha = Q;$$

and, in fact, as the velocity of the plane is, by hypothesis, invariable, the accelerations produced by gravity, which would have place in a direction perpendicular to its surface, must be continually destroyed by its resistance; consequently, this force must be equal and contrary to the component of the weight $\Pi$ in this same direction, to which component the pressure $Q$ should be equal.

We may remark that when the angle $\alpha$ is obtuse, the sign of the preceding value of $Q$ is minus. But, it was supposed above, that the direction of the pressure exerted on this plane was the same as that of the velocity $a$; and, if the contrary was the case, the sign of $Q$ should be changed in all the preceding equations. Now, as the pressure, in fact, takes place in a direction contrary to that of the velocity $a$, it follows, that when the angle $\alpha$ is obtuse, the value of $Q$ must be $-\Pi \cos \alpha$; or, in other words, this value must be always equal to $\Pi \cos \alpha$, abstracting from the consideration of the sign.

696. Equation (o), which is evident in itself, will enable us to determine the pressure on a plane AB (fig. 57), of a fluid vein which is actuated by a velocity equal to $a$, in a direction perpendicular to this plane, and which direction makes an angle equal to $\alpha$ with that of gravity.

For greater clearness, we shall suppose that the liquid issues from a vessel through a horizontal orifice, and that it forms below the contraction of the vein (No. 676), a vertical cylinder, all whose points are actuated by a common vertical velocity, which we shall denote by $\gamma$. We shall also suppose that the level of the liquid as kept at a constant height in the vessel; by which means the velocity $\gamma$ becomes independent of the time.

The vein retains its cylindrical form and the velocity $\gamma$ as far as CD, a horizontal section that is made at a short distance above the plane AB; it will then spread on this plane, and will eventually flow over it. After the lapse of a certain time, the fluid will attain to a permanent state, in which the velocity of each molecule will only depend on the place that it occupies, and where the pressure in any point whatever of the plane AB, will be also independent of the time. It is in this state that it is proposed to determine Q, the total pressure that is exerted on the entire surface of the plane.

The part of this pressure that is due to the weight of the liquid, will be the component of this weight estimated in a direction perpendicular to the plane AB, with the exception of the part of this same weight that is sustained by the sides of the vessel from which the liquid flows. As it will be always easy, in each particular case, to take it into account, we shall here abstract from the consideration of it, and consequently make $\Pi = 0$ in equation (o). It is evident that in the sums $\Sigma mk \cos \delta$ and $\Sigma mv \cos \varepsilon$, we may also omit the consideration of the molecules of the liquid that are situated above CD, since they always retain the same velocity, which causes the dif-

ference of these two sums to disappear. Finally, if the diameter of the fluid vein is very small, the thickness of the liquid stratum will be also very small, to an inconsiderable distance about the axis of the vein. At this distance, the relative velocities of the points of the stratum will be sensibly parallel to the plane AB, throughout the entire thickness of the stratum, or what comes to the same thing, their components perpendicular to this plane will be equal to $a$. Moreover, this part of the fluid that is comprised between AB and CD will be much more considerable than the part adjoining to the axis of the vein, if the surface of the plane AB is very great relatively to the section CD; we may therefore, without sensible error, assume $a$ to be equal to the $v\cos\varepsilon$, the component of $v$, the velocity of each point of the fluid contained between AB and CD.

This being established, let C′D′ be another section of the fluid vein, made above CD, and such that the volume comprised between CD and C′D′ may be equivalent to the volume of the fluid contained between AB and CD; and let the time that the first volume of the liquid takes to traverse the section CD, and to occupy the place of the second volume, be denoted by $\theta$, then if the sums $\Sigma mk\cos\delta$ and $\Sigma mv\cos\varepsilon$, of equation (o), extended to all the points of the second volume, be supposed to refer to the beginning and end of the time $\theta$, since at the commencement, all these points were situated above CD, and consequently, are actuated by a velocity, which makes an angle $\alpha$ with the velocity $a$, we have for any point whatever, such as $m$,

$$k=\gamma,\quad \delta=\alpha,\quad k\cos\delta=\gamma\cos\alpha.$$

At the end of the time $t$, we have, as has been already stated,

$$v\cos\varepsilon=a,$$

for any point whatever such as $m$, of the liquid contained be-

tween AB and CD. If therefore the mass of this liquid be denoted by $\mu$, there will result

$$\Sigma mk \cos\delta - \Sigma mv \cos\varepsilon = \mu(\gamma \cos a - a).$$

Moreover, as the pressure Q is constant, the integral $\int \mathrm{Q}dt$ is equal to the product $\mathrm{Q}\theta$, for the duration of the time $\theta$; therefore by equation (o) we shall have

$$\mu(\gamma \cos a - a) = \mathrm{Q}\theta.$$

If $n$ be the number of times that the time $\theta$ is contained in $t$ any time whatever; then $n\mu$ will be the mass of the liquid which traverses the section CD during the time $t$; but as this mass is also equal to $\rho c\gamma t$, in which $\rho$ denotes the density of the liquid, $c$ the area of the section CD, and $\gamma$ the constant velocity with which the liquid flows through this section, we shall consequently have

$$n\mu = \rho c\gamma t.$$

If the preceding equation be multiplied by $n$, and if we substitute in it this value of $n\mu$, and then suppress the factor $t$, which is common to all its terms, we have finally

$$\mathrm{Q} = \rho c\gamma\,(\gamma \cos a - a)$$

for the value of this pressure that it is proposed to determine. It should be observed that this formula implies that the angle $a$ is acute; when it is obtuse, the sign of Q should be changed, as has been stated above; so that then we shall have

$$\mathrm{Q} = \rho c\gamma(\gamma \cos a + a).$$

These two expressions for Q likewise suppose that the plane AB is entirely covered with the fluid spread over its surface; in order to this it is necessary that $a$ should not be a right angle, and in general, that it should differ sensibly from 90°. When the direction of the motion of the plane is vertical we have $a = 0$, or $a = 180°$, and, consequently,

$$\mathrm{Q} = \rho c\gamma(\gamma \pm a);$$

the upper sign has place when the plane moves in the direction of gravity, and the lower sign in the contrary case. If $a = 0$, and $\gamma$ be the velocity due to the height $h$, and $g$ the gravity, we shall have

$$Q = 2g\rho ch;$$

so that in this case the pressure Q will be equal to the weight of a portion of the cylindrical vein, the length of which is $h$.

# NOTES.

## CHAPTER I.

(*a*) From equation (1) we obtain

$$\left(\frac{m}{l}-\frac{m'}{l'}\right)ghdt = mdv - m'dv',$$

which becomes, by reducing the first terms of the first member to a common denominator, and putting $dv$ for $-dv'$ in the second,

$$\frac{(ml'-m'l)}{ll'}gh\,dt = (m+m')\,dv, \ \therefore \frac{dv}{dt} = \frac{ml'-m'l}{(m+m')ll'}gh\ ;$$

now if in the equation

$$\text{T} = m\left(\frac{gh}{l}-\frac{dv}{dt}\right),$$

we substitute this value of $\frac{dv}{dt}$, there results

$$\text{T} = m\frac{gh}{l} - \left(\frac{m^2l'-mm'l}{(m+m')ll'}\right)gh,$$

which is equal to

$$\text{T} = \left(\frac{m^2l'+mm'l'-m^2l'+mm'l}{(m+m')\,ll'}\right)gh = \frac{mm'}{m+m'}\,\frac{(l+l')}{ll'}gh\ ;$$

and when $m'l = ml'$, this expression becomes, as $l' = \frac{m'l}{m}$,

$$l\,.\left(\frac{m+m'}{m}\right).\,mm'gh \div (m+m')\frac{m'l^2}{m} = \frac{mgh}{l}.$$

(*b*) The quantities of motion lost are $m\,.\,(a-c) = m'\,(a'+c)$, from which we obtain

$$ma - m'a' = (m + m')c, \because c = \frac{ma - m'a'}{m + m'};$$

if this value of $c$ be substituted in $m \cdot (a - c)$, it becomes

$$ma - \left(\frac{m^2a - mm'a'}{m + m'}\right) = \frac{m^2a + mm'a - m^2a + mm'a'}{m + m'} = \frac{mm'(a + a')}{m + m'}.$$

($c$) Since in this equation $dv = \frac{d^2x}{dt}$, $dv' = -\frac{d^2x}{dt}$, and $x' = \lambda - x$, we obtain by substituting these values for $dv$, $dv'$, $x'$,

$$x\left(\frac{gh}{l} \cdot dt - \frac{d^2x}{dt}\right) = (\lambda - x)\left(\frac{gh}{l'} dt + \frac{d^2x}{dt}\right),$$

from which there results, by reducing,

$$\lambda \frac{d^2x}{dt^2} = \left(\frac{l + l'}{ll'}\right) ghx - \frac{\lambda gh}{l'},$$

and by substituting $\alpha^2$ for $\frac{l + l'}{\lambda ll'}$ and $\beta$ for $\frac{gh}{l'}$, becomes the equation in the text, the integral of which can be obtained by the common rules. Now when the entire chain exists on the same inclined plane, that is, when $x - x' = \pm \lambda$, we have $dx = dx'$, $v = \frac{dx}{dt} = v'$, consequently, the preceding equation becomes

$$x\left(\frac{gh}{l} dt - \frac{d^2x}{dt}\right) = (x \mp \lambda)\left(\frac{gh}{l'} dt - \frac{d^2x}{dt}\right),$$

which, as $l$ in this case is equal to $l'$ is reduced to $\left(\frac{d^2x}{dt^2} = \frac{gh}{l}\right)$, which is the equation in uniformly accelerated motion.

($d$) Since a line parallel to the base of a triangle divides the sides which it meets into segments $\div\!\!\div l$ to the sides, it is evident that as $x$, $x'$, the parts of the chain in equilibrio on $l$ and $l'$, are proportional to these lines, the line connecting the two extremities of the chain must be parallel to the base, and consequently horizontal.

($e$) By adding these two equations together, we obtain $mdv + m'dv' = 0$, and if the first members of these equations be multiplied by $2v$, and $2v'$ respectively, and then added together, there results

$$2\,mvdv + 2\,m'v'dv' + 2\,\text{R}\,(vdt - v'dt) = 0,$$

now

$$vdt - v'dt = dx - dx' = -dr,$$

consequently,

$$2\,mvdv + 2\,m'v'dv' = 2\,\mathrm{R}dr,\ \therefore\ mv^2 + m'v'^2 = 2\int \mathrm{R}dr + c'.$$

(*f*) By the law of Mariotte we have

$$\mathrm{R} : k\omega :: \alpha : r,\ \therefore\ \mathrm{R} = \frac{k\omega\alpha}{r}.$$

(*g*) As in general $f(r, \alpha) = \int \mathrm{R}dr + c'$, in this case we shall have

$$f(r, \alpha) = -\int k\omega\alpha \,.\, \frac{dr}{r} + c' = -k\omega \log r + c'.$$

Now when $r = \alpha, f(r, \alpha) = 0,\ \therefore\ -k\omega \log \alpha + c' = 0$, and $c' = k\omega \log \alpha$, consequently,

$$f(r, \alpha) = k\omega \log \frac{\alpha}{r}.$$

(*h*) See the discussion of this point of the question in the 21st Number of the Journal of the Polytechnique School, page 191.

(*i*) From the equation $mv + m'v' = 0$ we obtain $v' = -\frac{mv}{m'}$, and at the mouth of the piece where $v = \mathrm{v}$, $v' = \mathrm{v}'$, and $r = l$, we have

$$\mathrm{v}'^2 = \frac{m^2\mathrm{v}^2}{m'^2},\ \therefore\ m\mathrm{v}^2 + m'\mathrm{v}'^2 = m\mathrm{v}^2 + \frac{m^2\mathrm{v}^2}{m'} = \frac{(m+m')}{m'}m\mathrm{v}^2 = 2k\omega\alpha \log \frac{l}{\alpha}.$$

(*k*) The differential of $\mathrm{v}^2$ relatively to $\alpha$ is evidently equal to

$$\frac{2\,m'k\omega}{m\,(m+m')}\left(\log \frac{l}{\alpha} - 1\right) d\alpha,$$

which, since in the case of a maximum it is equal to cipher, gives $\log \frac{l}{\alpha} = 1,\ \therefore\ \frac{l}{\alpha} = e$, and as $e = 2{,}71828$, it follows that $\alpha$ is somewhat greater than $\frac{l}{3}$.

(*l*) When $m'$ is at rest, $u$ is equal $\frac{mv}{m+m'}$, and if $m'$ be considered as infinite relatively to $m$ this expression is cipher.

(*l*) In the first case the actual motion impressed on $m'$ is equal to $\frac{mm'v}{m+m'}$, which becomes, when $m'$ is so small that it may be neglected relatively to $m$, equal to $m'v$; in the second case the actual motion impressed on $m'$ is $\frac{2\,mm'v}{m+m'}$, which becomes, in the same supposition, equal to $2\,m'v$.

($m$) The integral of $\cos^3\theta \sin\theta\, d\theta = -\frac{1}{4}\cos^4\theta$, which taken between the limits $\frac{\pi}{2}$ and 0, is equal to $\frac{1}{4}$, hence the value of R is that given in the text.

($n$) See Newton's Principia, Book II. Prop. 10, and Scholium to Prop. 34, 3rd edit.; Lacroix's Differen. Calculus, Tom. 2, article 867; Woodhouse's Isoperimetrical Problems, page 115.

## CHAPTER II.

($a$) The expression for C is equal to

$$\text{C} = \rho \iiint x^2.\, dx\, dy\, dz + \rho \iiint y^2 dy\, dx\, dz,$$

and when these are integrated between the limits stated in the text, there results

$$\text{C} = \rho.\left(\frac{a^3bc}{3} + \frac{ab^3c}{3}\right) = \rho\frac{abc}{3}(a^2+b^2);$$

if the axis passed through the centre of gravity of this homogeneous parallellopiped, then the moment of inertia would be $\frac{\rho abc}{12}(a^2+b^2)$; it is evident from the expressions for the moments of inertia respecting the three sides, that the greatest moment of inertia belongs to the least side; and it appears from what is stated in the last paragraph of No. 370, that the rectangular axes passing through the centre of gravity of the parallellopiped parallel to its three sides are principal axes.

($b$) As $\sqrt{r^2-y^2} = \frac{r^2-y^2}{\sqrt{r^2-y^2}}$, we shall obtain by making $y = rz$,

$$\int_{-r}^{r}\sqrt{r^2-y^2}.\, dy = \int_{-r}^{r}\frac{r^2dz}{\sqrt{1-z^2}} - \int_{-r}^{r}\frac{r^2z^2dz}{\sqrt{1-z^2}};$$

now $\int\frac{dz}{\sqrt{1-z^2}} = \text{arc sin} = z$, i. e., by substituting for $z$, arc sin $= \frac{y}{r}$, and this taken between the limits $r, -r$, gives $\int_{-r}^{r}\frac{r^2dy}{\sqrt{r^2-y^2}} = r^2.\, 2\,\text{arc sin} = 1 = r^2\pi$, and the integral of $\frac{r^2z^2dz}{\sqrt{1-z^2}} = r^2(-\frac{1}{2}z \sqrt{1-z^2} + \frac{1}{2}\,\text{arc sin} = z)$, which when $\frac{y}{r}$ is substituted for $z$, and

then taken between the limits $r, -r$, is reduced to $\frac{1}{2}\pi r^2$, and consequently,

$$\frac{2\rho c x^2 dx}{b}\int\sqrt{r^2-y^2}.dy = \frac{\rho\pi c r^2 x^2\,dx}{b},$$

which becomes, by substituting for $r^2$ its value $b^2 - \frac{b^2}{a^2}x^2$, and reducing $\frac{\rho\pi bc}{a^2}(a^2x^2 - x^4)dx$, the integral of which is $\frac{\rho\pi bc}{a^2}\left(\frac{a^2x^3}{3} - \frac{x^5}{5}\right)$, and when this is taken between the limits $x = a$, $x = -a$, there results

$$\frac{2\pi\rho bc}{a^2}\left(\frac{a^5}{3} - \frac{a^5}{5}\right) = \frac{4\rho\pi bc.a^3}{15}.$$

(*c*) $\frac{8\pi\rho}{15}(a + da)^5 = \frac{8\pi\rho a^5}{15} + \frac{5.8\pi\rho a^4 da}{15}$, the other terms of the series being neglected as infinitely small.

(*d*) $(r + dr)^2 - r^2 = 2rdr + dr^2 \because \pi(r + dr)^2dx - \pi r^2dx = 2\pi r dr dx + \pi dr^2dx$; this last term is of the third order, and may be neglected.

(*e*) See No. 84.

(*f*) In this case we have

$$\mu = \frac{\pi\rho}{2}\int(2ax - x^2)^2\,dx = \frac{\pi\rho}{2}\int(4a^2x^2 - 4ax^3 + x^4)\,dx =$$
$$\frac{\pi\rho}{2}\left(\tfrac{4}{3}a^2x^3 - ax^4 + \frac{x^5}{5}\right) + \text{C},$$

which when taken between the limits $x = 0$, $x = \alpha$, and reduced, becomes the expression in the text, and when $\alpha = 2a$, it is equal to

$$\frac{\pi\rho}{2}\left(\frac{4.8}{3}a^5 - 16a^5 + \frac{32a^5}{5}\right) = \frac{8\pi\rho a^5}{15}.$$

(*g*) By multiplying and dividing the value of $\mu$ by $\beta - \alpha$, we obtain

$$\mu = \tfrac{1}{10}\pi\rho\theta^4.(\beta - \alpha)\left(\frac{\beta^5 - \alpha^5}{\beta - \alpha}\right) = \tfrac{1}{10}\pi\rho\theta^4.(\beta - \alpha).$$
$$(\beta^4 + \alpha\beta^3 + \alpha^2\beta^2 + \alpha^3\beta + \alpha^4) = (\text{as } \alpha\theta = a,\ \theta\beta = b),$$

the expression in the text.

(*h*) $p^2 = \text{D}^2 - (\text{D}\cos\delta)^2 = x^2 + y^2 + z^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2 = x^2(1 - \cos\alpha^2) + y^2(1 - \cos^2\beta) + z^2(1 - \cos^2\gamma) - 2xy\cos\alpha\cos\beta - 2xz\cos\alpha\cos\gamma - 2yz\cos\beta\cos\gamma$, which is equal to the expression in the text.

(*i*) For in consequence of equations (2), if the members of equations (1) be multiplied by $a, b, c$, respectively, and then added, the result will be equations (3).

(*k*) By No. 377 we have $x_1 = ax + a'y + a''z$, equal by substituting for $a, a', a''$, their respective values given in equation (5),

$$(\cos\theta.\sin\psi\sin\phi + \cos\psi\cos\phi)x + (\cos\theta\cos\psi\sin\phi - \sin\psi\cos\phi)y - \sin\theta\sin\phi.z,$$

which is evidently the same thing as

$$(x\cos\theta\sin\psi + y\cos\theta\cos\psi - z\sin\theta)\sin\phi + (x\cos\psi - y\sin\psi)\cos\phi$$

$=$ by substituting X and Y for their values

$$\text{Y}\sin\phi + \text{X}\cos\phi,$$

the expressions for $y_{\prime}$ and $z_{\prime}$ may be obtained in a similar manner; hence we evidently obtain

$$\int x_{\prime}y_{\prime}dm = 0 = \sin\phi\cos\phi.\int(\text{Y}^2 - \text{X}^2)dm + (\cos^2\phi - \sin^2\phi)\int \text{XY}\,dm,$$

in which if we substitute $\frac{\sin 2\phi}{2}$ for $\sin\phi\cos\phi$, and $\cos 2\phi$ for $\cos^2\phi - \sin^2\phi$, we obtain equation (b).

(*l*)
$$\text{Y} = x\cos\theta\sin\psi + y\cos\theta\cos\psi - z\sin\theta,$$
$$z_{\prime} = x\sin\theta\sin\psi + y\sin\theta\cos\psi + z\cos\theta,$$

$$\begin{aligned}\therefore \text{Y}z_{\prime} = {} & x^2\sin\theta\cos\theta\sin^2\psi + xy\sin\theta.\cos\theta\sin\psi\cos\psi - zx\sin^2\theta\sin\psi \\ & + xy\sin\theta\cos\theta\sin\psi\cos\psi + y^2\sin\theta\cos\theta\cos^2\psi - zy\sin^2\theta\cos\psi \\ & + zx\cos^2\theta\sin\psi + zy\cos^2\theta\cos\psi - z^2\sin\theta\cos\theta \\ = {} & x^2.\sin\theta\cos\theta\sin^2\psi + 2xy\sin\theta\cos\theta\sin\psi\cos\phi + zx\sin\psi(\cos^2\theta - \sin^2\theta) \\ & + y^2\sin\theta\cos\theta\cos^2\psi + zy\cos\psi(\cos^2\theta - \sin^2\theta) - z^2\sin\theta\cos\theta;\end{aligned}$$

consequently if we substitute $f$ for $\int x^2dm$, $g$ for $\int y^2dm$, &c., and reduce all the terms into two expressions, one of which has the factor $\sin\theta.\cos\theta$, and the other the factor $\cos^2\theta - \sin^2\theta$, the value of $\int \text{Y}z_1dm$ given in the text, will be obtained.

(*m*) Dividing both sides of this last equation by $\cos\theta$, we obtain

$$(h'(\cos^2\psi - \sin^2\psi) + (f-g)\sin\psi\cos\psi)\tan\theta = f'\sin\psi - g'\cos\psi;$$

in which if we substitute for $\cos\psi, \sin\psi$, their values, there results

$$\frac{[h'.(1-u^2) + (f-g).u.]\,\text{tang}.\theta}{1+u^2} = \frac{f'u - g'}{\sqrt{1+u^2}},$$

from which it is easy to obtain the expression for tang $\theta$, given in

text; and if in the first equation, their values be substituted for $\sin\psi$, $\cos\psi$, it becomes

$$\left(\frac{fu^2+2h'u+g}{1+u^2}-h\right)\sin\theta\cos\theta+\frac{(g'u+f')}{\sqrt{1+u^2}}(\cos^2\theta-\sin^2\theta)=0,$$

and if all these terms be divided by $\cos^2\theta$, there results

$$\left(\frac{fu^2+2h'u+g}{1+u^2}-h\right)\operatorname{tang}\theta+\frac{g'u+f'}{\sqrt{1+u^2}}=\frac{(g'u+f')}{\sqrt{1+u^2}}\operatorname{tang}^2\theta$$

$$=[(f-h)u^2+2h'u+g-h]\frac{\operatorname{tang}\theta}{\sqrt{1+u^2}}+(g'u+f')=(g'u+f').\operatorname{tang}^2\theta),$$

and by substituting for $\dfrac{\operatorname{tang}\theta}{\sqrt{1+u^2}}$, there results,

$$\frac{[(f-h)\,u^2+2h'u+g-h]\,(f'u-g')}{h'(1-u^2)+(f-g)\,u}+g'u+f'$$
$$=\frac{(g'u+f').(f'u-g')^2.(1+u^2)}{[h'(1-u^2)+(f-g)\,u]^2};$$

multiplying both sides by the denominator of the first term of the first member, and we obtain

$$[(f-h)u^2+2h'u+(g-h).](f'u-g')+(g'u+f').[h'(1-u^2)+(f-g)\,u]$$
$$=\frac{(g'u+f')(f'u-g')^2.(1+u^2)}{h'(1-u^2)+(f-g)\,u},$$

now it is evident from inspection of this equation, that its first member can be resolved into two factors, one of which is $1+u^2$, from which it is easy to obtain equation (d).

($n$) In this case we have by equations (1).

$$\textstyle\int xydm=aa'\int x_{,}^2dm+bb'\int y_{,}^2dm+cc'\int z_{,}^2dm$$
$$=(\text{as}\textstyle\int x_{,}^2dm=\int y_{,}^2dm),\ (aa'+bb')\int x_{,}^2dm+cc'\int z^2dm,$$

now, by equations (4), we have $aa'+bb'=-cc'$, consequently, $\int xydm$ becomes equal to $cc'(\int z_{,}^2dm-\int x_{,}^2dm)$.

($o$) $\mathrm{A}=\frac{1}{5}\mathrm{M}(b^2+c^2)$ No. 370, $\because$ when $b=c$, as in the present case, we have

$$\mathrm{A}=\tfrac{2}{5}\mathrm{M}b^2,\text{ and }\alpha^2=\frac{\mathrm{A}-\mathrm{C}}{\mathrm{M}}=\mathrm{M}\frac{(b^2-a^2)}{5\mathrm{M}}=\frac{b^2-a^2}{5}.$$

($p$) The distances of the sides of the parallellopiped from the

centre of gravity, are $\sqrt{a^2+b^2}$, $\sqrt{a^2+c^2}$, $\sqrt{b^2+c^2}$, and the moments of inertia with respect to the sides, are $\frac{4}{3}\text{M}.(a^2+b^2)$, $\frac{4}{3}\text{M}(a^2+c^2)$, $\frac{4}{3}\text{M}(b^2+c^2)$; so that when $\text{M}(a^2+b^2)$ is taken from the first, the remainder $=\frac{1}{3}\text{M}(a^2+b^2)$; and therefore in the present case we have,

$$\alpha=\pm\sqrt{\frac{\text{A}-\text{C}}{\text{M}}},=\sqrt{\frac{\text{M}(b^2-a^2)}{3\text{M}}}=\sqrt{\frac{b^2-a^2}{3}}.$$

See second edition of Mr. Whewell's Dynamics, Part II. page 271.

## CHAPTER III.

(*a*) As the velocity of $dm$ is equal to $r\omega$, when resolved parallel to these axes, it becomes $-y\omega$, $x\omega$.

(*b*) By substituting $\int y^2dm$ for $\text{M}y_1y'$, and $\int x^2dm$ for $\text{M}x_1x'$ and $\frac{\mu vf}{\int r^2dm}$ for $\omega$ in the equation

$$\omega\text{M}'y_1y'+\omega\text{M}x_1x'-\mu vf=0,$$

there results

$$\frac{\mu vf}{\int r^2dm}(\int y^2dm+\int x^2dm)-\mu vf=\mu vf-\mu vf=0.$$

(*c*) In the case of equations 2, the value of the force parallel to the axis of $x$, is $-\omega\int ydm$, while in the case of equations (3) the value of the force parallel to the same axis of $x$, is $\omega^2\int xdm$, for the centrifugal force acts in the direction of the radius $r$, while the force in the former case was at right angles to $r$, consequently the expressions for $z'$, $z''$, the distance from the plane of the axes of $x$ and $y$, must be the inverse, the one of the other.

With respect to the statement in page 73, namely, "that the value of the integrals, which the equation $(\cos^2\theta-\sin^2\theta)\int xydm+\sin\theta\cos\theta(\int x^2dm-\int y^2dm)=0$, contains, may change with the position of the point O," it appears to be incorrect; in fact, the direction of the principal axes which with O$z$ belong to the point O, must be parallel to the corresponding principal axes passing through the centre of gravity, for if a point such as P be assumed in the principal axis GY passing through the centre of gravity, at a distance from G equal to $\alpha$, then the coordinates of any element, such as $dm$, passing through the centre of gravity, are $x, z, y-\alpha$, and we have $\int x.(y-\alpha)\,dm=\int xydm-\alpha\int xdm=0$, $\because$

9

$$\int xydm = 0, \because \frac{\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \tfrac{1}{2}\tan 2\theta = \frac{\int xydm}{\int x^2dm - \int y^2dm} = 0, \because \theta = 0,$$

consequently, the two other principal axes are respectively parallel to those passing through the centre of gravity.

(*d*) $x$ and $y$ being considered as functions of the variable $t$, $dx = \frac{dx}{dt}dt$, $dy = \frac{dy}{dt}dt$, and as $\frac{dx}{dt} = -y\omega$, $\frac{dy}{dt} = x\omega$, $\because xdy = x^2\omega dt$, $ydx = -y^2\omega dt$, and consequently, $xdy - ydx = (x^2 + y^2)\omega dt = r^2\omega dt$.

(*e*) $\quad \mathrm{U} = \int\left(\mathrm{X} - \frac{d^2x}{dt^2}\right)dm$, and as $\frac{d^2x}{dt^2} = -y\frac{d\omega}{dt} - x\omega^2$,

$$\text{and } \frac{d\omega}{dt} = \int\left(\frac{x\mathrm{Y} - y\mathrm{X}}{\int r^2dm}\right)dm,$$

there results, by substituting for $\frac{d^2x}{dt^2}$ and observing, that in the integral relative to $dm$, $\frac{d\omega}{dt}$ must be taken from without the sign of integration,

$$\mathrm{U} = \int \mathrm{X}dm + \omega^2\int xdm + \int\left(\frac{x\mathrm{Y} - y\mathrm{X}}{\int r^2dm}\right)dm\int ydm,$$

hence, as $\int xdm = \mathrm{M}x_1$ and $\int ydm = y_1\mathrm{M}$, we have evidently the expression in the text.

$$(f)\quad \frac{dx}{dt} = a\cos\theta\frac{d\theta}{dt}, \text{ i. e., } -y_1\omega = y_1\frac{d\theta}{dt}, \because \omega = -\frac{d\theta}{dt}.$$

(*g*) From the value of $l$ it appears, that when the axis of rotation passes through the centre of gravity, $l$ must be infinite, and consequently the time of vibration, in fact, the oscillatory motion is then changed into one of revolution.

(*h*) See Philosophical Transactions, year 1818, and Whewell's Dynamics, page 240.

(*i*) By dividing both numerator and denominator by $a$, the value of $l = \frac{\mathrm{M}a^2 + \mathrm{A}}{\mathrm{M}a}$, becomes $\frac{\mathrm{M}a + \frac{\mathrm{A}}{a}}{\mathrm{M}}$, $\because \frac{dl}{da} = \frac{\mathrm{M} - \frac{\mathrm{A}}{a^2}}{\mathrm{M}}$, which in case of a minimum $= 0$, from which there results $a^2 = \frac{\mathrm{A}}{\mathrm{M}}$, this value is evidently that of the *minimum*. It is easy to show, without having recourse to the calculus, that $l$ is the minimum in this case, for as it is

equal to $\text{M}a + \frac{\text{A}}{a} \div \text{M}$, it must be least when the numerator is least, since the denominator is constant, and as the product of the two parts of which the numerator consists is constant, their sum is least when these parts are equal, that is, when $a^2 = \frac{\text{A}}{\text{M}}$.

(*i*) By multiplying each member of this equation by $dt$, then integrating, we obtain $\left(\text{as } d\omega = \frac{d\omega}{dt}dt,\right)$ the value of $\omega$.

(*k*) Versed sine of $\beta = \frac{b^2}{2c}$, $\because \cos\beta = \text{rad} - \text{versed sine} = c - \frac{b^2}{2c} =$ when the radius is unity $= 1 - \frac{b^2}{2c^2}$, and by substituting this value in the preceding equation, there results by putting $n$ for $\frac{\text{M}}{\mu}$, the value of $v^2$ given in the text.

(*l*) The quantity of motion impressed on the compound pendulum, is known from knowing that of the recoil, which may be easily ascertained, and then as this is equal to $\mu v$, when $\mu$ is given, $v$ can be immediately obtained.

(*m*) See Hutton's Course of Mathematics, Vol. III. page 269, and Whewell's Dynamics, No. 214, page 199.

## CHAPTER IV.

(*a*) $a$, $b$, $c$, &c. are the same, at each instant, for all points of the body; for wherever $dm$ is situated the axis of $x$ is fixed, and $x_{\prime}$, $y_{\prime}$, $z_{\prime}$ are parallel.

(*b*) $y_{\prime}^2 = \frac{q^2 x_{\prime}^2}{p^2}$, $z_{\prime}^2 = \frac{r^2 x_{\prime}^2}{p^2}$, $\because x_{\prime}^2 + y_{\prime}^2 + z_{\prime}^2 = \left(\frac{p^2+q^2+r^2}{p^2}\right)x_{\prime}^2$

$$\cos \text{IO}x_{\prime} = \frac{x_{\prime}}{\sqrt{x_{\prime}^2 + y_{\prime}^2 + z_{\prime}^2}} = \frac{p}{\sqrt{p^2+q^2+r^2}}, \text{ \&c.}$$

(*c*) In No. 411 it is proved that when $p$, $q$, $r$ are constant, the differentials of these numerators are cipher, from which it follows that they are independent of the time.

(*d*) $p^2dt^2 = (bdc + b'd'c' + b''dc'')^2$, $q^2dt^2 = (adc + a'dc' + a''dc'')^2$ $\because (p^2 + q^2)\,dt^2 = (a^2 + b^2)dc^2 + (a'^2 + b'^2)\,dc'^2 + (a''^2 + b''^2)\,dc''^2 + 2(aa' + bb')\,dcdc' + 2(aa'' + bb'')\,dcdc'' + 2(a'a'' + b'b')\,dc'dc''$;

now as $a^2+b^2=1-c^2$, $(a'^2+b'^2)=1-c'^2$, $a''^2+b''^2=1-c''^2$, $2(aa'+bb')=-2cc'$, $2(aa''+bb'')=-2cc''$, $2(a'a''+b'b'')=-2c'c''$, the expression given above becomes

$$(p^2+q^2)\,dt^2=dc^2-c^2dc^2+dc'^2-c'^2dc'^2+dc''^2-c''^2dc''^2-2cc'dcdc'-2cc''dcdc''-2c'c''dc'dc''=dc^2+dc'^2+dc''^2-(cdc+c'dc'+c''dc'')^2.$$

(*e*) By making this substitution there results

$$a\frac{dx}{dt}=ax_{,}\frac{da}{dt}+ay_{,}\frac{db}{dt}+az_{,}\frac{dc}{dt}$$

$$a'\frac{dy}{dt}=a'x_{,}\frac{da'}{dt}+a'y_{,}\frac{db'}{dt}+a'z_{,}\frac{dc'}{dt}$$

$$a''\frac{dz}{dt}=a''x_{,}\frac{da''}{dt}+a''y_{,}\frac{db''}{dt}+a''z_{,}\frac{dc''}{dt};$$

consequently by adding them together we obtain

$$a\frac{dx}{dt}+a'\frac{dy}{dt}+a''\frac{dz}{dt}=x_{,}\left(a\frac{da}{dt}+a'\frac{da'}{dt}+a''\frac{da''}{dt}\right)+y_{,}\left(a\frac{db}{dt}+a'\frac{db'}{dt}+a''\frac{db''}{dt}\right)+z_{,}\left(a\frac{dc}{d\theta}+a'\frac{dc'}{dt}+a''\frac{dc''}{dt}\right);$$

now the first term of the second member of this equation is cipher, the second term of the second member is equal to $-r'y_{,}$, and the third term of the second member is equal to $qz_{,}$, consequently we shall have

$$a\frac{dx}{dt}+a'\frac{dy}{dt}+a''\frac{dz}{dt}=qz_{,}-ry_{,};$$

the two other values may be obtained in the same manner.

(*f*) If $qz_{,}-ry_{,}$, $rx_{,}-pz_{,}$, $py_{,}-qx_{,}$, and their values, be respectively multiplied by $a$, $b$, $c$, there will result

$$a^2\frac{dx}{dt}+aa'\frac{dy}{dt}+aa''\frac{dz}{dt}=a(qz_{,}-ry_{,})$$

$$b^2\frac{dx}{dt}+bb'\frac{dy}{dt}+bb''\frac{dz}{dt}=b(rx_{,}-pz_{,})$$

$$c^2\frac{dx}{dt}+cc'\frac{dy}{dt}+cc''\frac{dz}{dt}=c(py_{,}-qx_{,});$$

consequently, by adding these equations together, we obtain

$$(a^2+b^2+c^2)\frac{dx}{dt}+(aa'+bb'+cc')\frac{dy}{dt}+(aa''+bb''+cc'')\frac{dz}{dt}$$
$$=a(qz_{,}-ry_{,})+b(rx_{,}-pz_{,})+c(py_{,}-qx_{,}),$$

which as $a^2+b^2+c^2=1$, and $(aa'+bb'+cc')$, $(aa''+bb''+cc'')$ are $=0$, gives for $\frac{dx}{dt}$ the value in the text; and when the differential of this value of $\frac{dx}{dt}$ is taken, $x_{,}$, $y_{,}$, $z_{,}$ must be considered as constant, because they do not vary with the time.

(*g*) If $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, and their respective values, be multiplied by $a$, $b$, $c$, we obtain

$$a\frac{d^2x}{dt^2}=a^2(z_{,}dq-y_{,}dr)+ab(x_{,}dr-z_{,}dp)+ac(y_{,}dp-x_{,}dq)+ada.$$
$$(qz_{,}-ry)+adb(rx_{,}-pz_{,})+adc(py_{,}-qx_{,}),$$

$$a'\frac{d^2y}{dt^2}=a'^2(z_{,}dq-y_{,}dr)+a'b'(x_{,}dr-z_{,}dp)+a'c'(y_{,}dp-x'dq)+$$
$$a'da'(qz_{,}-ry_{,})+a'db'(rx_{,}-pz_{,})+a'dc'(py_{,}-qx_{,}),$$

$$a''\frac{d^2z}{dt^2}=a''^2(z_{,}dq-y_{,}dr)+a''b''(x_{,}dr-z_{,}dp)+a''c''(y_{,}dp-x_{,}dy+$$
$$a''da''(qz_{,}-ry_{,})+a''db''(rx_{,}-pz_{,})+a''dc''(qy_{,}-qx_{,});$$

and by adding them together we obtain

$$a\frac{d^2x}{dt^2}+a'\frac{d^2y}{dt^2}+a''\frac{d^2z}{dt^2}=(a^2+a'^2+a''^2)(z_{,}dq-y_{,}dr)+(ab+a'b'$$
$$+a''b'')(x_{,}dr-z_{,}dp)+(ac+a'c'+a''c'')(y_{,}dp-x_{,}dq)+(ada+$$
$$a'da'+a''da'')(qz_{,}-ry_{,})+(adb+a'db'+a''db'')(rx_{,}-pz_{,})+$$
$$(adc+a'dc'+a''dc'')py_{,}-qx_{,});$$

hence as $a^2+a'^2+a''^2=1$, $ab+a'b'+a''b''=0$, $ac+a'c'+a''c''=0$, $ada+a'da'+a''da''=0$, $adb+a'db'+a''db''=-rdt$, $adc+a'dc'+a''dc''=qdt$, the value of $p_{,}dt$ is evidently that given in the text.

(*h*) If the value of $a$ given in No. 378 be differenced with respect to $\phi$, the resulting quantity will be equal to the value of $b$ given in the same number, and the same obtains for the values of $\frac{da'}{d\phi}$, $\frac{da''}{d\varphi}$, which are respectively equal to $b'$ and $b''$.

(*i*) Since by No. 378 we have

$dc = d\theta \cos\theta \sin\psi + d\psi \sin\theta \cos\psi$, $b = \cos\theta \sin\psi \cos\phi - \cos\psi \sin\phi$ $-bdc = -d\theta.(\cos^2\theta \sin^2\psi \cos\phi - \cos\theta \sin\psi \cos\psi \sin\phi) - d\psi(\sin\theta \cos\theta \sin\psi \cos\psi \cos\phi - \sin\theta \cos^2\psi \sin\phi)$, $dc' = d\theta \cos\theta \cos\psi - d\psi \sin\theta . \sin\psi$, $b' = \cos\theta \cos\psi \cos\varphi + \sin\psi \sin\phi$, $-b'dc' = -d\theta.(\cos^2\theta \cos^2\psi \cos\phi + \cos\theta \sin\psi \cos\psi \sin\phi) + d\psi(\sin\theta \cos\theta \sin\psi \cos\psi \cos\phi + \sin\theta \sin^2\psi \sin\phi)$, $dc'' = -d\theta . \sin\theta$, $b'' = -\sin\theta \cos\phi \because -b''dc'' = -d\theta . \sin^2\theta . \cos\phi$.

Hence we obtain by adding these quantities together, and obliterating those that destroy each other,

$$pdt = -bdc - b'dc' - b''dc'' = -d\theta . \cos\phi + d\psi . \sin\theta . \sin\phi;$$

the values of $qdt$ and $rdt$ may be obtained in the same manner.

(*k*) For example, if these three equations be multiplied by $a$, $b$, $c$ respectively, we obtain

$$a^2dc + aa'dc' + aa''dc'' = aqdt$$
$$b^2dc + bb'dc' + bb''dc'' = -bpdt$$
$$c^2dc + cc'dc' + cc''dc'' = 0$$
$$\because (a^2 + a'^2 + a''^2)dc + (aa' + bb' + cc')dc' + (aa'' + bb'' + cc'')dc'' = dc = (aq - bp)\,dt.$$

(*l*) If in equations (8), $da$, $db$, $dc$, and their values, be respectively multiplied by $p$, $q$, $r$, and then added together, the result is evidently equal to cipher; and it is evident from the equation $pda + qdb + rdt = 0$, that when $p$, $q$, $r$ are constant quantities, the numerators of equations (4) are constant, for their differentials are cipher.

(*m*) The first of these equations is evidently equal to

$$\int(x_{,}Y_{,} - y_{,}X_{,})\,dm + \int(y_{,}p_{,} - q_{,}x_{,})\,dm = 0,$$

substituting for $p_{,}$ and $q_{,}$ their values given in No. 408, we obtain

$$y_{,}p_{,}dt = z_{,}y_{,}dq - y_{,}^2dr + (py_{,}^2 - qx_{,}y_{,})qdt + (py_{,}z_{,} - rx_{,}y_{,})rdt$$
$$x_{,}q_{,}dt = x_{,}^2dr - z_{,}x_{,}dp + (qx_{,}z_{,} - rx_{,}y_{,})rdt + (qx_{,}^2 - px_{,}y_{,})pdt$$

$\because$ as $x_{,}$, $y_{,}$, $z_{,}$ are supposed to be principal axes, the value of

$$\int(y_{,}p_{,} - x_{,}q_{,})dm = -\int(x_{,}^2 + y_{,}^2)\frac{dr}{dt} + \int(y_{,}^2 - x_{,}^2)pqdm.$$

(*n*) $X_1 = ga''$, $Y_1 = gb'' \because \int(x_{,}Y_{,} - y_{,}X_{,})dm = R = g[b''\int x_{,}dm - a''\int y_{,}dm] = g(b''\alpha - a''\beta)M$.

f (*o*) If $A = B$, then the value of $dt$ is

$$dt = \frac{\mp ACdr}{k^2 - Ah + (A - C)Cr^2},$$

the integral of which is obtained below; if $A = C$, then the value of $dt$ is

$$dt = \frac{\pm\sqrt{B}.A^{\frac{3}{2}}dr}{(k^2 - Bh + (B - A).Ar^2)^{\frac{1}{2}}\sqrt{Ah - k^2}}$$

and a corresponding expression results when $B = C$. When $k^2 = Ah$, then the value of $dt$ is of the form

$$dt = \frac{\mp\sqrt{ABC}.dr}{\sqrt{(A-C).r^2.[(A-B)h + (B-C).Cr^2]}};$$

it is easy to show, that in all these cases, the integral of the value of $dt$ may be obtained in a finite form; for in the first case, that is when $A = B$, it is evident that the two factors in the denominator of the general value of $dt$ are equal, but affected with contrary signs, consequently the value of $dt = \frac{\mp AC\,dr}{k^2 - Ah + (A - C)Cr^2}$, $\because$ if we make $\frac{AC}{(A-C).C} = m$ and $k^2 - Ah = n^2(A-C).C$, it becomes $\frac{\mp m dr}{n^2 + r^2}$, the integral of which is $\frac{m}{n}$ arc tang $= \frac{r}{n}$.

When $A = C$, by putting $\sqrt{B.A^{\frac{3}{2}}} \div \sqrt{Ah - k^2} \times \sqrt{A(B-A)} = m$ and $\frac{k^2 - Bh}{A(B-A)} = n^2(Ah - k^2)$, the value of $dt$ becomes equal to $\frac{mdr}{\sqrt{n^2 + r^2}}$ and if $m$ and $n$ are real, the integral of this differential is equal to $t = m.\log.(r + \sqrt{n^2 + r^2})$; if the coefficient of $r^2$ is negative, then the expression for $dt$ may be reduced to $dt = \frac{mdr}{\sqrt{n^2 - r^2}}$, the integral of which is $m.\text{arc}\sin = \frac{r}{n} + C$; it may happen that $n$ or $m$ are negative, in which case the integrals are also easily computed. In the last case, i. e. when $k^2 = Ah$ or $Bh$, then the terms independent of $r^2$ in one of the factors of the denominator of equation (g) vanishes, so that the term $r^2$ comes from under the radical, and the value of

$$dt = \frac{\mp\sqrt{AB.C}dr}{r\sqrt{(A-C).[(A-B)k + (B-C)Cr^2]}},$$

which is evidently either of the form

$$dt = \frac{2mndr}{r\sqrt{n^2 + r^2}}, \text{ or } dt = \frac{2mndr}{r\sqrt{n^2 - r^2}};$$

the integral in the first case is

$$t = m \log . \frac{\sqrt{n^2 + r^2} - n}{\sqrt{n^2 + r^2} + n}, \text{ or } t = \log \frac{n - \sqrt{n^2 - r^2}}{n + \sqrt{n^2 - r^2}},$$

($p$) In consequence of the three first equations (8), we have $dc = (aq - bp)\,dt$, $db = (cp - ar)dt$, &c., ∵ substituting $dc$ for $(aq - bp)dt$, &c., we obtain

$$(cdr + rdc)\,\text{C} + (bdq + qdb)\,\text{B} + (adp + pda)\,\text{A} = 0.$$

($q$) By taking the squares of equations (h), and then adding them together, we obtain

$$\begin{aligned} &\text{C}^2 r^2 (c^2 + c'^2 + c''^2) + \text{B}^2 q^2 . (b^2 + b'^2 + b''^2) + \text{A}^2 p^2 . (a^2 + a'^2 + a''^2) \\ &\quad + 2\,\text{C}r.\text{B}q.(cb + c'b' + c''b'') + 2\,\text{C}r.\text{A}p.(ca + c'a' + c''a'') \\ &\qquad + 2\,\text{B}q.\text{A}p.(ba + b'a' + b''a'') = b^2 + b'^2 + b''^2, \end{aligned}$$

which as $c^2 + c'^2 + c'' = 1$, $b^2 + b'^2 + b''^2 = 1$, $a^2 + a'^2 + a''^2 = 1$, $cb + c'b' + c''b'' = 0$, $ca + c'a' + c''a'' = 0$, $ba + b'a' + b''a'' = 0$, becomes the expression in text.

($r$) If the first equation (7) be multiplied by $\sin\theta . \sin\phi$, and the second by $\sin\theta . \cos\phi$, there results

$$\sin\theta . \sin\phi\, pdt + \sin\theta . \cos\phi\, qdt = \sin^2\theta . d\psi,$$

i. e. by substituting for $\sin\theta \sin\phi$, $\sin\theta . \cos\phi$, their values given above,

$$-\left(\frac{\text{A}p^2 + \text{B}q^2}{k}\right) dt = \frac{k^2 - \text{C}^2 r^2}{k^2} d\psi.$$

($s$) By neglecting the products $\int x_i y_i dm$, $\int z_i x_i dm$, $\int y_i z_i dm$, this sum is evidently equal to

$$\int (q^2 z_i^2 + r^2 y_i^2) dm + \int (r^2 x_i^2 + p^2 z_i^2) dm + \int (p^2 y_i^2 + q^2 x_i^2) dm,$$

that is,

$$\begin{gathered} q^2 . \int (x_i^2 + z_i^2) dm + r^2 \int (x_i^2 + y_i^2) dm + p^2 \int (y_i^2 + z_i^2) dm = \\ q^2 . \text{B} + r^2 \text{C} + p^2 \text{A}. \end{gathered}$$

($t$) By No. 407 we have $\omega \cos \text{IO}z = a''p + b''q + c''r$, $=$ in this case $\theta$, and by substituting for $a''$, $b''$, $c''$, their respective values $\frac{\text{A}p}{k}$, $\frac{\text{B}q}{k}$, $\frac{\text{C}r}{k}$, we obtain

$$\theta = \frac{\text{A}p^2 + \text{B}q^2 + \text{C}r^2}{k} = \frac{h}{k}.$$

(*u*) By taking the value of $\frac{\beta'}{\beta}$ in each of these equations we obtain

$$\frac{\beta'}{\beta} = \frac{(\text{A}-\text{C})n}{\text{B}n'} = \frac{\text{A}n'}{(\text{B}-\text{C})n} \therefore n'^2 = \frac{n^2.(\text{A}-\text{C})(\text{B}-\text{C})}{\text{AB}}$$

and

$$\beta'^2 : \beta^2 :: (\text{A}-\text{C})\text{A} : (\text{B}-\text{C})\text{C}.$$

(*v*) Since $py_{,}-qx_{,}=0$, No. 404, tang $\zeta = \frac{y_{,}}{x_{,}} = \frac{q}{p}$.

(*x*) When the instantaneous axis coincides with the axis $\text{O}z_{,}$, we have $\sin \text{I}\text{O}z_{,} = 0$, and therefore $p=0$, and $q=0$, consequently we must have also $\beta$, $\beta'$, respectively equal to cipher, and $\therefore \alpha = 0$.

(*y*) This is evident from the known formulæ

$$\sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}, \quad \cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2};$$

in which if $x$ is of the form $x'\sqrt{-1}$, it is evident that each of these values of $\sin x$, $\cos x$, when expressed in a series, are composed of possible quantities.

(*z*) It appears from equations (i) that

$$p = -\frac{\sin\theta\sin\phi.k}{\text{A}}, \quad q = -\frac{\sin\theta\cos\phi.k}{\text{B}}, \quad r \text{ or } n = \frac{\cos\theta.k}{\text{C}};$$

hence by substituting for $p$ and $q$ their values given in equations (2), we shall obtain, by observing that $k = mvf$, equations (3).

(*a'*) The second members of the two first equations (3) become, when $t=0$, $-\alpha.\frac{\text{A}\sqrt{\text{B}.(\text{B}-\text{C})}}{mvf}\sin\gamma$, $\quad -\alpha.\frac{\text{B}.\sqrt{\text{A}.(\text{A}-\text{C})}}{mvf}\cos\gamma$. by means of which $\alpha$ and $\gamma$ can be determined when $\theta$ and $\phi$ are known at the commencement of the motion, and as $\theta$ is very small at the commencement, it must, by what has been established above, continue very small.

(*b'*) $\text{Cos}\,\theta = 1 - \frac{\theta^2}{2} + \&\text{c.}$, now as $\sin\theta$ and $\therefore \theta$ is a very small angle, we may neglect $\theta^2$, $\therefore$ we have in the third equation (3) $1 = \frac{\text{C}n}{mvf}$, and $\text{C}n = mvf$.

(*c'*) If B be supposed equal to A in each of the two first equations (3), there results, by adding their squares together,

$$\sin^2\varepsilon = \sin^2\theta = \frac{\alpha^2.\text{A}^3(\text{A}-\text{C})}{\mu^2 v^2 f^2}.$$

Now, $\omega = \sqrt{p^2+q^2+r^2} =$ (by substituting for $p$, $q$, $r$, their values given by equation (2)), $\sqrt{\alpha^2.\text{A}\,(\text{A}-\text{C})+n^2}$, $=$ (as $\alpha^2\text{A}.(\text{A}-\text{C}) = \frac{\mu^2 v^2 f^2 \sin^2\varepsilon}{\text{A}^2}$), the expression given in the text.

($d'$) It is evident from the value of $\delta$ given in No. 420, that when $\text{A}=\text{B}$, $1-\delta = \frac{\text{C}}{\text{A}}$, and as $\frac{n}{\cos\varepsilon} = \frac{\mu v f}{\text{C}}$, when these values are substituted in the expression for $\psi$, we shall obtain the value given in the text.

($e'$) By multiplying the two first equations (7) by $\sin\theta.\cos\phi$, $\sin\theta.\sin\phi$, respectively, we obtain

$$pdt.\sin\theta.\cos\phi = \sin\phi.\cos\phi\,\sin^2\theta d\psi - \sin\theta.\cos^2\phi d\theta$$
$$qdt.\sin\theta.\sin\phi = \sin\phi.\cos\phi\,\sin^2\theta d\psi + \sin\theta.\sin^2\phi.d\theta;$$

now it is evident that if we divide these equations by $dt$, and then take the second from the first, we shall have the value of $p\sin\theta.\cos\phi - q\sin\theta.\sin\phi$ given in the text. In like manner if the two first equations be multiplied by $\sin\theta\sin\phi$, $\sin\theta.\cos\phi$ respectively, we shall obtain

$$pdt.\sin\theta\sin\phi = \sin^2\phi.\sin^2\theta d\psi - \sin\phi\cos\phi.\sin\theta d\theta,$$
$$qdt.\sin\theta.\cos\phi = \cos^2\phi\sin^2\theta d\psi + \sin\phi.\cos\phi\sin\theta d\theta;$$

if these equations be divided by $dt$, and then added together, we shall obtain the value of $p\sin\theta\sin\phi + q\sin\theta\cos\phi$ given in the text. Also, as

$$p^2dt^2 = \sin^2\phi\sin^2\theta d\psi^2 - 2\sin\phi.\cos\phi\sin\theta.d\psi d\theta + \cos^2\phi d\theta^2$$
$$q^2dt^2 = \cos^2\phi\sin^2\theta d\psi^2 + 2\sin\phi.\cos\phi.\sin\theta d\psi d\theta + \sin^2\phi d\theta^2;$$

there results by adding these equations together, and dividing by $dt^2$,

$$p^2+q^2 = \sin^2\theta\frac{d\psi^2}{dt^2} + \frac{d\theta^2}{dt^2}.$$

($f'$) In the general expression of E of No. 281, $\varepsilon=\alpha$, $\varepsilon'=\alpha+90$, $\varepsilon''=90$, consequently, the value of E or $l$ in this case is

$$l = \text{c}n\cos\alpha + \mu\cos(\alpha+90) + m.\cos 90 = \text{c}n.\cos\alpha - \mu\sin\alpha.$$

($g'$) As $t$ is the independent variable, if this multiplication be performed, we shall have

$$\frac{\gamma^2 \sin^2\theta d\psi}{2} = \tfrac{1}{2}\gamma k \sin\alpha dt\,;$$

now as $\gamma \sin\theta$ is the horizontal projection of the radius vector GO, $\gamma \sin\theta d\psi$ is equal to the arc described by this projection in $dt$, $\therefore \frac{\gamma^2 \sin^2\theta d\psi}{2}$ is the area described in this time.

($h'$) From the first equation (6), we obtain

$$\sin^2\theta \frac{d\psi^2}{dt^2} = \left(\frac{\text{C}n}{\text{A}}\right)^2 \left(\frac{\cos\theta - \cos\alpha}{\sin\theta}\right)^2,$$

therefore, by substituting this value in the second equation (6), there results

$$\sin^2\theta \frac{d\theta^2}{dt^2} = \frac{2\text{M}g\gamma}{\text{A}}.\sin^2\theta.(\cos\theta - \cos\alpha) - \left(\frac{\text{C}n}{\text{A}}\right)^2.(\cos\theta - \cos\alpha)^2,$$

which, by substituting $\frac{1}{\lambda}$ for $\frac{\text{M}\gamma}{\text{A}}$, and $\frac{4g\beta^2}{\lambda}$ for $\frac{\text{C}^2 n^2}{\text{A}^2}$, and then concinnating, becomes the expression in the text.

($i'$) By substituting for $\sin\theta$, $\cos\theta$, $\cos\alpha$, their values expressed in series, and restricting the series, as is indicated in the text, equation (7) becomes

$$\theta^2 \frac{d\theta^2}{dt^2} = \frac{2g}{\lambda}\left[\theta^2 - 2\beta^2\left(1 - \frac{\theta^2}{2} - 1 + \frac{\alpha^2}{2}\right)\right]\left(1 - \frac{\theta^2}{2} - 1 + \frac{\alpha^2}{2}\right)$$

$$= \frac{2g}{\lambda}\left[\theta^2 - \beta^2.(\alpha^2 - \theta^2)\right]\left(\frac{\alpha^2}{2} - \frac{\theta^2}{2}\right),$$

which, by concinnating, becomes the expression in the text; and it is evident from an inspection of this expression, that if $\theta$ was greater than $\alpha$, $\theta^2 \frac{d\theta^2}{dt^2}$ would be negative, if $\theta$ was equal or less than $\frac{\beta\alpha}{\sqrt{1+\beta^2}}$, the corresponding value of $\theta^2 \frac{d\theta^2}{dt^2}$ would be cipher or negative. In order to resolve the first equation (9) for $dt$, we should multiply both sides by $dt^2$, by this means the first member becomes $\theta^2 d\theta^2$, and the second member is the factor by which $dt^2$ is multiplied.

($k'$) By substituting for $d\theta$ and $\theta$ in the value of $dt$, we obtain

$$\sqrt{\frac{g}{\lambda}}\,dt = \frac{\alpha^2 \sin u \cos u\, du}{\sqrt{[\alpha^2 \sin^2 u + \beta^2\alpha^2 \sin^2 u - \beta^2\alpha^2]\,(\alpha^2 - \alpha^2 \sin^2 u}}$$

$$= \frac{\alpha^2 \sin u \cos u\, du}{\sqrt{[\alpha^2 \sin^2 u - \beta^2\alpha^2.\cos^2 u]\,\alpha^2 \cos^2 u}} = \frac{\sin u du}{\sqrt{\sin^2 u - \beta^2.\cos^2 u}}$$

$$=\frac{\sin u du}{\sqrt{1-(1+\beta^2)\cos^2 u}}=(\text{by making } \cos u=-x)\frac{dx}{\sqrt{1-(1+\beta^2).x^2}};$$

i. e., by making $x=\frac{y}{\sqrt{1+\beta^2}}$, $\frac{1}{\sqrt{1+\beta^2}}.\frac{dy}{\sqrt{1-y^2}}$, of which the integral is $\frac{1}{\sqrt{1+\beta^2}}$ arc sin $= y + c$; i. e. $\frac{1}{\sqrt{1+\beta^2}}$. arc sin $=$ $\sqrt{1+\beta^2}.\cos u + c$. It is to be remarked, that when $dt$ is expressed in a function of $\theta$, that the signs with which the numerator is affected are the opposite of those by which it is affected when $dt$ is expressed in a function of $u$, because the $d\cos.u=-\sin u du$.

($l'$) From the preceding equation we obtain,

$$\sin.\left(t\sqrt{\frac{g}{\lambda}}.\sqrt{1+\beta^2}-i\pi\right)=\frac{\sqrt{1+\beta^2}}{\alpha}\sqrt{\alpha^2-\theta^2},$$

i. e.

$$\sin t\sqrt{\frac{g(1+\beta^2)}{\lambda}}=\sqrt{\frac{1+\beta^2}{\alpha^2}}\sqrt{(\alpha^2-\theta^2)},$$

hence we obtain the value of $\theta^2$; now when $\theta=\alpha$, $t=0$, and when $\theta=\frac{\beta\alpha}{\sqrt{1+\beta^2}}$, $t=\text{T}$, therefore we have

$$\frac{\beta^2\alpha^2}{1+\beta^2}=\alpha^2-\frac{\alpha^2}{1+\beta^2}\sin^2\text{T}\sqrt{\frac{g(1+\beta^2)}{\lambda}},$$

$$\therefore \sin^2\text{T}\sqrt{\frac{g(1+\beta^2)}{\lambda}}=1, \text{ and } \text{T}.\sqrt{\frac{g(1+\beta^2)}{\lambda}}=\frac{\pi}{2},$$

$\therefore$ T $=$ the expression given in the text.

($m'$) It is evident that the value of $\theta^2$ may be made to assume the form

$$\theta^2=\frac{\alpha^2\beta^2+\alpha^2.\left(1-\sin^2 t\sqrt{\frac{g(1+\beta^2)}{\lambda}}\right)}{1+\beta^2}$$

$$=\frac{\alpha^2\beta^2+\alpha^2.\cos^2 t\sqrt{\frac{g(1+\beta^2)}{\lambda}}}{1+\beta^2};$$

in like manner we have

$$\alpha^2-\theta^2=\frac{\alpha^2}{1+\beta^2}\sin^2 t\sqrt{\frac{g(1+\beta^2)}{\lambda}},$$

therefore, if these values of $\theta^2$ and $\alpha^2-\theta^2$ be substituted in the second equation (9), we obtain, by multiplying it by $dt$,

$$\frac{\alpha^2\beta^2+\alpha^2\cos^2 t\sqrt{\frac{g(1+\beta^2)}{\lambda}}}{1+\beta^2}d\psi=\frac{\alpha^2}{1+\beta^2}.\sin^2 t\sqrt{\frac{g(1+\beta^2)}{\lambda}}.\beta\sqrt{\frac{g}{\lambda}}.dt;$$

consequently,

$$\left(\beta^2+\cos^2 t\sqrt{\frac{g.(1+\beta^2)}{\lambda}}\right)d\psi=\left(1-\cos^2 t\sqrt{\frac{g(1+\beta^2)}{\lambda}}\right).\beta\sqrt{\frac{g}{\lambda}}dt,$$

$$\because d\psi=\frac{\beta\sqrt{\frac{g}{\lambda}}.dt}{\beta^2+\cos^2 t\sqrt{\frac{g(1+\beta^2)}{\lambda}}}-\frac{\cos^2 t\sqrt{\frac{g.(1+\beta^2)}{\lambda}}.\beta\sqrt{\frac{g}{\lambda}}dt}{\beta^2+\cos^2 t\sqrt{\frac{g(1+\beta^2)}{\lambda}}};$$

which is evidently reducible to the expression in the text.

($n'$) In order to perform this integration, let us make

$$\beta\sqrt{\frac{g}{\lambda}}(1+\beta^2)=m, \text{ and } \sqrt{\frac{g(1+\beta^2)}{\lambda}}=n,$$

then we shall have

$$d\psi=\frac{mdt}{\beta^2+\cos^2 tn}-\beta\sqrt{\frac{g}{\lambda}}.dt;$$

now if we assume tang $tn=x$, then

$$n.dt=\frac{dx}{1+x^2}, \text{ and } d\psi=\frac{m}{n}.\frac{dx}{1+x^2}, \div d \text{ by } \beta^2+\frac{1}{1+x^2}$$
$$=\frac{m}{n}.\frac{dx}{1+\beta^2+\beta^2x^2},$$

the integral of this last quantity is equal to

$$\frac{m}{n.\beta.\sqrt{1+\beta^2}}.\text{arc tang}=\frac{\beta x}{\sqrt{1+\beta^2}}=\frac{m}{n.\beta.\sqrt{1+\beta^2}}\text{ arc tang}$$
$$=\frac{\beta\,\text{tang}\,tn}{\sqrt{1+\beta^2}};$$

and if there be substituted for $m$ and $n$ their values, it is evident that we shall obtain for $\psi$ the expression in text. Now if in this expression of $\psi$, the value of $t$ at the end of the first, second, third, &c., intervals of time, which are (as appears from what is already established) equal to $\frac{1}{2}\pi\sqrt{\frac{\lambda}{g(1+\beta^2)}}$, $\pi\sqrt{\frac{\lambda}{g(1+\beta)^2}}$, &c., be substituted, we shall obtain the expressions given in the text; and it is by subducting the value of $\psi$ from the preceding one, and this again from the first, that the arcs traversed by the node N during the successive

intervals T are obtained, and it appears that they are all equal, and that their common value $= \frac{1}{2}\pi . \left(\frac{\sqrt{1+\beta^2}-\beta}{\sqrt{1+\beta^2}}\right)$, which by multiplying both numerator and denominator by $\beta + \sqrt{1+\beta^2}$ becomes the expression in the text.

($o'$) Since in this case $\cos u = 1 - \frac{u^2}{2}$, $\sin u = u$, we have

$$\sin\theta = \sin\alpha\cos u - \cos\alpha\sin u = \sin\alpha\left(1 - \frac{u^2}{2}\right) - \cos\alpha . u.$$

$\sin^2\theta = \sin^2\alpha\ (1-u^2) - 2\sin\alpha\cos\alpha . u + \cos^2\alpha . u^2$; $= \sin^2\alpha + (\cos^2\alpha - \sin^2\alpha)u^2 - 2\sin\alpha\ \cos\alpha . u = \sin^2\alpha + \cos 2\alpha . u^2 - \sin 2\alpha . u$; $\cos\theta = \cos\alpha\cos u + \sin\alpha\sin u = \cos\alpha\left(1 - \frac{u^2}{2}\right) + \sin\alpha . u$, $\because \cos\theta - \cos\alpha = \sin\alpha . u - \frac{u^2}{2}\cos\alpha$.

($p'$) Now if these values of $\frac{d\theta}{dt}$, $\sin\theta$, $\cos\theta$, be substituted in equation (7), there results

$$(\sin^2\alpha - u\sin 2\alpha + u^2\cos 2\alpha)\frac{du^2}{dt^2} = \frac{2g}{\lambda}\left[(\sin^2\alpha - u\sin 2\alpha + u^2\cos 2\alpha).\right.$$

$$\left.\left(u\sin\alpha - \frac{u^2}{2}\cos 2\alpha\right) - 2\beta^2\left(u\sin\alpha - \frac{u^2}{2}\cos 2\alpha\right)^2\right],$$

and as powers of $u$ higher than the second are neglected, the second member of this equation may be reduced to the form

$$\frac{2g}{\lambda}u\sin\alpha\ (\sin^2\alpha - u\sin 2\alpha) - \frac{2g}{\lambda}\left(\frac{u^2\cos\alpha}{2}\sin^2\alpha + 2\beta^2u^2\sin^2\alpha\right),$$

consequently, we shall have

$$\frac{\lambda}{g}.(\sin^2\alpha - u\sin 2\alpha + u^2\cos 2\alpha)\frac{du^2}{dt^2} = 2u\sin\alpha.(\sin^2\alpha - u\sin 2\alpha) -$$

$$u^2\sin^2\alpha\,(\cos\alpha + 4\beta^2);$$

now if both members of this equation be divided by $\sin^2\alpha - u\sin 2\alpha + u^2\cos 2\alpha$, it is evident that the first member will be $\frac{\lambda}{g}\frac{du^2}{dt^2}$, and we must divide the first term of the second member by $\sin^2\alpha - u\sin 2\alpha$, and we must divide the second term by $\sin^2\alpha$, in order that no powers higher than the second may occur; when these

operations are performed, we shall have the expression of $\frac{\lambda}{g}.\frac{du^2}{dt^2}$ given in the text; if the square root be taken of each side of this equation, and if then it be multiplied by $dt$, we shall obtain the value of $dt$, given in the text.

($q'$) This expression may be made to assume the form

$$\sqrt{\frac{g}{\lambda}}\,dt = \frac{du}{\sqrt{\cos\alpha + 4\beta^2}\ \sqrt{\frac{2\sin\alpha}{\cos\alpha + 4\beta^2}u - u^2}},$$

and if we make

$$z = u - \frac{\sin\alpha}{\cos\alpha + \beta^2}, \text{ and } \because u^2 - \frac{2\sin\alpha}{\cos\alpha + 4\beta^2}.u + \frac{\sin^2\alpha}{(\cos\alpha + 4\beta)^2} = z^2,$$

the value of $\sqrt{\frac{g}{\lambda}}\,dt$ will become

$$\sqrt{\frac{g}{\lambda}}\,dt = \frac{dz}{\sqrt{\cos\alpha + 4\beta^2}\left(\frac{\sin^2\alpha}{(\cos\alpha + 4\beta)^2} - z^2\right)}$$

and if we make $z = \frac{\sin\alpha}{\cos\alpha + 4\beta^2}y$, we shall have

$$dt.\sqrt{\frac{g}{\lambda}(\cos\alpha + 4\beta^2)} = \frac{dy}{\sqrt{1-y^2}}, \ \because\ t\sqrt{\frac{g}{\lambda}(\cos\alpha + 4\beta^2)} = \text{arc sin}$$

$$= y = \text{arc sin} = \frac{\cos\alpha + 4\beta^2}{\sin\alpha}z = \text{arc sin}\left(= \frac{\cos\alpha + 4\beta^2}{\sin\alpha}.u - 1\right),$$

and when we integrate for the cosine, we must take the opposite signs.

($r'$) When $\cos\alpha$ is neglected, the value becomes

$$u = \frac{\sin\alpha}{4\beta^2}\left(1 - \cos 2\beta t\sqrt{\frac{g}{\lambda}}\right), \text{and as } \cos 2\beta t\sqrt{\frac{g}{\lambda}} = 1 - 2\sin^2\beta t\sqrt{\frac{g}{\lambda}},$$

we have $u =$ the expression in the text.

($t'$) By making the substitutions indicated in the text, we obtain

$$(\sin^2\alpha - u\sin 2\alpha)\frac{d\psi}{dt} = 2\beta\sqrt{\frac{g}{\lambda}}\,u\sin\alpha,$$

and by dividing both sides of this equation by the factor of $\frac{d\psi}{dt}$, and neglecting the square of $u$, we obtain

$$\frac{d\psi}{dt} = 2\beta\sqrt{\frac{g}{\lambda}}\cdot\frac{u}{\sin\alpha},$$

now if in this equation there be substituted the value of $u$, given above, we obtain the expression of $\frac{d\psi}{dt}$ given in the text; and the value of $d\psi$ is obtained by multiplying both sides of this equation by $dt$, and noting that $\sin^2\beta t\sqrt{\frac{g}{\lambda}} = \frac{1}{2} - \frac{1}{2}\cos 2\beta t\sqrt{\frac{g}{\lambda}}$.

## CHAPTER V.

(*a*) When the same face is always presented to the central body, the angle which the moveable describes about its axis of rotation must be equal to the angle described about the central body; in addition to the condition of equality between the motions of rotation and revolution, it is also necessary that the axis of rotation should be perpendicular to the plane in which the body moves.

(*b*) As $p^2 + q^2 + r^2 = \omega^2$, we have by equations (3) of No. 405, $\cos \text{IO}x_1$ ($=$ in this case $\cos\alpha$) $= \frac{p}{\omega}$, and by equations (1) of No. 418, we have, (as $\cos zOx_1 =$ in this case $\cos a$), $p = (\omega\cos\alpha) = \frac{k\cos a}{\text{A}}$; now it is evident from the equations $\text{A}\omega\cos\alpha = k\cos a$, $\text{B}\omega\cos\beta = k\cos b$, $\text{C}\omega\cos\gamma = k\cos c$, that when $a = 90°$, $b = 90°$, $c = 0$, then $\alpha = 90°$, $\beta = 90$, $\gamma = 0$, in which case the value of $\omega$ is $\frac{k}{\text{C}}$, in which if we substitute $\text{MV}f$ for $k$, we obtain $\omega = \frac{\text{MV}f}{\text{C}}$.

(*c*) In fact, if the body when it commences to move, satisfies the conditions here specified, either all the three principal moments A, B, C, must be equal, or the perpendicular to the section HEK must coincide with one of the three principal axes, and, therefore, the other two must exist in the plane HEK, i. e., we must have $\alpha = a = 90$, $\beta = b = 90$, $\gamma = c = 0$.

(*d*) Since the instantaneous axis, if it does not exactly coincide with the axis of figure, deviates very little from this axis, which is that of the *greatest* moment of inertia, the constant $\delta$ of No. 420 must be real, and

the duration must depend on this quantity, i.e. on $\frac{(A-C).(B-C)}{AB} = \delta^2$; or on the differences of the moments of inertia of the earth; but, as no variations have ever been detected by the most accurate observations in the geographical latitudes of places on the earth's surface, it follows that the oscillations of the terrestrial axis, which depend on the *initial* state of motion, have long since been annihilated, so that whatever variations now exist, must have a permanent cause. See *Journal de l'Ecole Polytechnique*, Tome VIII., &c.

(*d*) As the arcs described in the several successive days constitute an arithmetical progression, the arc described in $t$ days is equal $[n + n(1 + (t-1)\alpha)]\frac{t}{2} = nt + \frac{1}{2}\alpha nt(t-1)$.

(*e*) As the length of the year is very nearly equal to 365,25, the number of days contained in a century is 36525, consequently, the number of days contained in $i$ centuries, i. e., in $t$ days, is equal to $(36525)\,i$, in like manner as $\alpha$ is the diminution of each day, the annual diminution is $(365{,}25)\,\alpha$, and the secular diminution is $(36525)\,\alpha$, in the same manner it may be shown, that the values of $m$ and $m'$ are respectively $(36525)\,n$, $(36525)\,n'$.

(*f*) By substituting in equation (1), for $\alpha, n, n', t$, their respective values, it becomes $\delta = \frac{1}{2}\frac{\beta}{(36525)^2}\cdot(m-m')i^2.(36525)^2 = \frac{1}{2}\beta.(m-m')i^2$; now in the most ancient eclipse recorded by the Chaldeans, 720 B.C., i. e. 2532 years ago, we have by hypothesis $\beta i = 36525 . 25{,}32\alpha = 0{,}0000001$; and therefore $\delta = \frac{1}{2}. 25{,}32.0{,}0000001.445210^\circ = 34'.qp$.

(*g*) There are three cases considered; 1st. When the axis of rotation is vertical, in which case the bullet deviates from the vertical plane passing through the axis of rotation, and consequently, describes a curve of double curvature. 2ndly. When the axis of rotation is horizontal and perpendicular to the direction in which the bullet moves, in this case, the force R acts in the vertical plane which is perpendicular to the axis of rotation; consequently, the curve described by the bullet will be a plane curve, but as the effect of the friction will be either to increase or diminish the weight, the range will be affected; if the velocity of the rotation be such as to render the resistance arising from the friction greater than the weight, the curve will be convex to the horizon, when the direction of the

motion of rotation is *towards* the horizon, consequently, in this case the curve described will have a point of contrary flexure. The third case is that in which the bullet turns about the diameter which coincides with the direction in which it moves, or with the line of flight, in which, as is stated in the text, the resistance has no effect; this advantage is secured by rifle-barrelled guns, whose effect is to render the axis of rotation of the bullet coincident with the line of flight.

## CHAPTER VI.

(*a*) By substituting for $x, y, z$ their values given in equation (6), there results

$$x\cos\lambda = x_{,}.\cos\lambda + \alpha a.\cos\lambda + \beta b\cos\lambda + \gamma c\cos\lambda,$$
$$y\cos\mu = y_{,}\cos\mu + \alpha.a'\cos\mu + \beta.b'\cos\mu + \gamma c'\cos\mu,$$
$$z.\cos\nu = z_{,}\cos\nu + \alpha.a''.\cos\nu + \beta.b''\cos\nu + \gamma c''\cos\nu,$$

consequently

$$x\cos\lambda + y\cos\mu + z\cos\nu = x_{,}\cos\lambda + y_{,}\cos\mu + z_{,}\cos\nu + \alpha.(a\cos\lambda + a'\cos\mu + a''\cos\nu) + \beta.(b.\cos\lambda + b'\cos\mu + b''\cos\nu) + \gamma.(c.\cos\lambda + c'\cos\mu + c''\cos\nu),$$

which, by putting $\cos\lambda'$, $\cos\mu'$, $\cos\nu'$, for their values furnished by equation (4), is evidently reducible to equation (7).

(*b*) Since when the moveable is terminated by a point that touches the given plane, the coordinates $\alpha$, $\beta$, $\gamma$ are constant, there are only ten unknown quantities to be determined, which can be obtained by means of the nine equations (1), (2), (3), and equation (7).

(*c*) When the given plane is fixed and horizontal, and is taken to be that of the axes of the coordinates $x$ and $y$, as we have then also, $\lambda = 90$, $\mu = 90$, also $z = 0$, we must have $\zeta = 0$; and as in this case we have also $\text{M}\frac{d^2x}{dt^2}$, $\text{M}\frac{d^2y}{dt^2}$, respectively equal to cipher, there results $\text{M}\frac{dx}{dt}$, $\text{M}\frac{dy}{dt}$ equal to constant quantities, consequently, the horizontal motion will be uniform and rectilinear, and the velocity must depend on the horizontal percussion that the moveable experiences at the origin of the motion.

(*d*) As $\text{A} = \text{B}$, $c'' = \cos\theta$, and $r = n$, the value of $l$ becomes,

by substituting $-\sin^2\theta\frac{d\psi}{dt}$ for $a''p+b''q$, the expression in text, and in the value of $h$, as $\frac{dz_1}{dt}=\gamma\sin\theta\frac{d\theta}{dt}$, it is equal to

$$\text{C}n^2+\text{A}.\left(\sin^2\theta\frac{d\psi^2}{dt^2}+\frac{d\theta^2}{dt^2}\right)+\text{M}.\left(\gamma^2\sin^2\theta\frac{d\theta^2}{dt^2}-2\gamma g\cos\theta\right).$$

(*e*) By substituting for $a, a', a'', b, b', b''$, their values given in No. 378, and observing that $\phi=nt$, we obtain

$a\cos\lambda+a'\cos\mu+a''\cos\nu=\cos\lambda.\cos\theta.\sin\psi.\sin nt+\cos\lambda.\cos\psi.\cos nt+\cos\mu.\cos\theta\cos\psi\sin.nt-\cos\mu.\sin\psi.\cos nt-\cos\nu.\sin\theta\sin nt$,

which becomes, by putting P and Q in place of their values $=\text{P}\sin.nt+\text{Q}.\cos nt$.

(*f*) Since the quantities P, Q, R, by hypothesis, vary very slowly, when the coefficient of $\sin nt$ contains a term which has $\cos mt$ for a factor, $m$ being a very small fraction of $n$, even in this case it may be rejected in the first approximation, for, as is stated in the text, the product of $\sin nt.\cos mt=\sin\left(\frac{n+m}{2}\right)t+\sin\left(\frac{n-m}{2}\right)t$.

(*g*) If in these equations we make $\frac{(\text{A}-\text{C})}{\text{A}}n=m$, $\frac{\text{R}\gamma}{\text{A}}(\text{P}\sin nt+\text{Q}\cos nt)=\text{T}$, $\frac{\text{R}\gamma}{\text{A}}(\text{Q}\sin nt-\text{P}\cos nt)=\text{T}'$, then they will become

$$dq+mpdt=\text{T}dt,\quad dp-mqdt=\text{T}'dt;$$

now if the second of these equations multiplied by $\theta_1$ be added to the first, we shall have

$$dq+\theta_1dp+m(p-\theta_1q)dt=(\text{T}+\theta_1\text{T}')dt,$$

let $q+\theta_1p=z$, and $\because$ $q=z-\theta_1p$, and $dq+\theta_1dp=dz-pd\theta_1$, by substituting this last quantity for $dq+\theta_1dp$, and observing that $\theta_1q=\theta_1z-\theta_1^2p$, we shall have

$$dz-pd\theta_1+mp.(1+\theta_1^2)dt-m\theta_1zdt=(\text{T}+\theta_1\text{T}')dt,$$

if the terms multiplied by $p$ be taken from this equation, there will remain an equation of the first degree between the two variables $z$ and $\theta_1$, from which there can be obtained a primitive equation between these variables.—See *Lacroix*, Tom. II. No. 562. Now this can be effected by making

$$d\theta_1-m.(1+\theta_1^2)dt=0,$$

this equation will enable us to determine the factor $\theta_1$, which in general depends on $t$; however in the present case, as $m$ is constant, this equation is satisfied by making $d\theta_1 = 0$, $m(1+\theta_1^2) = 0$, by means of this last, we obtain $\theta_1 = \pm\sqrt{-1}$, consequently, by substituting this value of $\theta_1$, there results

$$dz \mp m\sqrt{-1}.z\,dt = (\text{T} \pm \sqrt{-1}\,\text{T}')dt;$$

hence in the case of the upper signs we obtain by substituting their values for T, T′,

$$dz' - m\sqrt{-1}\,z'dt = \frac{\text{R}\gamma}{\text{A}}[\text{P}\sin nt + \text{Q}\cos nt + \text{Q}\sqrt{-1}.\sin nt -$$
$$\text{P}\sqrt{-1}\cos nt]dt$$
$$= \frac{\text{R}\gamma}{\text{A}}[\text{P}.(\sin nt - \sqrt{-1}.\cos nt) + \text{Q}.(\cos nt + \sqrt{-1}.\sin nt)]dt,$$

∵ by substituting their exponential values, there results

$$dz' - m\sqrt{-1}\,z'dt = \frac{\text{R}\gamma}{\text{A}}[e^{nt\sqrt{-1}}.\,dt.\,(\text{Q} - \sqrt{-1}.\text{P})],$$

consequently, by integrating, we obtain (Lacroix, No. 562)

$$z' = e^{mt\sqrt{-1}}.\left[\int \frac{\text{R}\gamma}{\text{A}}.e^{-mt\sqrt{-1}}.(\text{Q} - \sqrt{-1}.\text{P})\,e^{nt\sqrt{-1}}.dt + \text{const.})\right];$$

the value of the integral under the sign $\int$ is

$$\frac{\text{R}\gamma}{\text{A}}\frac{[e^{(n-m)t\sqrt{-1}}(\text{Q} - \sqrt{-1}.\text{P}) + \text{const.}]}{(n-m).\sqrt{-1}};$$

now if R′, P′, Q′ denote what R, P, Q become when $t=0$, we shall have

$$\text{const.} = \frac{\text{R}'\gamma}{\text{A}}.\frac{(\sqrt{-1}.\text{P}' - \text{Q}')}{(n-m).\sqrt{-1}};$$

consequently, the value of $z'$ will be equal to $\left(\text{because } \frac{1}{n-m} = \frac{\text{A}}{\text{C}n}\right)$

$$\frac{\text{R}\gamma}{\text{A}}\frac{\text{A}}{\text{C}n}.e^{nt\sqrt{-1}}\frac{(\text{Q} - \sqrt{-1}\,\text{P})}{\sqrt{-1}} + \frac{\text{R}'\gamma}{\text{A}}\frac{\text{A}}{\text{C}n}.e^{mt\sqrt{-1}}\frac{(\sqrt{-1}.\text{P}' - \text{Q}')}{\sqrt{-1}}.$$

In the same manner, if we suppose $\theta_1 = -\sqrt{-1}$, we have

$$dz'' + m\sqrt{-1}z''dt = (\text{T} - \sqrt{-1}\,\text{T}')\,dt$$
$$= \frac{\text{R}\gamma}{\text{A}}[(\text{P}(\sin nt + \sqrt{-1}.\cos nt) + \text{Q}(\cos nt - \sqrt{-1}.\sin nt)]\,dt,$$

$$= \frac{R\gamma}{A} e^{-nt\sqrt{-1}} . dt . (Q + \sqrt{-1}\, P);$$

$$\because z'' = e^{-mt\sqrt{-1}} \left[ \frac{R\gamma}{A} \int e^{mt\sqrt{-1}} . (Q + \sqrt{-1}\, P)\, e^{-nt\sqrt{-1}} . dt + \text{const.} \right],$$

the value of the quantity under the sign $\int$ is

$$\frac{R\gamma}{A} \left[ \frac{e^{(m-n)t\sqrt{-1}} . (Q + \sqrt{-1} . P) + \text{const.}}{(m-n)\sqrt{-1}} \right],$$

if $R'$, $Q'$, $P'$, denote as before the values of $R$, $Q$, $P$, when $t = 0$, we shall have

$$\text{const} = \frac{R'\gamma}{A} . \frac{A}{Cn} . \frac{(Q' + \sqrt{-1} . P')}{\sqrt{-1}},$$

$$\text{and } z'' = -\frac{R\gamma}{A} \frac{A}{Cn} . e^{-nt\sqrt{-1}} . \frac{(Q + \sqrt{-1}\, P)}{\sqrt{-1}}$$

$$+ \frac{R'\gamma}{A} \frac{A}{Cn} e^{-mt\sqrt{-1}} . \frac{(Q' + \sqrt{-1}\, P')}{\sqrt{-1}},$$

$$\because z' + z'' = 2q = \frac{R\gamma}{Cn} . Q . \frac{(e^{nt\sqrt{-1}} - e^{-nt\sqrt{-1}})}{\sqrt{-1}}$$

$$- \frac{R\gamma}{Cn} P . \sqrt{-1} . \frac{(e^{tn\sqrt{-1}} + e^{-nt\sqrt{-1}})}{\sqrt{-1}}$$

$$+ \frac{R'\gamma}{Cn} . Q' . \frac{(e^{-mt\sqrt{-1}} - e^{mt\sqrt{-1}})}{\sqrt{-1}} + \frac{R'\gamma}{Cn} P' \sqrt{-1} \frac{(e^{mt\sqrt{-1}} + e^{-mt\sqrt{-1}})}{\sqrt{-1}},$$

hence we obtain by substituting for the exponentials their values in functions of the sine and cosine, and observing that $m = \frac{(A-C)}{A} n$,

$$z' + z'' = 2q = \frac{R\gamma}{Cn} [Q . 2 \sin nt - P 2 \cos nt] - \frac{R'\gamma}{Cn} . Q' 2 \sin \frac{(A-C)}{A} nt + \frac{R'\gamma}{Cn} . P' 2 . \cos \frac{(A-C)}{A} nt,$$

which by putting D and E for their values, and dividing by two, gives the value of $q$, as it is stated in the text.

In a similar manner we might obtain the value of $p$, for as we have $\theta_1 = \pm \sqrt{-1}$, the original equation becomes

$$\left.\begin{aligned} q + \sqrt{-1} . p &= z' \\ q - \sqrt{-1}\, p &= z'' \end{aligned}\right\} \because 2q = z' + z'', \; 2p = \frac{z' - z''}{\sqrt{-1}},$$

consequently if we take the difference of the preceding values of $z'$ and $z''$, and then divide by $\sqrt{-1}$, we shall obtain the value of $2p$ given in the text.

(*h*) By substituting for $p$ and $q$ their respective values, we obtain

$$p\sin nt = \frac{\gamma}{\mathrm{C}n}\mathrm{R}'\left[\mathrm{P}'\sin\frac{(\mathrm{A}-\mathrm{C})}{\mathrm{A}}nt.\sin nt + \mathrm{Q}'.\cos\frac{(\mathrm{A}-\mathrm{C})}{\mathrm{A}}nt.\sin nt - \mathrm{R}(\mathrm{P}\sin^2 nt + \mathrm{Q}\sin nt.\cos nt)\right]$$

$$q.\cos nt = \frac{\gamma}{\mathrm{C}n}\mathrm{R}'\left[\mathrm{P}'.\cos\frac{(\mathrm{A}-\mathrm{C})}{\mathrm{A}}.nt.\cos nt - \mathrm{Q}'\sin\frac{(\mathrm{A}-\mathrm{C})}{\mathrm{A}}nt.\cos nt - \mathrm{R}(\mathrm{P}\cos^2 nt - \mathrm{Q}.\sin nt.\cos nt)\right]$$

$$\therefore p.\sin nt + q\cos nt = \frac{\gamma}{\mathrm{C}n}\mathrm{R}'.\mathrm{P}'\cos\frac{\mathrm{C}nt}{\mathrm{A}} + \mathrm{Q}.\sin\frac{\mathrm{C}nt}{\mathrm{A}} - \mathrm{R}.\mathrm{P};$$

for

$$\cos\left(\frac{\mathrm{A}-\mathrm{C}}{\mathrm{A}}\right)nt.\cos nt + \sin\left(\frac{\mathrm{A}-\mathrm{C}}{\mathrm{A}}\right)nt.\sin nt = \cos\left(1-\frac{\mathrm{A}-\mathrm{C}}{\mathrm{A}}\right)nt = \cos\frac{\mathrm{C}nt}{\mathrm{A}};$$

$$\cos\left(\frac{\mathrm{A}-\mathrm{C}}{\mathrm{A}}\right)nt.\sin nt - \sin\left(\frac{\mathrm{A}-\mathrm{C}}{\mathrm{A}}\right)nt\cos nt = \sin\left(1-\frac{(\mathrm{A}-\mathrm{C})}{\mathrm{A}}\right)nt = \sin\frac{\mathrm{C}nt}{\mathrm{A}};$$

it may be shown in the same manner that the value of $d\theta$ is that given in the text.

(*i*) By substituting for $\cos\nu$, $\cos\mu$, $\cos\lambda$, in the values of $\mathrm{P}'$ and $\mathrm{Q}'$, we obtain

$$\mathrm{P}' = \cos\theta'.\sin\varepsilon.\sin\varepsilon'\sin\psi' + \cos\theta'.\sin\varepsilon.\cos\varepsilon'\cos\psi' - \cos\varepsilon.\sin\theta',$$
$$\mathrm{Q}' = \sin\varepsilon.\sin\varepsilon'.\cos\psi' - \sin\varepsilon.\cos\varepsilon'.\sin\psi',$$

which are evidently reducible to the expressions for $\mathrm{P}'$ and $\mathrm{Q}'$ given in the text.

(*k*) Since the sine of an angle is the same as that of its supplement, we have in this case $\sin\theta$ very small, consequently $\sin^2\theta$ and $\therefore\ \theta^2$ may be neglected; hence we have $\cos\theta = -\left(1-\frac{\theta^2}{2}+\&c.\right)$ $= -1$, the value of $\cos\theta$ is affected with a negative sign, because it is by hypothesis an obtuse angle.

(*l*) Since $\cos^2\lambda + \cos^2\mu + \cos^2\nu = 1$, and R is nearly vertical, the squares of $\cos\lambda$ and $\cos\mu$ may be neglected, and therefore $\cos^2\nu = 1$.

(*m*) If we substitute for $d\theta$ and $\sin\theta d\psi$ in the value of $dc$, and observe that $\frac{\mathrm{M}g\gamma}{\mathrm{C}n} = m$, we obtain

$$dc = m[\cos\lambda\sin\psi\cos\psi - \cos\mu\sin^2\psi + \sin\theta\cos\psi - \cos\lambda\sin\psi\cos\psi - \cos\mu\cos^2\psi]\,dt = m\,[\sin\theta.\cos\psi - \cos\mu)\,dt = m(c'-\cos\mu)\,dt,$$ by

substituting $c'$ for its value; the second equation (15) may be obtained in the same manner.

($n$) As these equations are of the same form as those integrated in note ($g$), it will not be necessary here to show how they may be reduced to an integrable form, for, from a consideration of what is there stated, it is evident that they may be reduced to the form $dc' + \theta_i dc + m.\,(c - \theta_i c')dt = (\text{T} + \theta_i \text{T}').dt$, from which by the artifices there pointed out, we can obtain

$$dz \pm m\sqrt{-1}.zdt = m[\cos\lambda \pm \sqrt{-1}.\cos\mu]\,dt,$$

and therefore

$$z' = me^{-mt\sqrt{-1}}\,[\textstyle\int e^{mt\sqrt{-1}}([\cos\lambda - \sqrt{-1}.\cos\mu]\,dt] + \text{c},)$$

$$z'' = me^{mt\sqrt{-1}}\,(\textstyle\int e^{-mt\sqrt{-1}}([\cos\lambda + \sqrt{-1}\cos\mu]\,dt] + \text{c}'),$$

$\because$ by substituting for $e^{mt\sqrt{-1}}$, $e^{-mt\sqrt{-1}}$ their respective values, there results

$$z' = m\,[\cos mt - \sqrt{-1}.\sin mt]\textstyle\int[\cos mt + \sqrt{-1}.\sin mt]$$
$$[\cos\lambda - \sqrt{-1}.\cos\mu]dt,$$

equal by performing the multiplication

$$z' = m.\cos mt\textstyle\int\cos mt.\cos\lambda dt - m.\cos mt\textstyle\int\cos mt\sqrt{-1}.\cos\mu dt$$
$$+ m.\cos mt\textstyle\int\sqrt{-1}.\sin mt.\cos\lambda dt + m.\cos mt\textstyle\int\sin mt.\cos\mu.dt$$
$$- m\sqrt{-1}.\sin mt\textstyle\int\cos mt.\cos\lambda dt - m.\sin mt\textstyle\int\cos mt.\cos\mu.dt$$
$$+ m.\sin mt\textstyle\int\sin mt.\cos\lambda dt - m\sin mt\textstyle\int\sin mt\sqrt{-1}.\cos\mu.dt$$

$$z'' = m.(\cos mt + \sqrt{-1}.\sin mt)(\textstyle\int\cos mt - \sqrt{-1}.\sin mt).(\cos\lambda +$$
$$\sqrt{-1}.\cos\mu.)dt$$

$$= m.\cos mt.\textstyle\int\cos mt.\cos\lambda.dt + m.\cos mt\textstyle\int\cos mt\sqrt{-1}.\cos\mu.dt$$
$$- m.\cos mt\textstyle\int\sqrt{-1}.\sin mt.\cos\lambda dt + m.\cos mt\textstyle\int\sin mt.\cos\mu.dt$$
$$+ m\sqrt{-1}.\sin mt\textstyle\int\cos mt\cos\lambda dt - m.\sin mt\textstyle\int\cos mt\cos\mu\,dt$$
$$+ m.\sin.mt.\textstyle\int\sin mt.\cos\lambda.dt + m\sin.mt\textstyle\int\sin mt\sqrt{-1}.\cos\mu dt;$$

now as $c' = \frac{z' + z''}{2}$, if these values of $z'$ and $z''$ be added together, it appears from an inspection of the preceding expressions, that all terms multiplied by $\sqrt{-1}$, destroy each other, so that there remains

$$\frac{z'+z''}{2}=m.\cos mt\int\cos mt.\cos\lambda.dt+m.\cos mt\int\sin mt\cos\mu dt$$
$$-m.\sin mt\int\cos mt\cos\mu.dt+m.\sin mt\int\sin mt\cos\lambda dt;$$

in like manner as $c=\frac{z-z'}{2\sqrt{-1}}$, by taking the difference of the preceding values of $z'$ and $z''$, and dividing by $2\sqrt{-1}$, we obtain the value of $c$, or $\sin\theta.\sin\psi$, given in the text. With respect to the values of the arbitrary constants, as

$$z'=e^{-mt\sqrt{-1}}\left[\int me^{mt\sqrt{-1}}.(\cos\lambda-\sqrt{-1}.\cos\mu)\,dt+\text{C}\right],$$
$$z''=e^{mt\sqrt{-1}}\left[\int me^{-mt\sqrt{-1}}.(\cos\lambda+\sqrt{-1}.\cos\mu)\,dt+\text{C}'\right],$$

if $\text{A}_1$ and $\text{B}_1$ denote what $\cos\lambda$, $\cos\mu$, become when $t=0$, we shall have (as $z', z''=0$ when $t=0$,)

$$0=e^{mt\sqrt{-1}}.(\text{A}_1-\sqrt{-1}.\text{B}_1)+\text{C}]=0,\ 0=e^{-mt\sqrt{-1}}[(\text{A}_1+\sqrt{-1})\text{B}_1+\text{C}'],$$
$$\because\ \text{C}e^{mt\sqrt{-1}}+\text{C}'e^{-mt\sqrt{-1}}=\text{A}_1\,(e^{mt\sqrt{-1}}+e^{-mt\sqrt{-1}})$$
$$+\text{B}_1\sqrt{-1}\,(.e^{-mt\sqrt{-1}}-e^{mt\sqrt{-1}})=2\text{A}_1.\cos mt+2\text{B}_1.\sin mt.$$

($o$) In this case we must have $\theta'=0$, consequently the terms multiplied by $k$, $k'$, in the values of $\sin\theta\cos\psi$, $\sin\theta\sin\psi$ must disappear, and as $m$ is very small, the variations or deviations from the horizontal plane must continue very small during the continuance of the motion.

($p$) This series is that which arises from integrating $\int\cos\lambda\cos mtdt$ by parts, for if we put $dx$ for $\cos\lambda dt$, we have, by partial integration,

$$\int\cos mtdx=\cos mtx+m\int\sin mtxdt=(m\sin mt\int xdt-m^2\int\cos mt\int xdt)$$
$$=\cos m\int\cos\lambda dt+m\sin mt\iint\cos\lambda.dt^2-m^2\cos mt\iiint\cos\lambda dt^3-\text{\&c.};$$

and it is evident from its form, that when the variations of $\cos\lambda$ are very rapid relatively to those of $\sin mt$, $\cos mt$, that this series must converge rapidly, and may thus be reduced to its first term; but this is the same thing as if $\cos mt$ was regarded as constant in the integral $\int\cos\lambda\cos mt.dt$.

Now when $k$, $k'$, are respectively equal to cipher, we have, by considering $\sin mt$, $\cos mt$, as constant,

$$\sin\theta\sin\psi=-m.\sin^2 mt\int\cos\mu dt+m.\sin mt.\cos mt\int\cos\lambda dt$$
$$-m\cos^2 mt\int\cos\mu dt-m.\sin mt\cos mt\int\cos\lambda dt$$
$$=-m.\int\cos\mu dt.$$

The value of $\sin\theta\cos\psi$ can be obtained in the same manner.

($q$) We obtain by adding the two preceding equations

$$\sin^2\theta.(\sin^2\psi+\cos^2\psi)=\left(\frac{\text{M}\gamma}{\text{C}n}\right)^2.\left(\frac{dx_{\prime}^2+dy_{\prime}^2}{dt^2}\right)\therefore\sin\theta=\frac{\text{M}\gamma}{\text{C}n}.u.$$

($r$) Since the pulley turns on its axis in the direction in which the vertical forces act, the moments of the forces of the points of the pulley with respect to its axis must be added to the forces acting in the vertical direction; and, as the linear velocity of any point of the circumference of the pulley is by supposition equal to that of any point of the vertical string, its angular velocity must be equal $\frac{1}{c}\frac{dz'}{dt}$.

($s$) This equation becomes, by substituting $-\frac{d^2z}{dt^2}$ for $\frac{d^2z_{\prime}}{dt^2}$, and observing that $g\mu=\frac{wz}{l}$, $g\mu'=w-\frac{wz}{l}$,

$$-\left(\text{P}'+w-\frac{wz}{l}+\frac{mk^2}{c^2}\right)\frac{d^2z}{dt^2}-\left(\text{P}+\frac{wz}{l}\right).\frac{d^2z}{dt^2}$$
$$=g\left[\text{P}'+w-\frac{wz}{l}\quad h\text{P}-(a+a')\frac{dz^2}{dt^2}\right],$$

which is evidently reducible to

$$\left[\text{P}+\text{P}'+w+\frac{pk^2}{c^2}\right]\frac{d^2z}{dt^2}=g.\left[h\text{P}-\text{P}'-w+\frac{wz}{l}+(a+a')\frac{dz^2}{dt^2}\right]$$

($t$) If all the terms of this equation be multiplied by $dt^2$ and afterwards by $dz$, we shall obtain as $d^2z=\frac{d^2z}{dt^2}dt^2$

$$dzd^2z-\frac{g\beta}{l}zdzdt^2+gadzdt^2=0,$$

of which the integral is

$$dz^2-\frac{g\beta}{l}z^2dt^2+gazdt^2+\text{C}dt^2=0,$$

$$\therefore dt=\frac{dz}{\sqrt{g\frac{\beta}{l}z^2-gaz-\text{C}}},$$

which can be integrated by the ordinary rules.

($u$) Since $\frac{dz}{dt}=\sqrt{\frac{\beta\gamma}{l}}\left[\text{C}e^{t\sqrt{\frac{\beta\gamma}{l}}}-\text{C}'e^{-t\sqrt{\frac{\beta\gamma}{l}}}\right]$, when $t=0$, we have

$$\frac{dz}{dt}=0,=\sqrt{\frac{\beta\gamma}{l}}.(\text{C}-\text{C}')=0,\ \therefore\ \text{C}=\text{C}';$$

and since $z = 0$, when $t = \theta$ we have

$$\left(\frac{al-\gamma\beta}{2\beta}\right)\left[e^{\theta\sqrt{\frac{\beta\gamma}{l}}}+e^{-\theta\sqrt{\frac{\beta\gamma}{l}}}\right]=\frac{al}{\beta}.$$

(*v*) And since

$$e^{\theta\sqrt{\frac{\beta g}{l}}}=1+\theta\sqrt{\frac{\beta g}{l}}+\frac{\theta^2\frac{\beta\gamma}{l}}{1.2.}+\frac{\theta^3.\left(\frac{\beta\gamma}{l}\right)^{\frac{3}{2}}}{1.2.3}+\frac{\theta^4\left(\frac{\beta g}{l}\right)^2}{1.2.3.4}+\text{\&c.},$$

$$e^{-\theta.\sqrt{\frac{\beta g}{l}}}=1-\theta\sqrt{\frac{\beta\gamma}{l}}+\frac{\theta^2.\frac{\beta g}{l}}{1.2}-\frac{\theta^3.\left(\frac{\beta g}{l}\right)^{\frac{3}{2}}}{1.2.3}+\frac{\theta^4.\left(\frac{\beta g}{l}\right)^2}{1.2.3.4}-\text{\&c.},$$

$$\therefore (al-\beta g).2\left[1+\frac{\theta^2\frac{\beta g}{l}}{1.2}+\frac{\theta^4\left(\frac{\beta g^2}{l}\right)}{1.2.3.4}\right]=2al,$$

$$\therefore\ 2\beta\gamma=2(al-\beta g)\left[\frac{\theta^2\frac{\beta g}{l}}{1.2}+\frac{\theta^4\frac{\beta^2 g^2}{l^2}}{1.2.3.4}.\right],$$

therefore, dividing by $\beta$, we obtain the value of $\gamma$, given in the text.

(*x*) Since all forces $pd\sigma$ act in parallel directions, it is evident, as they cannot impress any rotatory motion on the body, we must have $\int xpd\sigma$, $\int ypd\sigma$, respectively equal to cipher, and as the moment of the resultant is equal to the sum of the moments of the components, we have $\text{H}x_1 = \int hxpd\sigma$, and $\therefore x_1 = 0$.

(*y*) It is evident from what is established in No. 57, that when $\int xd\sigma = 0$, $\int yd\sigma = 0$, the origin of the coordinates $x$ and $y$, coincides with the centre of gravity of $b$.

(*z*) When the two centres of gravity exist on the same vertical, $\int xd\sigma = 0$, $\therefore p$ must be a constant quantity, consequently,

$$p\int d\sigma = \text{P},\ \text{i.e.}\ pb = \text{P},\ \text{and}\ \therefore p = \frac{\text{P}}{b}.$$

(*a′*) We have

$$\text{M}\frac{d^2x}{dt^2} = -\frac{h\text{P}}{b}\int \cos\alpha d\sigma,$$

which by substituting for P its value $\text{M}g$, gives the expression in the text.

(*b′*) In this case, if each of equations (1) be squared and then added together, we shall have

$$v = r\frac{d\theta}{dt}; \text{ and } \because \cos\alpha = -\sin\theta,\ \cos\beta = \cos\theta.$$

($c'$) When this is the case, equation (3) becomes

$$k^2\frac{d\omega}{dt} = -\frac{hg}{b}\int(\cos^2\theta + \sin^2\theta)\,r d\sigma = -\frac{hg}{b}.\int r d\sigma = -\frac{hgc}{b},$$

$$\therefore \frac{d\omega}{dt}dt = -\frac{hgcdt}{bk^2}, \text{ and } \omega = \Omega - \frac{hgct}{bk^2},$$

hence when

$$\omega = 0,\ \Omega - \frac{hgct}{bk^2} = 0 \text{ and } t = \Omega\frac{bk^2}{hgc}.$$

($d'$) By means of this equation we have

$$\frac{1}{v} = \left[u^2 - 2\left(\frac{dx}{dt}\sin\theta - \frac{dy}{dt}\cos\theta\right)r\frac{d\theta}{dt}\right]^{-\frac{1}{2}},$$

which is equal to the value of $\frac{1}{v}$ given in the text, when we restrict the expansion of the radical to the two first terms; now by the first equation (1), we have

$$\cos\alpha = \frac{1}{v}\frac{dx}{dt} - \frac{1}{v}\left(r\sin\theta\frac{d\theta}{dt}\right),$$

and by substituting for $\frac{1}{v}$, and neglecting the square and higher powers of $r\frac{d\theta}{dt}$, there results

$$\cos\alpha = \frac{1}{u}\frac{dx}{dt} + \frac{1}{u^3}\left(\frac{dx^2}{dt^2}\sin\theta - \frac{dx}{dt}\frac{dy}{dt}\cos\theta\right)r\frac{d\theta}{dt} - \frac{1}{u}r\frac{d\theta}{dt}\sin\theta.$$

Now as $u^2 = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}$, we have evidently

$$-\frac{1}{u}r\frac{d\theta}{dt}\sin\theta = -\frac{1}{u^3}\left[\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right]r\frac{d\theta}{dt}\sin\theta;$$

consequently, by substituting this value of $-\frac{1}{u}r\frac{d\theta}{dt}\sin\theta$ in the value of $\cos\alpha$ given above, there will result the expression for $\cos\alpha$ given in the text; in the same manner the expression for $\cos\beta$ may be obtained.

($e$) In this case the second terms of the values of $\cos\alpha$, $\cos\beta$ are evidently cipher, and $\because \int\cos\alpha d\sigma = \frac{1}{u}\frac{dx}{dt}\int d\sigma = \frac{b}{u}\frac{dx}{dt}$; hence it is evident that the value of $\frac{d^2x}{dt^2}$ is that given in equation (4).

($f'$) Since $\cos\beta.\cos\theta$, $\cos\alpha.\sin\theta =$ by putting for $\cos\beta$, $\cos\alpha$ their respective values,

$$\frac{1}{u}\frac{dy}{dt}\cos\theta + \frac{1}{u^3}.\left(\frac{dx}{dt}.\cos^2\theta + \frac{dy}{dt}.\sin\theta.\cos\theta\right)\frac{dx}{dt}.\frac{rd\theta}{dt}$$

$$\frac{1}{u}\frac{dx}{dt}.\sin\theta - \frac{1}{u^3}\left(\frac{dx}{dt}.\sin\theta.\cos\theta + \frac{dy}{dt}\sin^2\theta\right)\frac{dy}{dt}.\frac{rd\theta}{dt};$$

if we substitute these expressions in equations (3), we obtain (as $\frac{1}{u}.\frac{dy}{dt}\int r\cos\theta d\sigma = 0$, $\frac{1}{u}.\frac{dx}{dt}\int .r\sin\theta d\sigma = 0$, $\int\frac{1}{u^3}.\frac{dx}{dt}.\frac{dy}{dt}\int r^2\sin\theta.$ $\cos\theta d\sigma = 0$, $\frac{1}{u^3}.\frac{dx^2}{dt^2}.\int r^2\cos^2\theta d\sigma = \frac{1}{u^3}.\frac{dx^2}{dt^2}.\,b\gamma^2$, $\frac{1}{u^3}\frac{dy^2}{dt^2}.\int r^2\sin^2\theta d\sigma$ $= \frac{1}{u^3}.\frac{dy^2}{dt^2}.b\gamma^2$), $k^2\frac{d\omega}{dt} = \frac{-hg}{b}.\frac{dx^2 + dy^2 b\gamma^2}{u^3}\frac{d\theta}{dt}$ = the expression in the text, by substituting $u^2$ for $dx^2 + dy^2$, and $\frac{d^2\theta}{dt^2}$ for $\frac{d\omega}{dt}$.

($g'$) It is evident, from a consideration of equations (4), that $\frac{d^2x}{dt^2}$ divided by $\frac{dx}{dt}$ being equal to $-\frac{hg}{u}$, is also equal to $\frac{d^2y}{dt^2}$ divided by $\frac{dy}{dt}$, consequently we have $\frac{dx}{dt} = \tan\varepsilon.\frac{dy}{dt}$, $\varepsilon$ being an arbitrary constant, and $u = \frac{dy}{dt}(1 + \tan g^2\varepsilon)^{\frac{1}{2}}$, hence there results

$$\frac{d^2y}{dt^2}dt = -\frac{hg.dt}{(1 + \tan^2\varepsilon)^{\frac{1}{2}}},\ \text{and}\ \therefore \frac{dy}{dt} = \frac{a - hg.t}{(1 + \tan g^2\varepsilon)^{\frac{1}{2}}};$$

in the same manner it may be shown that

$$\frac{dx}{dt} = \frac{(a - hg.t)\tan g\,\varepsilon}{(1 + \tan g^2\varepsilon)^{\frac{1}{2}}}.$$

These values of $\frac{dx}{dt}$, $\frac{dy}{dt}$ are evidently equal to those given in the text, and by squaring them, and then adding them together, we obtain the value of $u$ given in the text.

($h'$) If both sides of this equation be multiplied by $dt$, we shall have

$$\frac{d^2\theta}{dt} = -\frac{\gamma^2}{k^2}.\frac{hg.dt}{(a - hgt)}.\frac{d\theta}{dt};$$

hence there results

$$\frac{d^2\theta}{dt} = -\frac{\gamma^2}{k^2}\left(\frac{\frac{hgdt}{a}}{1-\frac{hgt}{a}}\cdot\frac{d\theta}{dt}\right), \because \log.\frac{d\theta}{dt} = \frac{\gamma^2}{k^2}.\log.\left(1-\frac{hgt}{a}\right) + \text{const};$$

hence if $\Omega$ be the initial angular velocity, we shall have const. $=$ $\log\Omega$, $\because$ $\log\frac{d\theta}{dt} = \frac{\gamma^2}{k^2}\log\left(1-\frac{hgt}{a}\right) + \log\Omega$, consequently $\frac{d\theta}{dt}$ equal to its value given in the text.

Now in obtaining the approximate values of $v$ and $\frac{d\theta}{dt}$, the square and higher powers of $\frac{rd\theta}{dt}$ are neglected; therefore at the commencement, when $\frac{d\theta}{dt} = \Omega$, and $r$ is a maximum, the smaller this product is, the more accurate will be the values of $v$ and $\frac{d\theta}{dt}$.

## CHAPTER VII.

(*a*) By multiplying the three first equations (1) by $\cos\alpha$, $\cos\beta$, $\cos\gamma$, respectively, and then adding them together, we obtain

$$\text{N}.(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) + \text{M}[(u - u_1)\cos\alpha + (v - v_1)\cos\beta + (w - w_1)\cos\gamma] = 0,$$

by multiplying the three first equations (2) by $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$, we obtain the second value of N.

(*b*) By substituting $\theta.\cos\text{HGL}$, $\theta'\cos\text{H}'\text{G}'\text{L}'$, in place of the quantities to which they are respectively equal, we shall have

$$\frac{\text{N}}{\text{M}} + \frac{\text{N}}{\text{M}'} + \theta\cos\text{HGL} + \theta'\cos\text{H}'\text{G}'\text{L}' = 0,$$

from which it is easy to obtain the value of N given in the text.

(*c*) From the first equation (6) we obtain

$$\frac{\text{N}}{\text{M}} + u.\cos\alpha + v\cos\beta + w\cos\gamma = u_1\cos\alpha + v_1\cos\beta + w_1\cos\gamma = \theta_1.\cos\text{HG}l,$$

that is, by substituting

$$-\text{M}'\frac{(\theta\cos\text{HGL} + \theta'\cos\text{H}'\text{G}'\text{L}')}{\text{M}+\text{M}'} + \theta.\cos\text{HGL} = \theta_1\cos\text{HG}l,$$

which by reducing becomes equal to the value of $\theta_1\cos\text{HG}l$.

(*d*) From the three first equations (4) we obtain, by multiplying them by $\cos\alpha$, $\cos\beta$, $\cos\gamma$ respectively,

$$\frac{2\text{N}}{\text{M}}+u\cos\alpha+v\cos\beta+w\cos\gamma=\text{U}\cos\alpha+\text{V}\cos\beta+\text{W}\cos\gamma=\theta_1\cos\text{HG}l,$$

that is,

$$-2\text{M}'\frac{(\theta\cos\text{HGL}+\theta'\cos\text{H}'\text{G}'\text{L}')}{\text{M}+\text{M}'}+\theta.\cos\text{HGL}=\theta_1\cos\text{HG}l.$$

(*e*) In the first equation (7), when $\theta'=0$, we have

$$\theta_1\cos\text{HG}l=\frac{\text{M}\theta\cos\text{HGL}}{\text{M}+\text{M}'},$$

now if M, in consequence of its density, may be neglected with respect to M′, the denominator of the preceding fraction may be considered as infinite, relatively to its numerator, consequently, we shall have $\theta_1=0$; when the bodies are perfectly elastic, the first equation (8) gives in this case,

$$\theta_1\cos\text{HG}l=\frac{(\text{M}-\text{M}')\,\theta.\cos\text{HGL}}{\text{M}+\text{M}'};$$

and the second member of this equation, when M is neglected relatively to M′, is reduced to $-\theta.\text{HGL}$; the value of the second equation (8), is in this case

$$\theta_1'\cos\text{H}'\text{G}'l'=-\frac{2\text{M}\theta.\cos\text{HGL}}{\text{M}+\text{M}'},$$

= to cipher when M is neglected relatively to M′. The preceding cases strictly obtain when M, whether perfectly elastic or soft, impinges on a fixed obstacle.

(*f*) Since $\text{M}a'=\text{M}a-2\text{M}hb$, consequently, $a'=a-2hb$, and by what is established in No. 386, we have

$$\tfrac{2}{5}\text{M}c^2\alpha=\text{M}ac,\ \tfrac{2}{5}\text{M}c^2\alpha'=\text{M}a'c,\ \therefore\ \tfrac{2}{5}\text{M}c^2(\alpha-\alpha')=\text{M}c(a-a')=2h\text{M}bc,$$

hence we obtain

$$\alpha'=\alpha-\frac{5hb}{c},\ \text{and}\ \therefore\ a'+c\alpha'=a+c\alpha-7hb.$$

(*g*) When the absolute velocity of the point K is *constantly* negative, its initial velocity $a+c\alpha$ and its final velocity $a'+c\alpha'$ are both negative; consequently, as $a'+c\alpha'=a+c\alpha+7hb$, $a+c\alpha$ must exceed $7hb$.

(*h*) Since the final velocity of K is supposed to be equal to cipher, we have

$$a' = -c\alpha' = 5h\frac{(b+b')}{2} - c\alpha = 5\frac{(a+c\alpha)}{7} - c\alpha = \frac{5a-2c\alpha}{7}.$$

(*i*) By equations (6), and on the hypothesis that $\gamma = 0$, we have

$$\frac{\text{N}}{\text{M}} + u\cos\alpha + v\cos\beta - u_{,}\cos\alpha - v_{,}\cos\beta + \frac{\text{N}}{\text{M}'} + u'\cos\alpha' + v'\cos\beta'$$
$$+ w'\cos\gamma' - u_1{}'\cos\alpha' - v_1{}'\cos\beta' - w_1{}'.\cos\gamma' = 0,$$

which, combined with this equation, gives the expression in the text.

(*k*) By multiplying the three first equations (a) by $\cos\alpha$, $\cos\beta$, $\cos\gamma$, respectively, we obtain, by adding them together,

$$\text{M}\cdot(a.\cos\alpha + b.\cos\beta + c.\cos\gamma) - \text{M}\,(u\cos\alpha + v\cos\beta + w\cos\gamma)$$
$$-\ \text{N}\,(\cos^2\alpha + \cos^2\beta + \cos^2\gamma),$$

from which by substituting $k$ for its value, and dividing by M, gives the expression in the text. In like manner, if the three last equations (a) be multiplied by $\cos\alpha$, $\cos\beta$, $\cos\gamma$, respectively, and then added together, there results,

$$\text{N}(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)+\text{N}'.(\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma')-$$
$$\mu.((u_1\cos\alpha + v_1\cos\beta + w_1\cos\gamma) = 0,$$

i. e. $\text{N} + \text{N}'\cos\delta - \mu\,(u_1\cos\alpha + v_1\cos\beta + w_1\cos\gamma) = 0.$

Hence the first equation (b) becomes

$$\frac{\text{N}+\text{N}'\cos\delta}{\mu} = k - \frac{\text{N}}{\text{M}}, \text{ and } \frac{\text{N}'+\text{N}.\cos\delta}{\mu} = k' - \frac{\text{N}'}{\text{M}'},$$

therefore, from the first equation, we obtain

$$\text{N}' = \left(k\mu - \frac{\text{N}\mu}{\text{M}} - \text{N}\right)\frac{1}{\cos\delta},$$

and from the second,

$$\text{N}' = \left(\frac{k'\mu - \text{N}.\cos\delta}{\text{M}'+\mu}\right).\text{M}',$$

hence, by comparing these two values of N′, we obtain the value of N given in the text.

## CHAPTER VIII.

(*a*) If the weight of the string be taken into account, the string can no longer be considered as rectilinear, for it will be then acted on by three forces, namely, the two tensions at its extremities, and this weight, acting at the centre of gravity of the string.

(*b*) Since the elements are proportional to the lengths, if the element at the point M be denoted by $\mu$, we have

$$\mu : \frac{p}{g} :: dx : l, \therefore \mu = \frac{pdx}{gl}.$$

(*c*) By substituting in the first equation (1) its value for T, we obtain, as $d\text{T} = q\frac{d^2u}{dx}$, and $d(x+u) = ds$,

$$q\frac{d^2u}{dx} = \frac{p}{gl}\frac{d^2u}{dt^2}dx,$$

and in the second equation (1), as $d\,.\,\text{T}\frac{dy}{ds} = d\text{T}\frac{dy}{ds} + \text{T}\frac{d^2y}{ds}$, there results, by substituting for T and $d$T,

$$q\frac{d^2u}{dx}\frac{dy}{ds} + w\frac{d^2y}{ds} + q\frac{d^2y}{ds}\frac{du}{dx} = \frac{p}{gl}\frac{d^2y}{dt^2}dx,$$

that is,

$$\underline{q\,.\,d\,.\left(\frac{du}{dx}\cdot\frac{dy}{ds}\right)} + w\frac{d^2y}{ds} = \frac{p}{gl}\frac{d^2y}{dt^2}dx,$$

hence, as by hypothesis, we may neglect $\frac{du}{dx}\frac{dy}{ds}$, we obtain

$$w\frac{d^2y}{ds} = \frac{p}{gl}\frac{d^2y}{dt^2}dx,$$

as we may substitute $\frac{dy}{dx}$ for $\frac{dy}{ds}$, this equation is evidently the same as the second of equations (2).

(*d*) See Lacroix *Traite Elementaire*, No. 319, the equation may be reduced to an integrable form, also in the following manner:

Since $\frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2}$, if to each member there be added $a\frac{d^2y}{dx\,.\,dt}$, we have

$$\frac{d}{dt}\cdot\left[\frac{dy}{dt} + a\frac{dy}{dx}\right] = a\cdot\frac{d}{dx}\left[\frac{dy}{dt} + a\frac{dy}{dx}\right],$$

hence, if we suppose $\frac{dy}{dt} + a\frac{dy}{dx} = z$, we have evidently

$$\frac{dz}{dt} = a\frac{dz}{dx}, \therefore \frac{dz}{dt}dt = \frac{dz}{dx}adt,$$

if to each side there be added $\frac{dz}{dx}.dx$, there results

$$\frac{dz}{dt}dt+\frac{dz}{dx}dx=\frac{dz}{dx}.d.(x+at), \because z \text{ or } \frac{dy}{dt}+a\frac{dy}{dx}=\Psi(x+at),$$

$\Psi$ being an arbitrary function of $x+at$. In like manner, if $a\frac{d^2y}{dx.dt}$ be taken from each side of the equation $\frac{d^2y}{dt^2}=a^2\frac{d^2y}{dx^2}$, we shall, by a similar process, obtain $\frac{dy}{dt}-a\frac{dy}{dx}=\psi(x-at)$, $\psi$ being another arbitrary function, therefore, by adding and subtracting these two equations, we obtain

$$\frac{dy}{dt}=\tfrac{1}{2}\Psi(x+at)+\tfrac{1}{2}\psi(x-at),$$

$$\frac{dy}{dx}=\frac{1}{2a}\Psi(x+at)-\frac{1}{2a}\psi(x-at)$$

consequently,

$$dy=\frac{dy}{dt}dt+\frac{dy}{dx}dx=\tfrac{1}{2}\Psi(x+at)dt+\tfrac{1}{2}\psi(x-at)dt+\frac{1}{2a}\Psi(x+at)$$
$$dx-\frac{1}{2a}\psi(x+at)\,dx=\frac{1}{2a}.\Psi(x+at).d.(x+at)+\frac{1}{2a}\psi(x-at)$$
$d.(x-at)$, $\because y=f(x+at)+\text{F}(x-at)$, $f$ and F being two functions depending on $\Psi$ and $\psi$.

(*e*) Since $\phi x=\frac{1}{a}\int\phi' x dx=fx-\text{F}x$, we have evidently

$$\phi x+\Phi x=2fx,\ \phi x-\Phi x=2\text{F}x.$$

(*f*) The form of $\phi$ is known, because $y=\phi x$ is the equation of the string at the commencement of the motion, the only restriction to $\phi$ is, that it must be equal to cipher when $x=0$, $x=l$; it may be, as is stated in No. 488, a discontinuous function, i. e. the form of the string may be made up of different parts, which are not represented by the same equation, in its extent from A to B. From the circumstance of the extremities A and B being fixed, the remarkable properties indicated in equations (5) are inferred; from the first equation (5) it follows, that reckoning from A, the curve represented by $y=\phi x$ is continued on each part of A, and has corresponding forms each side, the one being above and the other below AB; and the second equation (5) shows that the same thing is also true with respect to the point B; the curve is continued on each side, with similar forms, the one being above and the other below the axis AB.

(*g*) In this case it is evident, that the ordinate of the curve must be a maximum; and when the initial velocity is cipher, i. e. when $\phi' x = 0, \frac{d.fx}{dx} = \frac{d.\text{F}x}{dx}$, $\because fx = \text{F}x$, and the value of $y$ in equation (3) becomes

$$y = f(x + at) + f(x - at).$$

and the function $f$ expresses the original form of the string, i. e. the form when $t = 0$, consequently $f$ is known, and from equation (5) we have $f\zeta + f(-\zeta) = 0$.

(*h*) In like manner, after the time equal to $\frac{l}{a}$, the form of the curve is the same as at the commencement, with its position inverted, and after the lapse of the times, $\frac{3l}{a}, \frac{5l}{a}$, &c., the figure will be the same, if we suppose $at = \frac{l}{2}$ we have the figure the curve assumes in the middle of the times between the extreme positions, which have been just discussed; in this case when there is no initial velocity, $y = \frac{1}{2}\left[f\left(x + \frac{l}{2}\right) + f\left(x - \frac{l}{2}\right)\right]$, and since by equation (5) $f\left(x - \frac{l}{2}\right) = -f\left(\frac{l}{2} - x\right)$ there results

$$y = [f\left(x + \frac{l}{2}\right) - f\left(\frac{l}{2} - x\right),$$

the abscissæ indicated by $x + \frac{l}{2}$ and $\frac{l}{2} - x$ refer to points at equal distances from the middle point of AB, if in the original form of the curve, the ordinates for the portion between the middle point and B be greater than the ordinates for the portion the middle point and A, i. e. if it is not symmetrical, then the part of the curve which is to the left of the ordinate that passes through the middle point, will be above the axis AB, at the *middle* of the time of a vibration, and the position of the portion which is at the right of the middle point will be below AB, and similar to the other portion inverted, so that in this case the string is never rectilinear. If the ordinate raised at the middle of AB divides the curve, in its original position, into two equal and similar parts, then the ordinates corresponding to the ab-

scissæ $\frac{l}{2}+x$, $\frac{l}{2}-x$ are equal, therefore, in this case, whatever be the value of $x$, $y$ vanishes when $at=\frac{l}{2}$, consequently, in this case the curve becomes a right line at the middle of each vibration.

(*i*) In like manner for equation (d) we have

$$\frac{dy}{dt}=-\frac{2\pi a}{l^2}\Sigma\left(\int_0^l \sin\frac{i\pi x'}{l}\,\phi x'dx'\right) i\sin\frac{i\pi x}{l}.\sin\frac{i\pi at}{l}$$

$$+\frac{2}{l}\Sigma\left(\int_0^l \sin\frac{i\pi x'}{l}\phi' x'dx'\right)\sin\frac{i\pi x}{l}.\cos\frac{i\pi at}{l},$$

$$\frac{d^2y}{dt^2}=-\frac{2\pi^2a^2}{l^3}\Sigma\left(\int_0^l \sin\frac{i\pi x'}{l}\,\phi x'\,dx'\right) i^2\sin\frac{i\pi x}{l}.\cos\frac{i\pi at}{l}$$

$$-\frac{2\pi a}{l^2}\Sigma\left(\int_0^l \sin\frac{i\pi x'}{l}\,\phi' x'dx'.\right) i.\sin\frac{i\pi x}{l}.\sin\frac{i\pi at}{l},$$

$$\frac{dy}{dx}=\frac{2\pi}{l^2}\Sigma\left(\int_0^l \sin\frac{i\pi x'}{l}\,\phi x'dx'\right) i\cos\frac{i\pi x}{l}.\cos\frac{i\pi at}{l}$$

$$+\frac{2}{la}\Sigma\left(\int_0^l \sin\frac{i\pi x'}{l}\,\phi' x'\,dx'\right)\cos\frac{i\pi x}{l}\sin\frac{i\pi at}{l},$$

$$\frac{d^2y}{dx^2}=-\frac{2\pi^2}{l^3}\Sigma\left(\int_0^l \sin\frac{i\pi x'}{l}\,\phi x'dx'\right) i^2\sin\frac{i\pi x}{l}\cos\frac{i\pi at}{l}$$

$$-\frac{2\pi}{l^2a}\Sigma\left(\int_0^l \sin\frac{i\pi x'}{l}\phi' x'dx'\right) i\sin\frac{i\pi x}{l}.\sin\frac{i\pi at}{l},$$

hence it is evident that $\frac{d^2y}{dt^2}=a^2\frac{d^2y}{dx^2}$.

(*k*) When $x=\frac{l}{2m}$ the equation $\phi x=h\sin\frac{m\pi x}{l}=h\sin\frac{\pi}{2}$, which therefore is the greatest value of $\phi x$, or the height of the trochoid.

(*l*) As $\phi' x=0$, the second term, second member of equation (d) is cipher, and the first term is reduced to

$$y=\frac{2h}{l}\int_0^l\left(\sin^2\frac{m\pi x'}{l}dx'.\right)\sin\frac{m\pi x}{l}\cos\frac{m\pi at}{l};$$

as $\sin^2\frac{m\pi x'dx'}{l}=-\frac{1}{2}\cos\frac{2m\pi x'dx'}{l}+\frac{1}{2}dx'$, the integral is equal $\frac{l}{2}$ when taken between the limits $l$, 0, consequently the value of $y$ is that given in the text.

($l$) $\frac{ds-dx}{dx}=\frac{du}{dx}=\frac{\delta.(\text{T}-w)}{\Delta+\delta w}=\delta.(\text{T}-w)(\Delta+\delta w)^{-1}$, which when $\delta^2$ is neglected becomes the expression in the text, and as by No. 483, $\text{T}-w=q\frac{du}{dx}$, by substituting this value of $\text{T}-w$ we obtain $1=q.\frac{\delta}{\Delta}$, consequently, when $\delta=1$, $q=\Delta$. in which case the ratio $1+\delta:1$ becomes that of $2:1$.

*Notes to Paragraph II.*

($m$) As $\delta l$ is by hypothesis the entire increase of length of the rod when subjected to tension $\Delta$, $\delta dx$ is the increase of length of the part $dx$ of the same rod, No. 288, therefore we must have $\text{T}:\Delta::du:\delta.dx$, $\because\ \text{T}=\frac{\Delta}{\delta}.\frac{du}{dx}$, and, consequently, $q=\frac{\Delta}{\delta}$; now when $x=0$, the equation of motion is $\frac{d\text{T}}{dx}-\frac{p}{gl}.\frac{d^2u}{dt^2}=$ by substituting $q.\frac{d^2u}{dx}$, for $d\text{T}$, $q.\frac{d^2u}{dx^2}-\frac{pd^2u}{gl\,dt^2}=0$.

($n$) That this value of $u$ satisfies equation (1) is apparent at once by taking the values of $\frac{d^2u}{dx^2}$, $\frac{d^2u}{dt^2}$, as in note ($i$), and substituting them in equation (1); it is likewise evident from inspection of this equation, that $u=0$ when $x=0$, and $\frac{du}{dx}=0$, when $x=l$.

($o$) In taking the value of $\frac{du}{dt}$ in this case, it is evident that the two first terms in the value of $u$ need not be taken into account, and the third term becomes, when differenced with respect to $t$, $=\frac{1}{l}\int_0^l \phi' x' dx$, and the fourth term when differenced with respect to this same quantity is

$$\frac{2i\pi a}{\pi a}.\Sigma\left(\int_0^l \cos\frac{i\pi x'}{l}\phi' dx'\right)\frac{1}{i}.\cos i\pi x\cos\frac{i\pi at}{l},$$

which when $t=0$, is the second term in the value of $\frac{du}{dt}$.

($p$) Beyond $\alpha$ the extent of the original agitation, as long as $at \angle x-\alpha$, the functions $f(x-at)$, $\text{F}(x-at)$, are cipher, and consequently the quantities $v$ and $s$, they have a finite value, the moment that $at = x-\alpha$, then they become cipher again, when $at = x+\alpha$, so that the agitation of the points situated on the positive side of $x$ commences when $t = \frac{x-\alpha}{a}$, and it lasts from $t = \frac{x-\alpha}{a}$ to $t' = \frac{x+\alpha}{a}$, or during an interval of time $= \frac{2\alpha}{a}$, and the portion which is agitated at the *same* time, is comprised between $x = at-\alpha$, and $x = at+\alpha$, so that its length is $2\alpha$, $\because$ as the extent is constant, i.e. $2\alpha$, and the time the same in all, namely, $\frac{2\alpha}{a}$ the velocity $= a$. The proper velocities will depend on the equation $v = \frac{1}{2}f(x-at) - \frac{a}{2}\text{F}(x-at)$. See Nos. 660, 1, 2, 3, 4, 5, 6.

($q$) Since $\frac{dv}{dx} = \frac{ds}{dt}$, *as* $v = \frac{du}{dt}$, $s = \frac{du}{dx}$, $vdt + sdx$ must be an exact differential. Lacroix, 261.

*Notes to Paragraph III.*

($q$) As $\varphi' x = h$ from $x = 0$, to $x = c$, the integral of $\int \varphi' x' dx'$ in this interval $= hc$, and as from $x = c$ to $x = l$, $\varphi' x = h'$, the integral $\int \varphi' x' dx'$ in this same interval is $h'(l-c) = h'c'$, consequently, from $x = 0$ to $x = l$, $\int \phi' x' dx' = hc + h'c'$. In like manner the integral of $\varphi' x' \cos \frac{i\pi x'}{l} dx'$ from $x = 0$ to $x = c$, is $\frac{l}{i\pi} . h . \sin \frac{i\pi c}{l}$, and from $x = c$ to $x = l$, this integral is $\frac{l}{i\pi} . h' . \sin \frac{i\pi . (l-c)}{l} = -\frac{l}{i\pi} h' . \cos i\pi . \sin \frac{i\pi c}{l}$; $\because \int \varphi' x' \cos \frac{i\pi x'}{l} dx'$ from $x = 0$ to $x = l = \frac{l}{i\pi}(h-h') \sin \frac{i\pi c}{l}$.

($r$) See the expression for $u$ given in the third case of No. 495, and as $\text{T} = q . \frac{du}{dx}$ by differentiating this value of $u$ with respect to $x$, we obtain the expression for $\text{T}$ given in the text.

(s) $\int x'^2.\cos\frac{i\pi x'}{l}.\frac{i\pi dx'}{l}=x'^2\sin\frac{i\pi x'}{l}-2\int x'.\sin\frac{i\pi x'}{l}.dx'$, now— $\int x'\sin\frac{i\pi x'}{l}.dx'=\frac{l}{i\pi}.x'\cos\frac{i\pi x'}{l}-\frac{l}{i\pi}.\int\cos\frac{i\pi x'}{l}dx'\left(=-\frac{l^2}{i^2\pi^2}.\sin\frac{i\pi x'}{l}\right)$, $\therefore\int x'^2\cos\frac{i\pi x'}{l}.\frac{i\pi dx'}{l}=x'^2\sin\frac{i\pi x'}{l}+\frac{2l}{i\pi}.x'\cos\frac{i\pi x'}{l}-\frac{2l^2}{i^2\pi^2}\sin\frac{i\pi x'}{l}$, which when taken between the limits of $l$ and 0, is reduced to $\frac{2l^2}{i\pi}\cos i\pi$, $l$ being substituted for $x'$, hence by putting this value in the preceding equation, we obtain $x=-\frac{2l}{\pi}.\Sigma\frac{1}{i}\cos i\pi.\sin\frac{i\pi x}{l}$, and by making $\frac{\pi x}{l}=\theta$, $\frac{\theta}{2}=-\Sigma\frac{1}{i}\cos i\pi.\sin.i\theta$, therefore by substituting for $i$ all integer numbers from $i=1$, to $i=\infty$, we shall obtain (by remarking that $\cos i\pi$ is alternately $-1$, and $+1$), the value of $\frac{\theta}{2}$ furnished by equation (3).

(t) This value of $v$, when $t$ is either cipher or an even multiple of $\frac{l}{a}$, results from the equation $\sin\frac{i\pi c}{l}.\cos\frac{i\pi x}{l}=\frac{1}{2}\sin i\pi\left(\frac{c+x}{l}\right)+\frac{1}{2}\sin i\pi.\left(\frac{c-x}{l}\right)$; and if in these last terms, $l-c'$ be substituted for $c$, they become $\frac{1}{2}\sin i\pi\left(\frac{l+x-c'}{l}\right)$, $\frac{1}{2}\sin i\pi\left(\frac{l-(c'+x)}{l}\right)$ which are respectively $=\frac{1}{2}\cos i\pi.\sin i\pi.\left(\frac{c'-x}{l}\right)-\frac{1}{2}\cos i\pi.\sin i\pi.\left(\frac{c'+x}{l}\right)$.

(u) By substituting these values in equation (4) there results $v=\frac{1}{l}(hc+h'c')-\frac{1}{\pi}(h-h')\left[\pi.\left(\frac{2l-c'-x}{2l}\right)-\pi.\left(\frac{c'-x}{2l}\right)\right]=$ (by substituting $c$ for $l-c'$, and remarking that $l=c+c'$) $h'$.

(v) When $x=l$, then $\cos\frac{i\pi x}{l}=\cos i\pi=(-1)^i$, consequently the value of $v$ is that given in the text.

(x) This appears at once from the consideration $\sin\frac{i\pi c}{l}\cos\frac{i\pi x}{l}=\frac{1}{2}\sin\frac{i\pi(c+x)}{l}+\frac{1}{2}\sin\frac{i\pi(c-x)}{l}$, and that $\cos i\pi=(-1)^i$.

(y) In the second case of No. 495, when A is fixed and B is free, the first member of the value of $u$ vanishes by hypothesis, and the second becomes by substituting $-k$ for $\phi'x=$

$$-\frac{4k}{\pi a}\,\Sigma\left(\int_c^l \sin\frac{(2i-1)}{2l}\pi x'dx'\right)\frac{1}{2i-1}\sin\frac{(2i-1)}{2l}\pi x\sin\frac{(2i-1)}{2l}\pi at$$
$$=\frac{8kl}{\pi^2 a}\,\Sigma\cos\frac{(2i-1)}{2l}\pi.(l-c)\,\frac{1}{(2i-1)^2}\sin\frac{(2i-1)}{2l}\pi x.\sin\frac{(2i-1)}{2l}\pi at$$
$$=\left(\text{as }\cos\frac{(2i-1)}{2l}.\pi(l-c)=(-1)^{2i-1}\cos\frac{(2i-1)}{2l}\pi c\right),$$

the value of $u$ given in the text.

($z$) In equation (7) of No. 326, when $x$ is substituted for $\phi x$, and then differenced with respect to $x$, it becomes

$$dx=\frac{2}{l}\,\Sigma\left(\int_0^l x'\sin\frac{(2i-1)}{2l}\pi x'.dx'\right)\frac{(2i-1)}{2l}\pi dx.\cos\frac{(2i-1)}{2l}\pi x.$$

Now the integral of $x'.\sin\frac{(2i-1)}{2l}\pi x'dx'=\frac{-2l}{(2i-1)\pi}x'.\cos\frac{(2i-1)}{2l}\pi x'+\frac{4l^2}{(2i-1)^2\pi^2}\sin\frac{(2i-1)}{2l}\pi x'$, which, when taken between the limits $l$ and 0, becomes

$$\frac{4l^2}{(2i-1)^2\pi^2}.\sin(2i-1)\frac{\pi}{2}=\frac{4l^2}{(2i-1)^2\pi^2};$$

consequently, we shall have

$$dx=\frac{2}{l}\,\Sigma\frac{2l}{(2i-1)\pi}.\cos\left(\frac{2i-1}{2l}\right)\pi x dx,$$

that is, dividing by $dx$, and substituting $\theta$ for $\frac{\pi x}{2l}$, we have

$$1=\frac{4}{\pi}.\,\Sigma\frac{1}{2i-1}.\cos(2i-1).\theta,$$

therefore substituting their different values for $i$ we obtain equation (6).

($a'$) By substituting $l-c'$ for $c$, $\sin\frac{(2i-1)\pi.(x+c)}{2l}=\sin\frac{(2i-1)\pi.(l+x-c')}{2l}=\sin(2i-1)\frac{\pi}{2}.\cos\frac{(2i-1)\pi.(x-c')}{2l}$ for $\cos(2i-1)\frac{\pi}{2}=0$.

($b'$) $\mathrm{Cos}\frac{(2i-1)\pi.(2l-x-c')}{2l}=\cos(2i-1)\pi.\cos\frac{(2i-1)\pi.(x+c')}{2l}=-\cos\frac{(2i-1)\pi(x+c')}{2l}.$

*Notes to Paragraph IV.*

(*a*) As the variables are by hypothesis independent of each other, it is evident that when $L = 0$, is developed, as pointed out in the text, the coefficients of each of these variables may be put separately equal to cipher, and they must consequently contain one independent variable less than $L = 0$, the given equation of partial differences.

(*b*) If we suppose in equation (a), that $\theta = t - h$, and also that the coefficients P, Q, R, &c. are functions of $x$, then we have

$$u = P(t-h)^{\alpha} + Q.(t-h)^{\beta} + R.(\theta-h)^{\gamma} + \&c.,$$

and, by substituting for $u$ in equation (b), we obtain

$$\alpha P.(t-h)^{\alpha-1} + \beta.Q.(t-h)^{\beta-1} + \gamma R.(t-h)^{\gamma-1} + \&c. =$$

$$a\frac{d^2P}{dx^2}(t-h)^{\alpha} + a.\frac{d^2Q}{dx^2}(t-h)^{\beta} + \&c.;$$

now in order that these two series may be identical, we must have

$$\alpha = 0,\ \beta = 1,\ \gamma = 2,\ \&c.,\ \text{and } Q = a\frac{d^2P}{dx^2},\ 2R = a\frac{d^2Q}{dx^2}, 3S = a\frac{d^2R}{dx^2},$$

&c.; and it is evident from inspection, that all the coefficients depend on the first P, which alone remains independent, if we call it $\phi x$, then $Q = \frac{ad^2\phi x}{dx^2}$, $R = \frac{a^2d^4\phi x}{1.2.dx^4}$, and so on; hence we obtain the value of $u$ furnished by equation (c).

If now we suppose $\theta = (x-h)$, and ∵ the coefficients of series (a), namely, P, Q, R, &c. to be functions of $t$, we shall have

$$u = P(x-h)^{\alpha} + Q(x-h)^{\beta} + R(x-h)^{\gamma} + \&c.,$$

and by substituting them in equation (b), we obtain

$$\frac{dP}{dt}(x-h)^{\alpha} + \frac{dQ}{dt}(x-h)^{\beta} + \frac{dR}{dt}(x-h)^{\gamma} + \&c. = a\alpha(\alpha-1)P(x-h)^{\alpha-2}$$

$$+ a.\beta.(\beta-1).Q.(x-h)^{\beta-2} + a.\gamma.(\gamma-1)R(x-h)^{\gamma-2} + \&c.;$$

since the exponents $\alpha$, $\beta$, $\gamma$, $\delta$, &c. constitute always an increasing series, in order that these two series may be identical, the first term of the second must disappear, consequently we must have either $\alpha = 0$, or $\alpha = 1$; in the first case, the other exponents $\beta$, $\gamma$, $\delta$, &c. must be the even numbers 2, 4, 6, &c., and in the second case, they

must be the odd numbers 3, 5, 7, &c.; therefore in the first case we have

$$\frac{dP}{dt} = 1.2.aQ,\ \frac{dQ}{dt} = 3.4.aR,\ \frac{dR}{dt} = 5.6.as;$$

and in the second,

$$\frac{dP}{dt} = 2.3.aQ,\ \frac{dQ}{dt} = 4.5.aR,\ \frac{dR}{dt} = 6.7.as, \&c.;$$

consequently there will result for $u$ two series, one of which proceeds according to the even powers of $(x-h)$, and the other according to the odd powers; and in each of them the first coefficient P remains indeterminate, and all the other coefficients may be expressed in terms of P, which in this case is a function of $t$, ∵ if in the first case it be called $\psi t$, and in the second $\Psi t$, we shall have, by making $u$ equal to the sum of the two series, equation (d) of the text.

(*c*) By Taylor's theorem, we have

$$u = U + \left(\frac{x-h}{1}\right).U' + \left(\frac{x-h}{1.2}\right)^2 U'' + \left(\frac{x-h}{1.2.3}\right) U''' + \&c.,$$

in which U is the value of $u$ when $x = h$, $U' = \frac{du}{dx}$, $U'' = \frac{d^2u}{dx^2}$, &c. on the same hypothesis; now from the equation (b) we can obtain the values of $\frac{d^2u}{dx^2}, \frac{d^4u}{dx^4}$, &c. when $x = h$, in functions of $t$, consequently it is easy to conceive that the resulting value of $u$ may assume the form

$$u = \psi t + \frac{(x-h)^2}{1.2}.\frac{d\psi t}{dt} + \frac{(x^4-h)^4}{1.2.3.4}\frac{d^2\psi t}{dt^2} + \&c. + \frac{(x-h)}{1}\Psi t + \frac{(x-h)^3}{1.2.3}\frac{d\Psi t}{adt} + \frac{(x-h)^5}{1.2.3.4.5}\frac{d^2\Psi t}{a^2dt^2}, \&c.$$

which when the constant $h$ is made equal to cipher, i. e. when the series is developed according to powers of $x$, becomes series (d).

(*d*) In fact if in series (d) we assume

$$\psi t = A + \frac{Bt}{1} + \frac{Ct^2}{1.2} + \frac{Dt^3}{1.2.3} + \&c., \text{ and } \Psi t = A' + B't + \frac{C't^2}{1.2} + \frac{D't}{1.2.3} + \&c.,$$

by taking the values of $\frac{d\psi t}{dt}$, $\frac{d^2\psi t}{dt^2}$, &c., $\frac{d\Psi t}{dt}$, $\frac{d^2\Psi t}{dt^2}$, &c., and arranging them into series proceeding according to the powers of $x$, we can obtain the series ($c$).

($e$) By multiplying both sides of this equation by $dt$, there results

$$\frac{d\text{P}}{\text{P}} = a\alpha^2 dt, \text{ and } \therefore \frac{\log \text{P}}{\log \text{A}} = a\alpha^2 t, \text{ and } \therefore \text{P} = \text{A}e^{a\alpha^2 t}, \text{ \&c.}$$

($f$) This is evident by substituting for $e^{-\omega^2}$ its value $1 - \frac{\omega^2}{1} + \frac{\omega^4}{1.2} -$ &c., for then $e^{-\omega^2}\,\omega^{2n-1} = \omega^{2n-1} - \omega^{2n+1} + \omega^{2n+3} - \omega^{2n+5} +$ &c., and when these terms are respectively multiplied by $d\omega$, and integrated between the limits $\infty - \infty$, each of the terms is cipher, $\therefore \int_{-\infty}^{\infty} e^{-\omega^2}\,\omega^{2n-1}\,d\omega = 0$.

($g$) By performing this differentiation with respect to $g$, and dividing by $dg$, we obtain $\int_{-\infty}^{\infty} e^{-g\omega^2}\,\omega^2 d\omega = \frac{1}{2}\frac{k}{g^{\frac{3}{2}}}$, $\int_{-\infty}^{\infty} e^{-g\omega^2}\omega^4 d\omega = \frac{1.3}{2^2}\frac{k}{g^{\frac{5}{2}}}$, $\int_{-\infty}^{\infty} e^{-g\omega^2}\,\omega^6 d\omega = \frac{1.3.5}{2^3}\cdot\frac{k}{g^{\frac{7}{2}}}$, &c., consequently, the value of the $n^{th}$ term when $g = 1$, is that given in the text; this formula evidently gives $k = \frac{2^n}{1.3.5\ldots.2n-1}\int_{-\infty}^{\infty} e^{-\omega^2}\,\omega^{2n} d\omega$, where $n$ may be any positive integer.

($h$) If in formula (c) we suppose each term to be multiplied and divided by equal quantities, namely, $\frac{2n}{1.3.5\ldots.2n-1}\int_{-\infty}^{\infty} e^{-\omega^2}\,\omega^{2n} d\omega$ and $k$, we shall evidently obtain by concinnating

$$u = \frac{1}{k}\int_{-\infty}^{\infty}\left(\phi x + \frac{4\omega^2}{1.2}at\frac{d^2\phi x}{dx^2} + \frac{16\omega^4 a^2 t^2 d^4\phi x}{1.2.3.4.dx^4} + \text{\&c.}\right) e^{-\omega^2}\,d\omega,$$

but in consequence of what has been established in note ($f$), namely, that $\int_{-\infty}^{\infty} e^{-\omega^2}\,\omega^{2n-1}\,d\omega = 0$, we may evidently introduce in this value of $u$ the terms $2\sqrt{at}\,\frac{d\phi x}{dx}\int_{-\infty}^{\infty} e^{-\omega^2}\,\omega d\omega$, $8at\sqrt{at}\,\frac{d^3\phi x}{dx^3}\int_{-\infty}^{\infty}$

$e^{-\omega^2}\omega^3 d\omega$, &c., for they are respectively equal to cipher; consequently, the value $u$ may assume the form given in text.

(*i*) As $\int e^{-\omega^2}\omega d\omega = -\frac{e^{-\omega^2}}{2}$ by partially integrating the value of $\frac{du}{dt}$ we obtain $\frac{du}{dt} =$

$$-\frac{a}{k}.e^{-\omega^2}\varphi'(x+2\omega\sqrt{at})\frac{1}{2\sqrt{at}}+\frac{a}{k}\int_{-\infty}^{\infty}e^{-\omega^2}.\phi''(x+2\omega\sqrt{at})\,d\omega,$$

consequently, as the first term of the second member of this equation vanishes, we have the value of $\frac{du}{dt}$ equal to that of $a\frac{d^2u}{dx^2}$.

(*k*) When the radii of the interior and exterior surfaces of the cylindrical slices are respectively $r$ and $r+dr$, the base of the slice will be equal to $2\pi rdr$, and this multiplied into their height $z$, or its equivalent $e^{-r^2}$, gives the volume of the slice equal to $2\pi e^{-r^2}rdr$, and this when integrated between the limits 0 and $\infty$, gives the volume of the entire surface of revolution equal to $2\pi\int_0^{\infty}e^{-r^2}rdr = \pi$, for the value of $e^{-r^2}rdr$ when $r=\infty$ is 0, and when $r=0$, it is equal to $-\frac{1}{2}$.

This value of $k$ may be also obtained from the consideration that if $e^{-\omega^2}d\omega$ be expanded into a series, and if each of the terms be integrated between the limits $\infty$, $-\infty$, the result will be a series equal to the known value of $\sqrt{\pi}$.

(*l*) $e^{-\omega^2}=1-\frac{\omega^2}{1}+\frac{\omega^4}{1.2}-\frac{\omega^6}{1.2.3}$, $\sin 2a\omega = 2a\omega - \frac{(2a\omega)^3}{1.2.3}+$ &c., therefore,

$$e^{-\omega^2}.\sin 2x\omega d\omega=\left(1-\frac{\omega^2}{1}+\frac{\omega^4}{1.2}-\frac{\omega^6}{1.2.3}+\&c.\right)\left(\frac{2a\omega}{1}-\frac{(2a\omega)^3}{1.2.3}+\&c.\right)d\omega,$$

$\because$ when these two series are multiplied together, and then integrated between the limits $\infty$, $-\infty$, the result will be evidently cipher. From similar considerations it may be shown that

$$\int_{-\infty}^{\infty}e^{-\omega^2}\cos 2\alpha\omega d\omega = 2\int_0^{\infty}e^{-\omega^2}d\omega\cos 2\alpha\omega d\omega.$$

(*m*) As $\cos(x+2a\omega)=\cos x.\cos 2a\omega - \sin x\sin 2a\omega$, by substituting this expression in the value of $u$, as the function of $x$ can be

taken from under the sign of integration, and $\int_{-\infty}^{\infty} e^{-\omega^2} \sin 2\alpha\omega d\omega = 0$, we have evidently $u = \frac{2\cos x}{\sqrt{\pi}} \int_0^{\infty} e^{-\omega^2} \cos 2\alpha\omega d\omega = e^{-\alpha^2} \cos x$, ∵

$$\int_0^{\infty} e^{-\omega^2} \cos 2\alpha\omega d\omega = \frac{\sqrt{\pi}}{2} e^{-\alpha^2}$$

($n$) Since $\cos 2\alpha\omega = \frac{e^{2\alpha\omega\sqrt{-1}} + e^{-2\alpha\omega\sqrt{-1}}}{2}$, when $\alpha\sqrt{-1}$ is substituted for $\alpha$, the preceding equation becomes

$$\int_0^{\infty} e^{-\omega^2} \left(\frac{e^{2\alpha\omega} + e^{-2\alpha\omega}}{2}\right) d\omega = \frac{\sqrt{\pi}}{2} e^{\alpha^2}.$$

($o$) As by supposition L $= 0$, is not higher than the second degree, when series ($g$) is substituted in place of $u$ in L $= 0$, the result must be the value of L given in the text; and as the equation L $= 0$ obtains whatever may be the value of $t$, the coefficients of $e^{\alpha t}$, $e^{\beta t}$, $e^{\gamma t}$, &c., must be separately equal to cipher, from which it appears that equations (h) obtain. +

($p$) By making this substitution we obtain

$$u = \left(\frac{p}{2} \pm \frac{p'}{2}\sqrt{-1}\right) e^{\pm\lambda t\sqrt{-1}} - \left(\frac{q}{2} \pm \frac{q'}{2}\sqrt{-1}\right) e^{\pm\lambda t\sqrt{-1}} + \text{\&c.}$$

$$= \frac{p}{2} \cdot \left(e^{(\lambda\sqrt{-1}} + e^{-\lambda\sqrt{-1})}\right)^t + \frac{p'}{2}\sqrt{-1} \cdot \left(e^{\lambda\sqrt{-1}} - e^{-\lambda\sqrt{-1}}\right)^t$$

$$+ \frac{q}{2} \cdot \left(e^{\mu\sqrt{-1}} + e^{-\mu\sqrt{-1}}\right)^t + \frac{q'}{2}\sqrt{-1}\left(e^{(\mu\sqrt{-1}} - e^{\mu\sqrt{-1})}\right)^t + \text{\&c.},$$

$=$ (by substituting for the exponentials their values) $p \cos \lambda t + q \cos \mu t +$ &c., $+ p' \sin \lambda t + q' \sin \mu t +$ &c.

($q$) When $\lambda, \mu, \nu$, &c., are real, then $\alpha, \beta, \gamma$, &c., are imaginary, and *vice versa*. $\alpha = \pm\lambda\sqrt{-1}$

($r$) In equation (i) when $t = 0$, in whch case $u = f(x, y, z)$ by hypothesis, we must have (as $\cos \lambda t$, $\cos \mu t$, &c. are respectively equal to unity, and $\sin \lambda t$, $\sin \mu t$, &c., are respectively equal to cipher) $f(x, y, z,) = p + q + r +$ &c., in a similar manner it may be shown that when we take the value of $\frac{du}{dt}$ in this equation and then suppose $t = 0$, that we will have F $(x, y, z)$ equal to $\lambda p' + \mu q' + \nu r' +$ &c.

*Notes to Paragraph V.*

(*s*) As $10 = 9 + 1 \dfrac{2}{\sqrt{10}} = \dfrac{2}{\sqrt{9+1}} = \left(\text{as } (9+1)^{-\frac{1}{2}} = \dfrac{1}{3} - \dfrac{1}{2.27}\right)$ $q.p.$, $\dfrac{2}{3} - \dfrac{1}{27}$, which differs from the result obtained by Chladni by a $\frac{1}{27}$th part q. p.

(*u*) See the different cases which have been discussed in No. 320, the quantity $b^2 = \dfrac{\beta}{\gamma\omega}$ of that number.

(*v*) Since $p$ is the weight of the rod, $\dfrac{p}{g}$ is equal to the quantity of matter, and $\omega$ being the area of its normal section $\dfrac{p}{gl\omega}$ is equal to the density; now as $b^2 = \dfrac{\beta}{\gamma.\omega}$ by substituting $\alpha \int_{-k}^{k} vu^2du$ for $\beta$, and $\dfrac{p}{gl}$ for $\omega\gamma$ we obtain equation (5).

(*x*) In this case $v$ is constant and equal to the base of the rectangle, and $\because \omega = 2\varepsilon v$, likewise as the centre of gravity is equally distant from the opposite sides of the rectangle, we have $k = k' = \varepsilon$, and the integral $\int_{-k}^{k} vudu^2$, becomes $v \int_{-\varepsilon}^{\varepsilon} u^2du = \frac{2}{3}v\varepsilon^3 = \omega h^2$, therefore, $\dfrac{\varepsilon^2}{3} = h^2$, and $b = ah = \dfrac{a\varepsilon}{\sqrt{3}}$.

(*y*) In the case of a cylindrical rod by substituting for $v$, $k$, $k'$, $\omega h^2 = \int_{-\varepsilon}^{\varepsilon} 2\sqrt{\varepsilon^2 - u^2}.u^2du$, now if we put $u = \varepsilon y$, the quantity $2\sqrt{\varepsilon^2 - u^2}.u^2du$ may be made to assume the form $2\varepsilon^4\left(\dfrac{y^2dy}{\sqrt{1-y^2}} - \dfrac{y^4dy}{\sqrt{1-y^2}}\right)$ the integral of these two expressions are respectively equal to

$$2\varepsilon^4.(-\tfrac{1}{2}y\sqrt{1-y^2} + \tfrac{1}{2}(\text{arc sin} = y)), 2\varepsilon^4\left[\left(\tfrac{1}{4}y^3 + \frac{1.3}{2.4}y\right)\right]\sqrt{1-y^2} - 2\varepsilon^4.$$
$$\frac{1.3}{2.4}\text{ arc sin} = y.$$

Now when $u = \varepsilon$, $y = 1$, $\because$ when the preceding integrals are taken between the limits 1, $-1$, the parts multiplied by $\sqrt{1-y^2}$ vanish, and the circular parts become respectively $\varepsilon^4\pi$, $-\dfrac{3}{2}\,\varepsilon^4\dfrac{\pi}{2}$, conse-

quently the value of $\omega h^2$ taken between $\varepsilon$, $-\varepsilon$, is $\varepsilon^4\frac{\pi}{4}$, $\because$ as $\omega = \pi\varepsilon^2$, $h = \frac{\varepsilon}{2}$, and $b = ah = \frac{a\varepsilon}{2}$.

($z$) In the case of a triangle, as its area is equal to the base multiplied by half the height which is equal to $2\varepsilon$, we must have $\omega = \lambda\varepsilon$, the integral of $vu^2du$ is supposed to be taken between the limits $k$, $-k'$, reckoning from the centre of gravity as the origin, $\because$ as its distance from the vertex is two-thirds, and from the base one-third of the height $2\varepsilon$, we must, in the *first* case, have $k = \frac{2}{3}\varepsilon$, $k' = \frac{4}{3}\varepsilon$, and in the second $k = \frac{4}{3}\varepsilon$, $k' = \frac{2}{3}\varepsilon$.

($a'$) Therefore in this first case we must have $v : \lambda :: \frac{4\varepsilon}{3} + u : 2\varepsilon$ $\because vu^2du = \frac{l}{2\varepsilon}\left[\frac{4\varepsilon u^2du}{3} + u^3du\right]$ of which the integral is

$$\frac{l}{2\varepsilon}\left[\frac{4\,\varepsilon u^3}{9} + \frac{u^4}{4}\right]$$

which taken between the limits $\frac{2\varepsilon}{3}$ and $\frac{4\varepsilon}{3}$ becomes equal to

$$\frac{2l}{9}\cdot\left(\frac{2^3}{3^3} + \frac{4^3}{3^3}\right)\varepsilon^3 + \frac{l}{2\,\varepsilon}\cdot\left(\frac{2^4}{4.3^4} - \frac{4^4}{4.3^4}\right)\varepsilon^4,$$

which after all reductions is equal to $\frac{2\lambda\varepsilon^3}{9} = \omega h^2$, $\because$ as $\omega = \lambda\varepsilon$, we have $h^2 = \frac{2\,\varepsilon^2}{9}$, and consequently $b = \frac{a\varepsilon\sqrt{2}}{3}$.

($b'$) When the convexity is turned downwards, we have $vu^2du = \frac{l}{2\varepsilon}\cdot\left(\frac{2\,\varepsilon u^2du}{3} + u^3du\right)$ of which the integral is $\frac{l}{2\varepsilon}\cdot\left(\frac{2\,\varepsilon u^3}{9} + \frac{u^4}{4}\right)$, and when taken between the limits $k = \frac{4\,\varepsilon}{3}$, $k' = \frac{2\,\varepsilon}{3}$ it becomes

$$\frac{l}{9}\left[\frac{4^3}{3^3} + \frac{2^3}{3^3}\right]e^3 + \frac{l}{2.4.\varepsilon}\cdot\left(\frac{4^4}{3^4} - \frac{2^4}{3^4}\right)\varepsilon^4,$$

which also, after all reductions, becomes equal to $\frac{2\lambda\varepsilon^3}{3}$, $\because h^2 = \frac{2}{3}\varepsilon^2$, and $b = a\varepsilon\sqrt{\frac{2}{3}}$.

($c'$) As $p$ and $q$ are functions of $x$, if this equation be differenced with respect to $x$, we obtain $\frac{d^4y}{dx^4} = \frac{d^4p}{dx^4}\cdot\sin m^2bt + \frac{d^4q}{dx^4}\cos m^2bt$; and

in like manner, if this equation be differenced twice successively with respect to $t$, we obtain $\frac{d^2y}{dt^2} = - m^4b^2p \sin m^2bt - m^4bq^2 \cos m^2 bt$; now if we suppose $m^2bt = \frac{\pi}{2}$, we have $\frac{d^4y}{dx^4} = \frac{d^4p}{dx^4}$, and $\frac{d^2y}{dt^2} = -m^4b^2p$ $=$ by equation (1) $- b^2 \frac{d^4p}{dx^4}$, $\therefore m^4p = \frac{d^4p}{dx^4}$.

($d'$) For the integration of this equation, see examples of the differential and integral calculus, page 393.

($e'$) By differentiating the value of $p$ twice successively, we obtain

$$\frac{d^2p}{dx^2} = - \text{A}m^2 \sin x - \text{A}'m^2 \cos mx + \tfrac{1}{2}\text{B}m^2(e^{mx} - e^{-mx}) + \tfrac{1}{2}\text{B}'m^2(e^{mx} + e^{-mx})$$

which because it is equal to cipher when $x = 0$, gives $\text{A}' = \text{B}'$, in like manner by taking the value of $\frac{d^3p}{dx^3}$ when $x = 0$, we obtain $\text{A} = \text{B}$; now as the values of $\frac{d^2p}{dx^2}, \frac{d^3p}{dx^3}, \frac{d^2q}{dx^2}$, &c., are also cipher when $l$ is put for $x$ in these values, we can obtain the equation

$$\text{A}\,(2 \sin ml - e^{ml} + e^{-ml}) = \text{A}'(e^{ml} + e^{-ml} - 2 \cos ml), \text{ \&c.}$$

($f'$) By expanding this product we obtain

$$4 \sin^2 ml - e^{2ml} + 2 - e^{-2ml} + 4 \cos m^2l - 4 \cos ml.(e^{ml} + e^{-ml}) + e^{2ml}$$
$$+ 2 + e^{-2ml} = 0, \text{ i. e. } 8 - 4 \cos ml\,(e^{ml} + e^{-ml}) = 0.$$

($g'$) If in the expression for $y$ we substitute for $p$ and $q$, their values, and observe to put for A, A$'$, C, &c., the quantities to which they are equal, we shall obtain equation (b); now the value of

$$\frac{d^2\text{X}}{dx^2} = (e^{ml} + e^{-ml} - 2 \cos ml)\,(- m^2 \sin mx + \tfrac{1}{2} m^2\,(e^{mx} - e^{-mx}) +$$
$$(2 \sin ml - e^{ml} + e^{-ml})\,(-m^2 \cos mx + \tfrac{1}{2} m^2\,(e^{mx} + e^{-mx}),$$

but when $x = 0$, this expression is evidently equal to cipher, and when $x = l$, it becomes

$$[(e^{ml} + e^{-ml}) - 2 \cos ml]\,[-m^2 \sin ml + \tfrac{1}{2} m^2(e^{ml} - e^{-ml}) +$$
$$(2 \sin ml - e^{ml} + e^{-ml})\,(- m^2 \cos ml + \tfrac{1}{2} m^2\,(e^{ml} + e^{-ml}),$$

which is also equal to cipher; in the same manner it may be shown, that $\frac{d^3\text{X}}{dx^3} = 0$, in the same circumstances.

$(h')$ $\frac{d^3\mathrm{x}}{dx^3}=(e^{ml}+e^{-ml}-2\cos ml)\left[-m^3\cos mx+\tfrac{1}{2}m^3(e^{mx}+e^{-mx})+\right.$

$(2\sin ml-e^{ml}+e^{-ml})\,(m^3\sin mx+\tfrac{1}{2}m^3.(e^{mx}-e^{-mx})$;

and, therefore,

$$\frac{d^4\mathrm{x}}{dx^4}=(e^{ml}+e^{-ml}-2\cos ml)\,(m^4\sin mx+\tfrac{1}{2}m^4\,(e^{mx}-e^{-mx})+$$
$$(2\sin ml-e^{ml}+e^{-ml})\,(m^4.\cos mx+\tfrac{1}{2}m^4\,(e^{mx}+e^{-mx}),$$

which is evidently equal to $m^4\mathrm{x}$.

$(i')$ If in equation (a) we substitute $\frac{e^{ml\sqrt{-1}}+e^{-ml\sqrt{-1}}}{2}$ for $\cos ml$, then it is immediately evident, that if $m$ is a root of equation (a), so will $-m$ and $\pm m\sqrt{-1}$.

$(k')$ It is evident from the value of x that it does not involve $t$, therefore, $\frac{d^2\mathrm{x}y}{dt^2}=\frac{\mathrm{x}d^2y}{dt^2}$, consequently, when equation (1) is multiplied by $\mathrm{x}dx$, and then integrated between the limits $l$, $o$, it may be made to assume the form of this equation.

$(l')$ $\int_0^l \mathrm{x}\frac{d^4y}{dx^4}=\mathrm{x}\frac{d^3y}{dx^3}-\int\frac{d\mathrm{x}}{dx}\cdot\frac{d^3y}{dx^3}dx$, and this last quantity is equal to $\frac{d\mathrm{x}}{dx}\cdot\frac{d^2y}{dx^2}-\int\frac{d^2\mathrm{x}}{dx^2}\cdot\frac{d^2y}{dx^2}dx$, and this quantity is equal to $\frac{d^2\mathrm{x}}{dx^2}\cdot\frac{dy}{dx}-\int\frac{d^3\mathrm{x}}{dx^3}\cdot\frac{dy}{dx}dx,=\frac{d^3\mathrm{x}}{dx^3}\cdot y-\int\frac{d^4\mathrm{x}}{dx^4}ydx$.

$(m')$ Since by equation (4), $y=\varphi x$, $\frac{dy}{dt}=\varphi' x$, when $t=0$, if in the equation $\int_0^l \mathrm{x}ydx=\mathrm{H}\cos m^2bt+\mathrm{H}'\sin m^2bt$, and in its differential with respect to $t$, we suppose $t=0$, we must have, as is stated,

$$\int_0^l \mathrm{x}\varphi x dx=\mathrm{H},\quad \int_0^l \mathrm{x}\varphi' x dx=m^2b\mathrm{H}';$$

now if in the first member of equation (e), we substitute for $y$ its value derived from formula $(b)$, it is evident that as the second member of equation (e) contains only $\sin m^2bt$ $\cos m^2bt$, if such a root as $m'$ is stated to be, occurs in the value of $y$, and if $\mathrm{x}'$ denotes what x becomes when $m$ is changed into $m'$, we must have $\int_0^l \mathrm{x}\mathrm{x}'dx=0$; and in the case of $m=m'$, when formula (b) is substituted for $y$ in equation (e), then

as this equation obtains for all values of $t$, we must have the coefficients of the corresponding circular functions equal, i. e.

$$\mathrm{E}\int_0^l \mathrm{X}^2 dx. = \frac{1}{bm^2}\int_0^l \mathrm{X}\phi' x dx.,\ \&c.$$

($n'$) In differentiating the second equation ($h$), it is evident that the differentials of its second number must, in this case, be cipher, and when we obtain a differential of the form $\Sigma\left(\int_0^l \frac{\mathrm{X}\phi' x dx}{\int_0^l \mathrm{X}^2 dx}\cdot\frac{d^{4i}\mathrm{X}}{dx^{4i}}\right.$, we can by means of equation (d) substitute $m^{4i}\mathrm{X}$ for $\frac{d^{4i}\mathrm{X}}{dx^{4i}}$, and thus obtain the expression in the text.

($o'$) If we substitute for $\sin m^2bt$, its value in a series, it is equal to $\frac{m^2bt}{1}-\frac{m^6b^3t^3}{1.2.3}+\frac{m^{10}.b^5t^5}{1.2.3.4.5}-$ &c., $\because$ when this is multiplied into $\Sigma.\left(\frac{\int_0^l \mathrm{X}\phi' dx}{\int_0^l \mathrm{X}^2 dx}\cdot\frac{\mathrm{X}}{m^2b}\right.$, the first term is equal to $t\Sigma\left(\frac{\int_0^l \mathrm{X}\phi' x dx.}{\int_0^l \mathrm{X}^2 dx}\right)\mathrm{X}$, and the coefficients of the subsequent powers of $t$ are of the form $\Sigma\left(\frac{\int_0^l \mathrm{X}\phi' x dx}{\int_0^l \mathrm{X}^2 dx}\right.\mathrm{.X}m^{4i}$, and therefore equal to cipher.

($p'$) Since $e^{\pm ml}=1\pm\frac{ml}{1}+\frac{m^2l^2}{1.2}\pm\frac{m^3l^3}{1.2.3}+$ &c., $\sin mx=\frac{mx}{1}+\frac{m^3x^3}{1.2.3}+$ &c., $\cos mx = 1-\frac{m^2x^2}{1.2}+$ &c., if their values in series be put for $e^{ml}$, $e^{-ml}$, $e^{mx}$, $e^{-mx}$, $\sin. mx$, $\cos mx$, &c., in the expression given for X in page 305, we will obtain by restricting ourselves to the third power of $m$,

$$\mathrm{X}=\left(1+\frac{ml}{1}+\frac{m^2l^2}{1.2}+1-ml+\frac{m^2l^2}{1.2}-2+\frac{2.m^2l^2}{1.2}+\&c.\right).\left(mx+\tfrac{1}{2}-\tfrac{1}{2}+\tfrac{1}{2}mx+\tfrac{1}{2}mx+\&c.\right)+\left(\frac{2ml}{1}-\frac{2m^3l^3}{1.2.3}-1-\frac{ml}{1}-\frac{m^2l^2}{1.2}-\frac{m^3l^3}{1.2.3}+1-\frac{ml}{1}+\frac{m^2l^2}{1.2}\quad\frac{m^3l^3}{1.2.3}\right).\left(1-\frac{m^2x^2}{1.2}+\tfrac{1}{2}+\frac{mx}{2}+\tfrac{1}{2}\frac{m^2x^2}{1.2}+\tfrac{1}{2}-\tfrac{1}{2}mx+\tfrac{1}{2}.\frac{m^2x^2}{1.2}\right),$$

which, by obliterating quantities that destroy each other, and concinnating, becomes $2m^2l^2.2mx-\frac{4m^3l^3}{3}=4m^3l^2.\left(x-\frac{l}{3}\right)$, now when this

expression is substituted in equation (g), it becomes (by taking $4m^3l^2$ from under the sign of integration) $\left(\frac{4m^3l^2}{3}\right)^2\left[\int_0^l(3x-l)\,\phi x\,dx+\int_0^l\frac{(3x-l)\,\phi' x\,dx\,m^2bt}{m^2b}\right](3x-l)$ divided by $(4m^2l^2)^2\int_0^l\left(x^2dx-\frac{4}{3}lxdx+\frac{l^2}{9}dx\right)$, and this divisor when integrated between the limits $l$, $o$, is equal to $\left(\frac{4m^2l^2}{3}\right)^2.l^3$; consequently, the value of the term in question will be that given in the text. The reason why we restricted ourselves to the third power of $m$ in the expansion of the exponential and circular functions was, because if higher powers were retained, they would not be obliterated by corresponding powers of $m$ in the value of $\int_0^l x^2dx$, consequently, when $m$ is supposed to be infinitely small they would vanish.

($q'$) In the case of an *entire* vibration, in which the vibrating body returns to the point from which it set out, we must have $\frac{\lambda^2}{l^2}b\tau=2\pi \because \tau=\frac{2\pi l^2}{b.\lambda^2}$, and as $b$ is the thickness, it is evident when $\lambda$ is given that $n=$ varies as $\frac{b}{l^2}$.

($r'$) In the case of a rectangle, if the base be $2\varepsilon'$ and height $2\varepsilon$, it is evident from the expression given for $b$ in page 302, that the values of $n$ will, every thing else being the same, be in the ratio of $\varepsilon:\varepsilon'$.

($s'$) When the normal section is a triangle, the value of $b^2$ in one case is $\frac{a\varepsilon\sqrt{2}}{3}$, and in the other $a\varepsilon\sqrt{\frac{2}{3}}$, so that in the two successive semi-vibrations the values of $\tau$ will be different; however, as it is evident that in these two vibrations, they interchange values, the entire vibrations will be always isochronous.

($t'$) In the preceding analysis the rod was assumed, as stated in page 300, to be free at its two extremities, here it is supposed to be firmly fixed at the extremity A, and free at the extremity B, in consequence of which we have $y=0$, and $\frac{dy}{dx}=0$, but this expresses in general the angle which the tangent at the point A makes with the axis of $x$, consequently when it is cipher, the mean filament must

coincide with the tangent at the point A; and in determining the constants A, B, &c. we must make use of the equations $x=0$, $y=0$, $\frac{dy}{dx}=0$, instead of equations (2).

($u'$) It is evident that neither $\delta$ nor $\delta'$ can surpass $\frac{\pi}{2}$, for if they did, then we would have, for instance, $\delta=\frac{\pi}{2}+\delta_1$, in which the expression for $\lambda$ could be reduced to a form in which $\delta$ did not surpass $\frac{\pi}{2}$; now it is evident that $\cos(\frac{1}{2}(2i+1)\pi\mp\delta)=\pm\sin\delta$, it follows that if in equation (a) we substitute for $ml$, we can obtain immediately the first equation (k).

($v'$) When $\delta=\frac{\pi}{2}$ we must have $e^{\frac{1}{2}(2i+1)\pi\mp\delta}+e^{-\frac{1}{2}(2i+1)\pi\pm\delta}=2$, consequently these exponents must be respectively equal to cipher, and as $\delta=\frac{\pi}{2}$ we must have $i=0$.

($x'$) By supposing $\delta=0$ in the second member of the first equation $k$, we obtain a value for $\sin\delta$, which is evidently only an approximation; but by substituting this approximate value in the second member of this equation, we obtain one still more accurate; now as it appears that the values of $\delta$ relative to $i=2$, $i=3$, &c., are less than 0,01765, it is evident, that the values of $\lambda$ will, as is stated in text, differ little from the $\frac{1}{2}(2i+1)\pi$, or the odd multiples of $\frac{\pi}{2}$. The least value of $\lambda$ taken into account corresponds to $i=1$, and $\because$ $\lambda=\frac{3}{2}\pi\mp\delta$.

($y'$) In second equation (k), if we approximate to the value of $\delta'$, by means of the expression for $\sin\delta'$ when $i=0$, we obtain the expression in the text; and, as in this case, $\delta'$ is not $=\frac{\pi}{2}$ when $i=0$, the least value of $\lambda'$ which gives the gravest tone is $\frac{1}{2}\pi+\delta'$; and as the other quantities which express the ratio of $n:n'$, are the same, we have $\frac{n'}{n}=\frac{\lambda'^2}{\lambda^2}=0,15715$; now if the values of $\delta'$ be determined, as in the former case, when $i=2$, $=3$, &c., it is found that they continually diminish, hence it follows that the corresponding values of $\lambda'$ will be $q.p$ odd multiples of $\frac{1}{2}\pi$.

($z'$) Since $a = n_1.2l$ and $b = ah$, we have $n = 2.(3,56082)\frac{hn_1}{l}$, and by substituting $\frac{\varepsilon}{2}, \frac{\varepsilon}{\sqrt{3}}$ respectively for $h$, we obtain the values of $n$ given in the text; it is evident from the expression $\int_{-k}^{k} vu^2 du = \omega h^3$, that, every thing else being the same, $h$ depends on the thickness.—See page 301, No. 520.

## CHAPTER IX.

($a$) See Nos. 122, 126, 128.

($b$) See Memoires De l'Academie Royale des Science, tome 1, une memoire sur la variation des constantes arbitraires par M. Poisson.

($c$) In this case $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}, \frac{d^2x'}{dt^2}$, &c., are respectively equal to ✓ cipher, and as from the values of L, L′, L″, &c. $\frac{d\text{L}}{dx} = \frac{x-a}{l}, \frac{d\text{L}}{dy} = \frac{y-b}{l}$, &c., it is evident that when $x = \alpha$, $y = \beta$, $z = \gamma$, the equations given in the text will result.

($d$) If $\alpha + u$, $\beta + v$, $\gamma + w$ be respectively substituted for $x, y, z$ in the value of L, there results

$$\zeta = \sqrt{(\alpha + u - a)^2 + (\beta + v - b^2) + (\gamma + w - c)^2} - \sqrt{(\alpha - a)^2 + (\beta - b)^2 + (\gamma - c)^2}$$

$$= \sqrt{(\alpha-a)^2+2(\alpha-a)u+(\beta-b)^2+2(\beta-b)v+(\gamma-c)^2+2(\gamma-c)w} - \sqrt{(\alpha-a)^2+(\beta-b)^2+(\gamma-c)^2}$$

$$= \sqrt{(\alpha-a)^2+(\beta-b)^2+(\gamma-c)^2} + \frac{(\alpha-a)u+(\beta-b)v+(\gamma-c)w}{l} - \sqrt{(\alpha-a)^2+(\beta-b)^2+(\gamma-c)^2}$$ ✓ [illegible]

when the squares and higher powers of $u, v, w$ are neglected.

($e$) Now in equations (4), we have in this case $\frac{d^2x}{dt^2} = \frac{d^2u}{dt^2}$, and $\frac{d\text{L}}{dx} = \left(\frac{\alpha + u - a}{l}\right)$, by substituting $a + u$ for $x$, consequently, by making similar substitutions for $y$, $z$, L′, &c., we obtain, as

$$\lambda = -\frac{gm\zeta}{\varepsilon}, \quad \lambda' = -\frac{gm\zeta'}{\varepsilon'}, \text{ \&c.,}$$

$$m\frac{d^2u}{dt^2} = -mg.\left(\frac{\alpha + u - a}{l\varepsilon}\right)\zeta - m'g.\left(\frac{\alpha + u - a'}{l'\varepsilon'}\right)\zeta'$$

$$-m''g\left(\frac{a+u-a''}{l''\varepsilon''}\right)\zeta''-\text{\&c.};$$

hence as $\zeta u$, $\zeta' u$, $\zeta'' u$, must be neglected, as is evident from the values of $\zeta$, $\zeta'$, $\zeta''$, &c., the value of $\frac{d^2u}{dt^2}$, given in the text, may be deduced.

(*f*) When $u$, $v$, $w$ are cipher, the accelerating forces, which in general are expressed by $\frac{d^2u}{dt^2}$, $\frac{d^2v}{dt^2}$, &c., vanish.

(*g*) From the first of these equations we obtain

$$\delta x=-\frac{\delta y\cos\mu+\delta z\cos\nu}{\cos\lambda};$$

and from the second,

$$\delta x=\frac{-y\delta y-z\delta z}{x};$$

hence a comparison of these two values of $\delta x$ gives the equation $\delta y.(y\cos\lambda-x\cos\mu)=\delta z.(x\cos\nu-z\cos\lambda)$; $\because$ $\delta y:\delta z::$ as these factors.—See No. 543.

(*h*) If in this equation their values be substituted for $\delta x$, $\delta y$, $\delta z$, $\delta x'$, &c., there will result

$$x'(z\cos\mu-y\cos\nu)\varepsilon+y'(x\cos\nu-z\cos\lambda)\varepsilon+z'(y\cos\lambda-x\cos\mu)\varepsilon$$
$$x(z'\cos\mu-y'\cos\nu)\varepsilon'+y(x'\cos\nu-z'\cos\lambda)\varepsilon'+z(y'\cos\lambda-x'\cos\mu)\varepsilon'=0,$$

which by concinnating can evidently be reduced to the expression in the text.

(*i*) By making this substitution there results

$$\Sigma m.(\text{A}-a)(z\cos\mu-y\cos\nu)\varepsilon+\Sigma m.(\text{B}-b)(x\cos\nu-z\cos\lambda)\varepsilon+\Sigma m(\text{C}-c)(y\cos\lambda-x\cos\mu)\varepsilon=0;$$

now as $\cos\lambda$, $\cos\mu$, $\cos\nu$, are the same for all the terms, we must have the quantities by which each of them is multiplied equal to cipher, as for instance, the quantity of which $\cos\nu$ is a factor is

$$\Sigma mx.(\text{B}-b)-\Sigma my.(\text{A}-a);$$

hence then we have

$$\Sigma m(x\text{B}-y\text{A})=\Sigma m(xb-ya),\ \text{\&c.}$$

(*k*) By substituting for $\delta x$, $\delta y$, in the expression

$$\sqrt{\frac{\delta x^2+\delta y^2+\delta z^2}{dt^2}},$$

it becomes, as $\delta z=0$,

$$= \sqrt{\frac{z^2\varepsilon^2(\cos^2\lambda + \cos^2\mu)}{dt^2}} = \sqrt{\frac{z^2\varepsilon^2(1-\cos^2\nu)}{dt^2}} = \frac{z\varepsilon\sin\nu}{dt}.$$

($l$) The expressions for $\delta x$, $\delta y$, $\delta z$, &c. are respectively grouped into quantities, whose respective multipliers are the increments of the several variables $u$, $v$, $w$, $u'$, &c.

($m$) It is evident, that as Q is constant, we can by a suitable variation of $u$, $v$, &c. cause this quantity to disappear; the values of $\alpha$, $\beta$, $\gamma$, &c. in this case belong to a state of equilibrium, because the accelerating forces vanish when $u = 0$, $v = 0$, &c.

($n$) $\frac{du}{dt} = \text{RN}\sqrt{\rho}\cos(t\sqrt{\rho} - r)$, $\frac{d^2u}{dt^2} = -\text{RN}.\rho.\sin(t\sqrt{\rho} - r)$, $\therefore$ if their values be substituted for $u\ \frac{d^2u}{dt^2}$, $v, \frac{d^2v}{dt^2}$, &c. in equation (a), there will result a common factor $\text{R}\sin(t\sqrt{\rho} - r)$, which may be struck out, and the terms on one side will be $\text{DN}\rho$, $\text{EN}'\rho$, $\text{FN}''\rho$, &c.; and on the other, GN, HN′, KN″, &c.

($o$) When $t\sqrt{\rho}$ is increased by $2\pi$, the value of $\sin(t\sqrt{\rho} - r)$ becomes the same as before, and the actual amplitude of the oscillations will be $(a\text{N} + b\text{N}' + c\text{N}'' + \&c.)\text{R}$.

($p$) In this case, in the expression for the amplitudes, the quantities $\sin\frac{i\pi x}{l}$, $\sin\frac{i\pi x'}{l}$, are the only terms which differ in these expressions, consequently the amplitudes are as these terms.

($q$) $\frac{du}{dt} = \text{RN}\sqrt{\rho}\cos(t\sqrt{\rho} - r)e^{-\omega t} - \omega\text{RN}.\sin(t\sqrt{\rho} - r)e^{-\omega t}$;

$$\frac{d^2u}{dt^2} = -\rho\text{RN}.\sin(t\sqrt{\rho} - r)e^{-\omega t} - \omega\sqrt{\rho}.\text{RN}\cos t(\sqrt{\rho} - r)e^{-\omega t}$$
$$+ \omega^2\text{RN}\sin(t\sqrt{\rho} - r)e^{-\omega t} - \omega\sqrt{\rho}\,\text{RN}\cos(t\sqrt{\rho} - r)e^{-\omega t};$$

now, if in equation (e), we substitute for $\frac{du}{dt}$ its value given here, as $\omega\text{D}'$, $\omega\text{E}'$, &c., are supposed to be neglected, the second term of the value of $\frac{du}{dt}$ must be neglected; in like manner, as $\omega^2$ is neglected, the third term of the value of $\frac{d^2u}{dt^2}$ may be omitted; and since this equation (e) always obtains, the terms multiplied by the sines must be identi-

cally equal, and likewise the terms multiplied by the cosines, this, therefore, gives at once

$$\text{D}'\text{N}.\sqrt{\rho}.\cos(t\sqrt{\rho}-r)\,e^{-\omega t}+\&\text{c}.,=-2\text{DN}\omega.\sqrt{\rho}.\cos(t\sqrt{\rho}-r)e^{-\omega t},$$

consequently,

$$\text{D}'\text{N}=-2\text{DN}\omega,\ \&\text{c}.;$$

$\omega$ must be positive, as is observed in the text, in order that when $t$ increases, the values of $u, v, w$, &c., may become less.

($r$) As all the coordinates are in this case independent variables, and as their number is triple of $n$ that of the material points, or bodies, the number of simple oscillations will be $3n$.

($s$) By substituting $c-z$ for $z$ in the equation of the ellipsoid, which must be done when the origin of the coordinates is transferred to the lowest point of the vertical diameter, this equation becomes

$$\frac{c^2-2cz+z^2}{c^2}+\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

which, when $z^2$ is neglected, gives the value of $z$. Now as the oscillations about the lowest point are supposed to be very small, $x$ and $y$ must be very small, and *à fortiori*, $z$ must be very small with respect to $x$ and $y$.

($t$) $x=\text{R}.\sin\left(t\sqrt{\frac{g}{h}}-r\right)\therefore\frac{dx}{dt}=\text{R}\sqrt{\frac{g}{h}}.\cos\left(t\sqrt{\frac{g}{h}}-r\right)=$ (when $t=0$) $\text{R}.\sqrt{\frac{g}{h}}.\cos r=p'$, in the same way, the values of $\text{R}\sin r$, $\text{R}\sin r'$, $\text{R}\cos r'$ may be obtained, and $x=\text{R}.\cos r.\sin t\sqrt{\frac{g}{h}}-\text{R}\sin r.\cos t\sqrt{\frac{g}{h}}=\frac{p'a}{\sqrt{gc}}\sin t\frac{\sqrt{gc}}{a}+p.\cos t\frac{\sqrt{gc}}{a}$, by substituting $\frac{a^2}{c}$ for $h$, $\frac{b^2}{c}$ for $k$.

($u$) On these suppositions the first equation becomes when $t=0$, $a\alpha=p$; and its differential coefficient becomes $-a\theta\frac{d\psi}{dt}.\sin\psi=p'.\sqrt{\frac{a}{g}}$, but as $\sin\psi=0$, we must have $p'=0$, $q$ is evidently equal to cipher when $t=0$; and as $a\theta\frac{d\psi}{dt}=q'$ in this case, we have $q'=\beta.\sqrt{ga}$.

($v$) By squaring and adding the corresponding members of these equations, we obtain

$$\theta^2.(\sin^2\psi + \cos^2\psi) = \alpha^2 \cos t\sqrt{\frac{g}{a}} + \beta^2 \sin^2 t\sqrt{\frac{g}{a}},$$

which as $\cos^2 t\sqrt{\frac{g}{h}} = \frac{\cos 2t\sqrt{\frac{g}{h}}+1}{2}$, $\sin^2 t\sqrt{\frac{g}{h}} = \frac{1-\cos 2t\sqrt{\frac{g}{h}}}{2}$ becomes the expression in the text.

($x$) See Whewell's History of the Inductive Sciences, vol. i. p. 175. If $v$, $v'$ be the velocities with which $m$ and $m'$ are respectively actuated along the line connecting these two points, the space described by them in $dt$ are $\frac{1}{2}dvdt$, $\frac{1}{2}dv'dt$.

($y$) Since it is evident that these points, when only subject to their mutual action, must eventually meet, and as the entire spaces described by each at the point of junction are in the inverse proportion of their masses, this point must be their common centre of gravity; indeed this is evident also from the consideration, that the motion of the centre of gravity of any number of points subjected to their mutual attraction, is not affected by this action, consequently as all the bodies must meet, their point of junction is the common centre of gravity.

($z$) By substituting $\alpha + x_1$ for $x$, and $\beta + y_1$, in the first equation (a) it becomes

$$\Sigma m\left[(\alpha + x_1).d^2\left(\frac{\beta + y_1}{dt^2}\right) - (\beta + y_1)d^2\left(\frac{\alpha + x_1}{dt^2}\right)\right] =$$

$$\Sigma m\left(\alpha.\frac{d^2\beta}{dt^2} + \alpha.\frac{d^2y_1}{dt^2} + x_1.\frac{d^2\beta}{dt^2} + x_1\frac{d^2y_1}{dt^2} - \beta.\frac{d^2\alpha}{dt^2} - \beta.\frac{d^2x_1}{dt^2} - y_1\frac{d^2\alpha}{dt^2} - y_1\frac{d^2x_1}{dt^2}\right),$$

which is evidently reducible to

$$\Sigma m\left(x_1\frac{d^2y_1}{dt^2} - y_1\frac{d^2x_1}{dt^2}\right) = 0.$$

($a$) By substituting for $x$, $y$, $\frac{dx}{dt}$, $\frac{dy}{dt}$, the first equation (c) becomes

$$\Sigma dm(x_1 + x_{\prime})d\left(\frac{y_1 + y_{\prime}}{dt}\right) - (y_1 + y_{\prime})d.\left(\frac{x_1 + x_{\prime}}{dt}\right) = \Sigma dm\left[\left(x_1\frac{dy_1}{dt} + x_{\prime}\frac{dy_1}{dt} + x_1\frac{dy_{\prime}}{dt} + x_{\prime}\frac{dy_{\prime}}{dt}\right) - y_1\frac{dx_1}{dt} - y_{\prime}\frac{dx_1}{dt} - y_{\prime}\frac{dx_1}{dt} - y_{\prime}\frac{dx_{\prime}}{dt}\right)\right] = c,$$

i. e. (as $\int dm = \text{M}$) $\text{M}x_1\frac{dy_1}{dt} + \frac{dy_1}{dt}\cdot\int x_, dm + x_1\int dm.\frac{dy_,}{dt} + \int dm x_,\frac{dy_,}{dt}$

$$-\text{M}y_1\frac{dx_1}{dt} - \frac{dx_1}{dt}\int y_, dm - y_1\int dm\frac{dx_,}{dt} - \Sigma dm y_,\frac{dx_,}{dt},$$

which evidently becomes equal to the value of $c$ given in the text.

(*b*) Since $\text{M} = \frac{4\pi\rho h^3}{3}$, the value of the moment of inertia of the sphere which is equal to $\frac{8\pi}{15}\cdot\rho h^5 = \frac{2\text{M}h^2}{5}$; and as $\theta$ is the mean angular velocity, and $\rho$ the mean radius of the earth, $\text{M}\rho\theta^2$ must be equal to $c$, the first term of the second member of formula (f).

(*c*) By making this substitution, the first equation (g) becomes

$$c = \Sigma\text{M}\left[(x-g).d\left(\frac{y-h}{dt}\right) - (y-h)d.\left(\frac{x-g}{dt}\right)\right] = \Sigma\text{M}.\left[x.\frac{dy}{dt} - x.\frac{dh}{dt}\right.$$

$$\left.-g.\frac{dy}{dt} + g.\frac{dh}{dt} - y\frac{dx}{dt} + h.\frac{dx}{dt} + y.\frac{dg}{dt} - h.\frac{dg}{dt}\right] = \Sigma\text{M}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

$$+\left(g.\frac{dh}{dt} - h\frac{dg}{dt}\right)\Sigma\text{M} - g\Sigma\text{M}.\frac{dy}{dt} + h\Sigma\text{M}\frac{dx}{dt} + \frac{dg}{dt}\Sigma\text{M}y - \frac{dh}{dt}\Sigma\text{M}x,$$

for $g$, $\frac{dg}{dt}$, $h$, $\frac{dh}{dt}$, can be taken from under the sign $\Sigma$, inasmuch as they are the same for all the bodies $\text{M}$; now as $g\Sigma\text{M} = \Sigma\text{M}x$, $h\Sigma\text{M} = \Sigma\text{M}y$, it follows that $g.\frac{dh}{dt}\Sigma\text{M} = \frac{dh}{dt}\Sigma\text{M}x$, $h.\frac{dg}{dt}\Sigma\text{M} = \frac{dg}{dt}\Sigma\text{M}y$; and $g\Sigma\text{M}\frac{dy}{dt} = \frac{1}{\Sigma\text{M}}\cdot\Sigma\text{M}x.\Sigma\text{M}\frac{dy}{dt}$, $h\Sigma\text{M}\frac{dx}{dt} = \frac{1}{\Sigma\text{M}}\Sigma\text{M}y\,\Sigma\text{M}\frac{dx}{dt}$.

By means of these equations, it is easy to obtain the value of $c$, given in the first equation (h).

(*d*) $m(\text{X}dx + \text{Y}dy + \text{Z}dz) = \text{N}.\text{V}\left(\frac{d\text{L}}{dx}dx + \frac{d\text{L}}{dy}dy + \frac{d\text{L}}{dz}dz\right)$,

but this last quantity is equal to $-\text{NV}.\frac{d\text{L}}{dt}dt$, consequently $-2\int\text{NV}\frac{d\text{L}}{dt}dt$ is the variation of the living force produced by the force N.

(*e'*) In this case to obtain the term of the second member of equation (a), that arises from the friction, the components must be multiplied by $dx$, $dy$, $dz$ respectively, and it will be

$$-\int N.\left(\frac{dx}{ds}dx+\frac{dy}{ds}dy+\frac{dz}{ds}dz\right)=-\int N.ds.$$

(*f*) $\Sigma m.\frac{dx^2}{dt^2}=\Sigma m.\left(\frac{dx_1+dx_{\prime}}{dt^2}\right)^2=\Sigma m.\frac{dx_1^{\,2}}{dt^2}+2\Sigma m.\frac{dx_1}{dt}.\frac{dx_{\prime}}{dt}+$

$$\Sigma m.\frac{dx_{\prime}^{\,2}}{dt^2}=\frac{dx_1^{\,2}}{dt^2}\Sigma m+2\frac{dx_1}{dt}\Sigma m\frac{dx_{\prime}}{dt}+\Sigma m.\frac{dx_{\prime}^{\,2}}{dt^2},$$

from which as $\Sigma m\frac{dx_{\prime}}{dt}=0$, it is easy to obtain the value of $\Sigma mv^2$ given in the text.

(*g*) Since $\Sigma m\frac{dx_{\prime}}{dt}=0$, we must have $\Sigma m\frac{dx}{dt}=\frac{dg}{dt}\Sigma m$, and also $\frac{dg^2}{dt^2}\Sigma m=\frac{1}{\Sigma m}\left(\Sigma m\frac{dx}{dt}\right)^2$, and $\frac{dg}{dt}\Sigma m\frac{dx}{dt}=\frac{1}{\Sigma m}\left(\Sigma m\frac{dx}{dt}\right)^2$, consequently as $\Sigma m\frac{dx_{\prime}^{\,2}}{dt^2}=\Sigma m\frac{dx^2}{dt^2}-2\Sigma m.\frac{dx}{dt}.\frac{dg}{dt}+\Sigma m\frac{dg^2}{dt^2}$, if we substitute for $\Sigma m\left(\frac{dx_{\prime}^{\,2}}{dt^2}\right)$, and also for $2\frac{dg}{dt}\Sigma m\frac{dx}{dt}$, and for $\frac{dg^2}{dt^2}\Sigma m$, we will obtain the expression in the text.

(*h*) By substituting $dm$ for $m$, and $\int$ for $\Sigma$, in equation (e), we obtain $\int(Aa+Bb+Cc)dm=\int(a^2+b^2+c^2)dm=h$.

(*i*) By substituting their values for $a$, $b$, $c$, in equation (f), we obtain

$$\int[(q.Az_{\prime}-rAy_{\prime})+(r.Bx_{\prime}-pBz_{\prime})+(pCy_{\prime}-qCx_{\prime})]dm=\int[(r.(Bx_{\prime}-Ay_{\prime})$$
$$+q.(Az_{\prime}-Cx_{\prime})+p(Cy_{\prime}-Bz_{\prime})]dm=k(r\cos\gamma+q\cos\beta+p.\cos\alpha)=h.$$

(*k*) This is evident by substituting $2\,\Sigma m(a^2+b^2+c^2)$ for $2\,\Sigma m(Aa+Bb+Cc)$.

(*m′*) If the equation $\Sigma m\frac{d^2x}{dt^2}=0$ be integrated twice successively we will obtain an equation of the form $\Sigma mx=a\Sigma m+At$.

# BOOK V.

## CHAPTER II.

(*a*) $dm = \rho dx\,dy\,dz$, therefore the motive force arising from this particle must be of the third order, for we have $\text{X}dm = \frac{dp}{dx}\,dx\,dy\,dz$, consequently $\gamma$ is also of the same order.

(*b*) $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ are equal to the cosines of the angles which the tangent at the point of the surface, whose coordinates are $x, y, z$, makes with the axes of $x, y, z$, and $\frac{\text{X}}{\text{R}}, \frac{\text{Y}}{\text{R}}, \frac{\text{Z}}{\text{R}}$ are the respective cosines of the angles which the resultant R makes with the same axes, consequently, $\frac{dx}{ds}\frac{\text{X}}{\text{R}} + \frac{dy}{ds}\frac{\text{Y}}{\text{R}} + \frac{dz}{ds}\frac{\text{Z}}{\text{R}}$ is equal to the cosine of the angle between the direction of R and the tangent, but by what is stated in page 422 this is cipher, hence the tangent is perpendicular to R.

(*c*) In this case we have $dp = (\text{X}dx + \text{Y}dy + \text{Z}dz) = -\rho g a^2 \frac{dr}{r^2}$, and $p = \frac{\rho g a^2}{r} + \beta$, when $p = 0$, $\beta = -\rho g a$, $\because p = \Pi + \frac{\rho g a^2}{r} - \rho g a$.

(*d*) In the case of a repulsive force it is evident from the expression for $p$ that its least value is when $r$ is greatest or equal to $c$, in which case it becomes $= \Pi - \frac{g\rho a(a-c)}{c}$, hence the least value of $\Pi$ in the case of a repulsive force is $g\rho a\frac{(a-c)}{c}$.

(*e*) If $\rho = f.\phi$, then $dp = f\phi.d\phi$, if this be integrated from any internal point to the surface, and if $\phi'$ be the value of $\phi$ at the surface, we shall have $p = \text{F}\phi - \text{F}\phi'$, and as this value of $p$ is the same whatever point on the surface is taken, $\text{F}\phi'$ must be constant for all points on the surface (because it is by hypothesis the same throughout it), $\because d.\text{F}\phi' = 0$, $\because \rho$ is constant at surface; hence then since $p = \text{F}\phi - \text{F}\phi'$, and $\text{F}\phi'$ is constant, $\phi$ must be a function of $p$, and is constant when $p$ is so, and surfaces of equal pressures are also surfaces of equal densities, but in homogeneous fluids $\rho$ is necessarily

constant, and is no longer a function of $p$; so that the preceding condition that when $p$ is the same, $\rho$ is the same, does not hold. When the fluid is incompressible, $\rho$ may be any function, continuous or discontinuous, of $p$.

($f$) If the central force be equal to $\frac{\mu}{r^n}$, we shall have

$$X = \frac{\mu}{r^n} \cdot \frac{x}{r},\ Y = \frac{\mu}{r^n} \cdot \frac{y}{r},\ Z = \frac{\mu}{r^n} \cdot \frac{z}{r},$$

and by No. 583 we have for each stratum $dp = \rho . (Xdx + Ydy + Zdz)$ $=$ in this case $\frac{\mu\rho}{r^{n+1}}(xdx + ydy + zdz)$, now as $p$ is constant for each stratum, we must have $xdx + ydy + zdz = 0$, and $\therefore$ for each stratum $x^2 + y^2 + z^2 =$ a constant quantity.

($g$) Let $p$, P, denote the pressures corresponding to the two functions $\varphi$, $\Phi$, then we have $\log p = \frac{\phi}{k} + c$, $\log P = \frac{\Phi}{k} + c$, consequently, $\frac{p}{P} = e^{\frac{\phi - \Phi}{k}}$, and as P, $\varphi$, are supposed to be given, if we suppose $\Pi = \frac{P}{e^{\frac{\Phi}{k}}}$, then there results the expression $p = \Pi . e^{\frac{\phi}{k}}$.

($h$) By substituting for $z$ its value $\frac{\alpha^2}{2g} . r^2 + c$, we have

$$a^2 b = 2\frac{\alpha^2}{2g}\int . r^3\, dr + c \int r dr = \frac{\alpha^2}{4g} r^4 + cr^2 + C,$$

and when this integral is taken between the limits $a$ and $o$, it becomes $a^2 b = \frac{\alpha^2}{4g} a^4 + ca^2 = \left(\text{by substituting } \frac{h}{a^2} \text{ for } \frac{\alpha^2}{2g}\right)\left(\frac{h}{2} + c\right)a^2 \therefore c = b - \frac{h}{2}$; now as in this case the general expression $Xdx + Ydy + Zdz = -gdz$, we shall have $dp = -gdz + \alpha^2(xdx + ydy)$ and $\therefore$ $p = -gz + \frac{1}{2}\alpha^2 . (x^2 + y^2) + C'$, for $\rho = 1$ by hypothesis, and at that circumference of the cylinder in which it is met by the most *elevated* section of the fluid, we have $z = b + \frac{1}{2}h$, and, as $p$ is necessarily equal to cipher at the surface, we have $0 = -g(b + \frac{1}{2}h)$ $+ \frac{1}{2}\alpha^2 . (x^2 + y^2) + C'$, and $C' = g . (b + \frac{1}{2}h) - \frac{1}{2}\alpha^2 . (x^2 + y^2)$, consequently $p = g . (b + \frac{1}{2}h - z)$; now as the diameter of cylinder $= 2a$; the area of any section of it is $2\pi a$, and the differential of the cylindrical area is equal to $2\pi adz$, consequently, $dp$ the pressure on this differential is equal to $2\pi adz \times p = 2\pi ag . (b + \frac{1}{2}h - z) . dz$, $\therefore$ the

pressure on the side $= 2\pi ag.(b + \frac{1}{2}h - \frac{1}{2}z)z + c_{,}$, and when taken between the limits $z = b + \frac{1}{2}h$, $z = 0$, it becomes $\pi ag.(b + \frac{1}{2}h)^2$.

(*i*) In No. 106, the value of A corresponds to that of Z in this number, $e$ corresponds to $\gamma$, $a$ to $c$, and $\alpha$ to $z$; consequently if in the expression for A of No. 106, we substitute for $e$, $a$, $\alpha$, and observe that $m = \frac{4}{3}\rho c^3.(1+\gamma^2)$, we shall obtain by changing the signs, the expression for Z; it likewise appears from No. 106 that the value of $A'^2$ in that number is equal to $X^2 + Y^2$ of this No., now by making the same substitutions as in the case of the expression for Z, we obtain

$$A' = \sqrt{X^2+Y^2} = \frac{3f\frac{4}{3}.\pi\rho c^3.(1+\gamma^2)}{2c^3.\gamma^3}\sqrt{x^2+\gamma^2}.\left(\frac{\gamma}{1+\gamma^2} - \text{arc tang} = \gamma\right) = \frac{2f.\pi\rho}{\gamma^3}.\sqrt{x^2+y^2}\,(\gamma - (1+\gamma^2)\,\text{arc tang} = \gamma),$$

now the values of X and Y are respectively $= \dfrac{A'.x}{\sqrt{x^2+y^2}}, \dfrac{A'y}{\sqrt{x^2+y^2}}$, from which it is easy to obtain the expressions for X and Y in the text.

(*k*) From the equation

$$\tfrac{1}{2}\gamma - \tfrac{1}{2}(1+\gamma^2)\,\text{arc}(\text{tang} = \gamma) + \varepsilon\gamma^3 = \text{arc}\,(\text{tang} = \gamma) - \gamma,$$

we obtain $\frac{3}{2}\gamma + \varepsilon\gamma^3 = \left(\dfrac{3+\gamma^2}{2}\right)$ arc tang $= \gamma$, from which it is easy to obtain equation (d).

(*l*) By substituting in equation (d) for tang $\gamma$, its value in a series, it is evident in the first place that it vanishes when $\gamma = 0$, and secondly, that its roots are equal two by two, but affected with contrary signs, and the differential of equation (d) is

$$\frac{9 + 3.(6\varepsilon - 1)\gamma^2 + 2\varepsilon\gamma^4}{(3+\gamma^2)^2} - \frac{1}{1+\gamma^2} = 0,$$

when this expression is reduced and concinnated, it becomes equation (e).

(*m*) This is evident from the consideration, that when any equation and its first differential coefficient are satisfied by the same value, the given equation must have two roots equal to this value.

(*n*) Equation (e) is evidently equal to $\varepsilon.(\gamma^4 + 10\gamma^2 + 9) - 2\gamma^2$, but the quantity within the brackets $= (\gamma^2 + 1)(\gamma^2 + 9)$, now if this value is substituted for $\varepsilon$, we evidently obtain the expression for arc tang $= \gamma$ given in the text.—See *Mechanique Celeste*, Book III. No. 18.

($o$) Since agreeably to what is stated in the text, the axis of the positive abscissæ intersects the curve only in two points, from $\gamma = 0$, to the value of $\gamma =$ to the distance of the first intersection from the origin, the ordinates are $+$, and from this to the second intersection, the ordinates are $-$, and they afterwards increase indefinitely on the side of the positive coordinates. It is evident from the equation $\varepsilon = \frac{2}{(\frac{1}{\gamma^2}+1)(\frac{9}{\gamma^2}+1)}$ that when $\varepsilon$ is $\propto$ $\gamma = 0$. Since for values of $\varepsilon \angle 0{,}1123$ there are two distinct intersections, the approximate values of equation (d) which belong to a value of $\varepsilon \angle 0{,}1123$, of which there are evidently two, determine the values which are competent to a figure of equilibrium.

($p$) By making these substitutions we obtain

$$(3\gamma + 2\varepsilon\gamma^3)\left(\frac{1}{3} - \frac{\gamma^2}{9} + \frac{\gamma^4}{27}\right)\left(=\gamma - \frac{\gamma^3}{3} + \frac{\gamma^5}{9} + \frac{2\varepsilon\gamma^3}{3} - \frac{2\varepsilon\gamma^5}{9}\right) = \gamma - \tfrac{1}{3}\gamma^3 + \tfrac{1}{5}\gamma^5,$$

(higher powers of $\gamma$ are neglected, because in the required expression we neglected powers of $\gamma$ higher than $\gamma^2$), $=$ by obliterating terms common to both sides of the equation, and reducing

$$\frac{2\varepsilon}{3}\left(1 - \frac{\gamma^2}{3}\right) = \frac{4\gamma^2}{45} \because \varepsilon = \frac{2\gamma^2}{15}\cdot\left(1 - \frac{\gamma^2}{3}\right)^{-1}$$

which, because we neglect powers of $\gamma$ higher than $\gamma^2$, becomes $\frac{2\gamma^2}{15}$.

($q$) Since $\frac{\alpha^2}{4\pi f\rho} = \varepsilon$, the ratio of $\alpha^2 c$: to $\frac{4}{3}\pi f\rho c$ is the same as that of $4\pi\rho c.\varepsilon : \frac{4}{3}\pi f\rho\, c$, i. e. : : $3\varepsilon : 1$ : now $3\varepsilon = \frac{2\gamma^2}{5}$, $\because \frac{\gamma^2}{2} = 5\alpha^2 c \div$ four times $\frac{4}{3}\pi f\rho c$, which is the proposition in the text.

($r$) See Harte's Translation of the *Systeme du Monde*, Vol. II. Chap. VIII. notes.

($s$) By making $z = c$, and substituting its values in a series for arc tang $= \gamma$, we obtain $z = 4\pi f\rho . c\left(\frac{1+\gamma^2}{\gamma^3}\right)(\gamma - \frac{1}{3}\gamma^3 + \frac{1}{5}\gamma^5 + \&c. - \gamma)$ $= 4\pi f\rho c(1 + \gamma^2)(-\frac{1}{3} + \frac{1}{5}\gamma^2)$ ($=$ when $\gamma^4$ is neglected) $4\pi f\rho c$ $\left(-\frac{1}{3} - \frac{2\gamma^2}{15}\right) = -\frac{4\pi f\rho c}{3}\left(1 + \frac{4\gamma^2}{10}\right)$; in like manner we obtain

$$(x^2 + y^2)^{\frac{1}{2}} = 2\pi f\rho c\frac{\sqrt{1+\gamma^2}}{\gamma^3}\left(\gamma - \gamma + \frac{\gamma^3}{3} - \frac{\gamma^5}{5} - \gamma^3 + \frac{\gamma^5}{3}\right) =$$

$2\pi f\rho c.\sqrt{1+\gamma^2}\left[-\tfrac{2}{3}+\frac{2\gamma^2}{15}+\&c.\right]=-\frac{4\pi f\rho c}{3}.\left(1-\frac{\gamma^2}{5}\right)\sqrt{1+\gamma^2}$, $\left(=\text{ because } \sqrt{1+\gamma^2}=1+\frac{\gamma^2}{2}q.p.,\right)-\frac{4\pi f\rho c}{3}\left[1+\frac{\gamma^2}{2}-\frac{\gamma^2}{5}\right]=-\frac{4\pi f\rho c}{3}\left[1+\frac{3\gamma^2}{10}\right]$.

(*t*) Since $\frac{\alpha^2}{4\pi f\rho}=\varepsilon$, we have $\alpha^2 c=4\pi f\rho c\varepsilon$, but $\varepsilon=\frac{2\gamma^2}{15}$, $\because \alpha^2 c=\tfrac{4}{3}\pi f\rho c.\frac{2\gamma^2}{5}=\tfrac{4}{3}\pi f\rho c.\frac{4\gamma^2}{10}$, $\because \sqrt{X^2+Y^2}+\alpha^2 c=-\frac{4\pi f\rho c}{3}.\left[1+\frac{3\gamma^2}{10}-\frac{4\gamma^2}{10}\right]=-\frac{4\pi f\rho c}{3}\left(1-\frac{\gamma^2}{10}\right)$, and $\because \frac{\sqrt{X^2+Y^2}+\alpha^2 c-Z}{Z}=\left[1-\frac{\gamma^2}{10}-1-\frac{4\gamma^2}{10}\right]\div$ by $-\left(1+\frac{4\gamma^2}{10}\right)=\frac{5\gamma^2}{10}\left(1+\frac{4\gamma^2}{10}\right)^{-1}=\frac{\gamma^2}{2}$ &c.

(*u*) Consequently, by taking the sum of $\frac{\sqrt{X^2+Y^2}+\alpha^2 c-Z}{Z}$ and $\frac{\gamma^2}{2}$ the compression, it comes out equal to $\gamma^2$. i.e. $\frac{5\alpha^2 c}{2.\frac{4}{3}.\pi f\rho c}$.

(*v*) Arc tang $\gamma$ + arc cot $\gamma=\frac{\pi}{2}$, but cot $\gamma=\frac{1}{\text{tang}=\gamma}$, $\because$ arc tang $=\gamma=\frac{\pi}{2}-$ arc tang $=\frac{1}{\gamma}$; now if in the equation $\frac{3\gamma+2\varepsilon\gamma^3}{\gamma^2+3}=\tfrac{1}{2}\pi-\frac{1}{\gamma}+\frac{1}{3\gamma^3}$ we multiply both sides by $\gamma^2+3$, there results

$$3\gamma+2\varepsilon\gamma^3=\gamma^2\frac{\pi}{2}-\gamma+\frac{1}{3\gamma}+3\frac{\pi}{2}-\frac{3}{\gamma}+\frac{1}{\gamma^3},\ \because$$

$$2\varepsilon\gamma^3=\frac{\pi}{2}\gamma^2-4\gamma+\frac{3\pi}{2}-\frac{8}{3\gamma}+\frac{1}{\gamma^3},$$

dividing by $2\varepsilon\gamma^2$, we obtain

$$\gamma=\frac{\pi}{4\varepsilon}-\frac{2}{\varepsilon\gamma}+\frac{3\pi}{4\varepsilon\gamma^2}-\frac{8}{6\varepsilon\gamma^3}+\&c.;$$

now by means of Lagrange's formula, Lacroix, Tome I. No. 107, we can obtain the value of $\gamma$ in this equation; for by that formula, if we have $\gamma=a+F(\gamma)$, then in general

$$\psi(\gamma)=\psi(a)+Fa.\psi'(a)+\frac{1}{1.2}.d.\frac{[F(a)^2.\psi'(a)]}{da}+\frac{1}{1.2.3}d^2\frac{[F(a)^3.\psi'(a)]}{da^3}+\&c.,$$

in this case $\psi(\gamma)=\gamma$, $\psi(a)=a$, $\psi'a=1$, and $\because$

$$\gamma = a + \text{F}a + \frac{1}{1.2}.d.\frac{\text{F}(a)^2}{da} + \frac{1}{1.2.3}.d.\frac{\text{F}(a^3)}{da^2} + \&c.,$$

now if this value of $\lambda$ be compared with that given above, we obtain

$a = \frac{\pi}{4\varepsilon}$, $\text{F}(\gamma) = -\frac{2}{\varepsilon\gamma} + \frac{3\pi}{4\varepsilon}\frac{1}{\gamma^2} -$ &c., consequently, $\text{F}a = -\frac{2}{\varepsilon}.\frac{1}{a}$ $+ \frac{3\pi}{4\varepsilon}.\frac{1}{a^2} - \&c., = \left(\text{by substituting } \frac{\pi}{4\varepsilon} \text{ for } a\right) - \frac{8}{\pi} + \frac{12\varepsilon}{\pi} - \&c.,$

$$\text{F}(a)^2 = \frac{4}{\varepsilon^2}.\frac{1}{a^2} - \&c., \; d.\frac{(\text{F}(a)^2)}{1.2.da} = -\frac{4}{\varepsilon^2}.\frac{1}{a^3} + \&c. = -\frac{4}{\varepsilon^2}.\left(\frac{4\varepsilon}{\pi}\right)^3 = -\frac{256\varepsilon}{\pi^3} + \&c.,$$

by substituting these expressions in the preceding value of $\gamma$, we obtain

$$\gamma = \frac{\pi}{4\varepsilon} - \frac{8}{\pi} + \frac{12\varepsilon}{\pi} - \frac{256\varepsilon}{\pi^3} + \&c.$$

($x$) In this case of a force varying as the distance, the motions are not deranged, however numerous the bodies are which compose the system; likewise in the case of this law, the orbits described are ellipses, whose centres coincide with the centre of force, and the times of describing different ellipses about the *same* centre are equal. See No. 235, notes.

($y$) Since by what is established in page 442 the resultant of the action of all the forces passes through the common centre of gravity, $x_1, y_1, z_1$, are respectively equal to cipher, consequently this equation must be reduced to equation (f), which, when $\varepsilon = 0$ is, as we know it ought to be, the equation of a sphere.

($z$) In this case, referring to the general expression of the ellipsoid, given in page 434, namely, $\frac{4}{3}\pi c^3(1+\gamma^2)$, it is evident that $c$ in in equation (c) corresponds to $\sqrt{c}$ in this equation and $\frac{1}{1-\varepsilon}$ to $1+\gamma^2$, consequently, the expression for the volume must be, as stated in text, $\frac{4\pi c\sqrt{c}}{3.(1-\varepsilon)}$.

## CHAPTER III.

($a$) It is evident from what is stated in the text, that in the application of the siphon, the distance of the surface of the fluid, in

which the lesser leg is immersed, from the highest point of the bending, must be less than the height to which the fluid would rise in a vacuum by the pressure of the atmosphere; and it is because the weight of the water in the longer leg is greater than in the shorter, that it flows out at the longer, and the pressure of the air keeps up the supply; this instrument is employed to raise water over a height less than thirty-three feet (which is about the height to which water ascends in a vacuum), when the fluid is to descend below the level of the water on the other side; the velocity of the ascending water depends on the difference between the length of the shorter leg and thirty-three feet; and when this is inconsiderable, it may not be such as to afford a sufficient supply to the water descending in the longer leg.

(*b*) What is termed the hydrostatic paradox depends on the principle of the hydraulic press, which is this, that any quantity of water or other fluid, how small soever, may be made to balance and support any quantity or weight however great; for it is evident from a consideration of the hydraulic press, that when the tube DE is very narrow, compared with AB, the addition of a small quantity of water in it may increase the pressure on AB in a great proportion.

(*c*) Since when all the points of a horizontal base experience equal and parallel pressures, the resultant of these forces passes through the centre of gravity of this base; in all fluids in which the pressure varies with the depth, the centre of pressure must be below the centre of gravity, when the pressed plane is not horizontal; and it is evident from the general expression for the pressure on a plane inclined to the horizon, that as long as its centre of gravity remains the same, the prism, the weight of which is equal to this pressure, remains the same.

(*d*) If $\alpha$ be the angle contained between the plane of the trapezium and a vertical plane, it is evident that the vertical distance between horizontal sections passing through AB and MN is equal to $x\cos\alpha$.

(*e*) From the second proportion we obtain $a-b:b::h:k$, and from the first, $u=b\left(\frac{k+h-x}{k}\right)=b.\left(\frac{bh}{a-b}+h-x\right)\frac{a-b}{b.h}$, equal by reducing the value of $u$ given in the text.

(*f*) By substituting for $u$ the preceding equation becomes

$$x'c\int_0^h a dx - x'c\int_0^h \frac{(a-b)}{h}x dx + x'\cos\alpha\int_0^h a x dx - x'\cos\alpha\int_0^h \frac{(a-b)}{h}x^2 dx$$

$$= c\int_0^h a x dx - c\int_0^h \frac{(a-b)}{h}.x^2 dx + \cos\alpha\int_0^h a x^2 dx - \cos\alpha\int_0^h \frac{(a-b)}{h}x^3 dx;$$

$\therefore$ by performing the integration between the prescribed limits, we have

$$x'c\,ah - x'c.\frac{(a-b)}{2}h + x'\cos\alpha.\frac{ah^2}{2} - x'\cos\alpha.\frac{(a-b)}{3}h^2$$

$$= c.\frac{ah^2}{2} - c.\frac{(a-b)}{3}h^2 + \cos\alpha.\frac{ah^3}{3} - \cos\alpha\,\frac{(a-b)}{4}h^3,$$

from which it is easy to obtain the value of $x'$.

(*g*) It is evident from the expression for $x'$, that when the trapezium revolves about its centre of gravity, though the magnitude of the pressure remains the same, the point, where the resultant of the pressures meets the surface, varies with the position of the pressed surface.

(*h*) It is easy also to show from No. 601, that when the sides of a vessel are perpendicular to its base, the entire pressure on the sides is equal to the weight of a triangular prism whose height is the same as that of the fluid, and whose base is a rectangular parallellogram, one of whose sides is the altitude of the fluid, and the other the perimeter of the vessel; so that if the vessel be a cube, the entire lateral pressure is twice the weight.

(*i*) The machine called Barker's Mill is constructed on the principle adverted to in the text.

(*k*) In all the cases discussed in this chapter, it is implied that the sides of the containing vessel are destitute of flexibility, but, strictly speaking, this is never the case, and when the flexibility is at all appreciable, the vessel must acquire some curvature; now if $p$ be as usual the pressure on the unit of surface, and $ds$ the element of the surface, the pressure on this element $= pds$, and if T denotes the tension which each of the extremities of the element $ds$ experience, and $m$ the angle which tangents drawn at the extremities of this element make with each other, it is evident that the resultant of the two tensions $= 2\text{T}.\sin\frac{m}{2}$; but $\sin\frac{m}{2} = \frac{ds}{r}$, ($r$ being the radius of curvature, No. 169, and $m$ being indefinitely small), consequently

when there is an equilibrium, we have $2\,\text{T}.\frac{ds}{r} = pds$, $\because \text{T} = \frac{r.p}{2}$; and when $p$ and $r$ are known we can determine T.

## CHAPTER IV.

(*a*) It is in consequence of what is stated in the text, that insects are enabled to walk on the surface of water, as is often observed to be the case.

(*b*) Since the immersed body is at rest, the line GF which connects the centres of the whole body and of the part immersed, must be vertical, and, consequently, perpendicular to MN a line drawn on the free surface of the fluid at rest.

(*c*) Since the area of MNC the immersed triangle is given, and since when the asymptotes are drawn to a conical hyperbola, a tangent to the hyperbola, terminated by the asymptotes, always cuts off a constant area and is bisected at the point of contact; it follows that MN touches a *given* hyperbola, whose asymptotes are the sides CA and CB of the triangle, that E is the point where MN touches the curve, and DE is perpendicular to the curve; if an ordinate be drawn from any point of a hyperbola, parallel to one asymptote, and terminated by the other, we know that the rectangle under this ordinate, and the part of the other asymptote, intercepted between it and the centre, is always constant; in the present case it is equal to $rab$; it is evident that the equation $x^2 - 2hx \cos\alpha = y^2 - 2hy \cos\beta$ is also the equation of an hyperbola, in which the origin of the ordinates is in the curve itself, and, in fact, the intersection of this hyperbola with the one whose asymptotes are CA, CB, will give a geometrical determination of the different values of CM, CN, and therefore of DF.

(*d*) When each of three angles is solely immersed, they give respectively, at most *three* positions of equilibrium, consequently, nine positions for the three angles; in like manner, in each of the three separate cases in which *two* of the three angles are immersed, we may have three positions of equilibrium, consequently, nine in all, therefore, for any given prism, we may have eighteen different positions of equilibrium, at most.

(*e*) If in the equation $y + x = \frac{4a^2 - c^2}{2a}$ we substitute $\frac{ra^2}{x}$ for $y$, we shall obtain

$$x^2 - \left(\frac{4a^2 - c^2}{2a}\right)x = -ra^2,$$

and by solving this equation we obtain the two roots given in the text, namely, $\frac{1}{4a}\cdot\left[4a^2 - c^2 \pm \sqrt{(4a^2 - c^2)^2 - 16ra^4}\right]$; and it is evident from the equation $y + x = \frac{4a^2 - c^2}{2a}$, that when one of the roots is taken for $x$ the other will be the value of $y$.

($f$) In the case then, of an isosceles triangle, when only one angle is immersed, the greatest number of positions of equilibrium is three, and the least number one; and also when the base is immersed, the greatest number is three and least one; so that when both cases are considered, the greatest number of positions of equilibrium is six, and the least two. It is also evident from a consideration of equations (5), that when the specific gravity of the prism is equal to that of the fluid in which case $r = 1$, that then $x = y = a$, so that when the vertical angle is immersed, the base of the triangle coincides with the surface of the fluid; and when the base is immersed, as $1 - r = 0$, we have $xy = (1 - r)ab = 0$, $\because$ $x$ and $y$ are respectively cipher, so that in this case the vertex of the triangle coincides with the surface of the fluid; it likewise appears from the general values of $x$ and $y$, namely, $\frac{1}{4a}\cdot\left[4a^2 - c^2 \pm \sqrt{(4a^2 - c^2)^2 - 16ra^4}\right]$, that in this case, all other positions are impossible.

($g$) The first condition, i. e. $r \angle \frac{9}{16}$ is necessary in order that the two values may be possible; and the second condition is required, in order that neither value may surpass $a$, for if $r = \frac{1}{2}$, then one of them will be equal to $a$; the same observations are applicable to the second case.

($h$) As the prism is equilateral, the same observations are precisely applicable, whichever of the three angles is immersed; consequently in this case of one sole angle being immersed, the entire number of positions of equilibrium is a multiple of three; when both formulæ are admissible, the number of positions will be eighteen as in the general case; when neither of the formulæ, there are only two positions for each angle, namely, one when the angle and the other when the opposite base is immersed, so that in this case, there are only six positions of equilibrium.

(*i*) Since in this case the weight of the prism is equal to the weight of a prism of the fluid, equal in magnitude to the part immersed, and the prisms have the same base, their volumes are as their heights, and as the weights are by hypothesis equal, the densities must be inversely as the volumes or heights.

(*k*) By substituting for z we shall have

$$\frac{2\pi}{3}abc - \frac{\pi ab}{c^2} \cdot \int_0^u (c^2 - z^2)\, dz = \frac{4\pi}{3}abcr,$$

therefore, by integrating and dividing by $\pi ab$ we obtain, by concinnating,

$$u^3 - 3c^2u - 2(2r - 1)c^3 = 0,$$

now when $r$ is $> \frac{1}{2}$, the third term is negative, consequently the value of $u$ is positive; and when $r \angle \frac{1}{2}$ the contrary is the case. In the extreme cases $r = 0$, $r = 1$, it is evident that $u = c$, $u = -c$ respectively satisfy the given equation; the first indicates that the part immersed is cipher, and the second that the entire body is immersed.

(*l*) When the body is supposed, as in the text, to be perfectly symmetrical, if it be slightly disturbed from its position of equilibrium, as the centre of gravity of the plane of *floatation* (see No. 613), and centre of gravity of body in this case exist previously in the same vertical, the volume of the disturbed fluid will be the same as before, so that the motion of the centre of gravity of the body will be horizontal and *q.p.* rectilinear; but if the centre of gravity of the plane of floatation does not exist in the vertical passing through the centre of gravity of the body, i.e. if the body is not symmetrically arranged with respect to the vertical passing through its centre of gravity, then the centre of gravity passing through the plane of the floatation, will not exist in this line, so that when the body is slightly disturbed, the volume of the fluid that is displaced will be altered, in consequence of which, as the weights of the body and fluid were equal before the disturbance, they must be now unequal, and thus the two forces that act at the centre of gravity of the body, namely, the weight and pressure, of the fluid, will be unequal, and so produce a vertical motion in the centre of gravity; in this case then there is both a motion of translation and rotation; however, the only force that is considered in the text, is that by which in consequence of its

new position, the pressure of the fluid causes the body to turn about a perpendicular to the section ABCD passing through G. When the metacentre coincides with G the centre of gravity of the body, the body will remain in whatever position it is placed.

($m$) The integral $\int z dz$ must be taken between the limits $z = 0$, $z = y$, in order to obtain the integral $\int z dv$; this is equal to $\frac{1}{2}\cos\theta y^2 d\lambda$; and in order to obtain $k$, we must take the sum of all these expressions $\frac{1}{2}\cos\theta y^2 d\lambda$ for every element $d\lambda$, or the expression $\frac{1}{2}\cos\theta \int y^2 d\lambda$ for the entire section ABCD.

($n$) By substituting for $y^2$ its value $\zeta^2 + 2\zeta l.\sin\theta + l^2\sin^2\theta$, and observing that $\zeta$ and $\sin\theta$ remain the same, we obtain the value given in the text, and since AC the common intersection of the planes ABCD, AB″CD″ passes through the centre of gravity of ABCD, and $l$ is perpendicular to AC, we must have $\int l d\lambda = 0$.

($o$) The integral of $dz d\lambda.\cos\theta$ is $z d\lambda.\cos\theta$, now when $\theta = 0$, in which case $y = \zeta$, all the cylinders $z d\lambda.\cos\theta$, of which the entire body is made up are equal to $\zeta d\lambda$, the difference is $q.p.\ z d\lambda \frac{\theta^2}{2}$.

($p$) Since terms of the third order with respect to $\theta$ and $\zeta$, are neglected, $g\rho a\mathrm{v}\cos\theta = g\rho a\mathrm{v} - g\rho\mathrm{v}\frac{\theta^2}{2}$, $\frac{1}{2}l.\cos\theta(\zeta + \gamma^2\sin^2\theta) = \frac{1}{2}l\left(1 - \frac{\theta^2}{2}\right)$ $(\zeta + \gamma^2\theta^2) = \frac{1}{2}l(\zeta + \gamma^2\theta^2)$.

($q$) As $b\gamma^2 = \int l^2 d\lambda$ is the moment of inertia of the plane of floatation of the fluid, it appears from the limits of $\theta$ given in text, that the stability of the body depends on this moment, on the relative position of the centre of gravity of the body and of the displaced fluid, and on the quantity of fluid that is displaced.

($r$) In this case when G is lower than H, the equilibrium must be stable, and the lower G is relatively to H, the greater will be stability, so that when the body is drawn from the position of equilibrium, the force that tends to reestablish it is so much the greater; it is on this principle that in order more effectually to prevent ships from upsetting, the heaviest loading is stowed in the lowest part of the vessel; in this case it is evident that the metacentre is higher than the centre of gravity of the body; on the contrary, it is evident that when $\mathrm{v}a$ is $> b\gamma^2$ and its sign is negative, the metacentre is lower than G, and when $\mathrm{v}a$ is negative, and equal to $b\gamma^2$, the metacentre coincides with G.

($s$) Since, by hypothesis, the body is symmetrical on each side of the vertical plane, the intersection of this plane with a horizontal plane, such as the surface at rest, must be symmetrical with respect to the horizontal section, consequently the centre of gravity of ABCD must be in AC.—See Note ($g$), p. 672.

($t$) Since K is the centre of gravity of ABCD, and the line AKD always meets the contour of ABCD in the same points as A and C, the distance of $d\lambda$ from AC is constant, and as the angle between the planes AB$''$CD$''$, ABCD is equal to the angle $\theta$, the distance of $d\lambda$ from AB$''$CD$''$ $= l\sin\theta$, consequently we shall have $y = \zeta + l\sin\theta$; in like manner the distance of $d\lambda$ from A the vertical plane passing through AKC $=$ $l\cos\theta$, and the distance of this last plane from the plane passing through G, and parallel to AKC is equal to $h.\sin\theta$, $\because x = l\cos\theta + h\sin\theta$.

($u$) V being the volume of displaced fluid, and $\rho$ its density, its weight is V$\rho$, which in the case of a floating body at rest, is equal to M, consequently by substituting V$\rho g$ for M$g$, the motive force of the displaced fluid comes out equal to $-\rho g$U.

($v$) Cos $\theta d\lambda$ is the projection of $d\lambda$ on the horizontal plane, and therefore equal to the base of the vertical cylinder; now if its value be substituted for $y$, we have $\int y\cos\theta d\lambda = \int\zeta\cos\theta d\lambda + \sin\theta.\cos\theta.\int l d\lambda$, hence as $\int l d\lambda = 0$, and $\int d\lambda = b$, we obtain $\int y\cos\theta d\lambda = b\zeta\cos\theta$; also as $\theta^2$ is neglected, $z_1 = \zeta + h$, $\because \frac{d^2 z_1}{dt^2} = \frac{d^2\zeta}{dt^2}$, $\because$ by substituting its values for M, U, and $\frac{d^2 z_1}{dt^2}$, we shall obtain equation (2).

($x$) In equation (3) of 392, $\int r^2 dm = \text{M}k^2$ and $\int x\text{Y} - y\text{X})dm = \mu$.

($y$) As $x = l\cos\theta + h\sin\theta$, $y = \zeta + l\sin\theta$, $xy = \zeta l.\cos\theta + \zeta h \sin\theta + l^2\sin\theta.\cos\theta + lh\sin^2\theta$; $\int xyd\lambda.\cos\theta = \zeta\cos^2\theta\int l d\lambda + \zeta h.\sin\theta.\cos\theta.\int d\lambda + \sin\theta.\cos^2\theta\int l^2 d\lambda + h\sin^2\theta.\cos\theta\int l d\lambda$; which in consequence of equations (1), is evidently reduced to $\sin\theta\cos^2\theta b\gamma^2 + \zeta h\sin\theta\cos\theta.b$. Now the value of $\mu$, when unity and $\theta$ are substituted for $\cos\theta$ and $\sin\theta$, and $\theta\zeta$ is neglected, is $\mu = (b\gamma^2 \pm \text{V}a)\rho g.\theta$; consequently as $\frac{d\omega}{dt} = -\frac{d^2\theta}{dt^2}$, $\text{M}k^2 = \rho\text{V}k^2$, the equation $\text{M}k^2.\frac{d\omega}{dt} = \mu$, becomes

$$-\rho.\text{V}k^2.\frac{d^2\theta}{dt^2} = (b\gamma^2 \pm \text{V}a)\rho g\theta.$$

($z$) See Note ($l$), page 720.

($a'$) The integral of equation (2) determines the vertical motion of the centre of gravity; and the integral of equation (3) determines the oscillatory motion on each side of the vertical GE, when $b\gamma^2 \pm va$ is positive; when this last expression is negative, the value of $\theta$ is of the form $\cos \sqrt{-1}.t$, and when expressed in exponentials, it is evident $\theta$ may increase indefinitely. See No. 421, notes.

($b'$) See Nos. 548, 549.

## CHAPTER V.

($a$) The medium height of the barometer expressed in inches is 29.75 inches, $\because$ as a cubic inch of mercury is equal to 8 ounces q. p. $29.75 \times 8$, i. e. 238 ounces, or 15 pounds nearly, is the pressure on each square inch at this height; from which it is easy to infer the total pressure on the entire convex surface of the earth.

($b$) As each square metre of the earth's surface may be regarded as the base of a prism of the atmosphere, whose height is that of this fluid, by multiplying this height into s, or, what is the same thing, by multiplying $m \times 0{,}76$ into s, we obtain the total mass of the atmosphere.

($c$) The quantity $l$ is called the height of the homogeneous atmosphere; it is evident that at any *given* latitude and elevation above the level of the sea, it is not varied by any difference in the weight of the air, when the effects of a variation of the temperature are not considered, but if while $\Pi$ remains the same, the density undergoes any change, $l$ will be changed in the same proportion; however the density does decrease not indefinitely, but has a limit, as appears from the following note; see also the Philosophical Transactions for 1822.

($d$) The limit is less at the equator than at any parallel, because the centrifugal force is here the greatest.

($e$) As the centrifugal force is $::l$ to the radii of the circles described in the same time, the force at the distance $r+z$ from the centre is to $\frac{g}{289} :: r+z : r$, $\because$ it is equal to $\frac{(r+z)g}{289.r}$; from the equation $z = r\left(\sqrt[3]{289}-1\right)$, it is evident that $z$ is somewhat more than $5r$.

(*f*) See Theorie de Chaleur, par Poisson, No. 203. There are two causes of diminution of temperature as we ascend in the air, the increased distance from the earth, the principal source of heat, and also the greater power of absorbing which the air acquires by being less compressed.

(*g*) Since the quantity of water that ascends in the pump by raising the piston is $bx$, and as it is equal to the quantity by which the water descends in the reservoir, namely to $\beta y$, we must have $y = \frac{bx}{\beta}$.

(*h*) By substituting for $y$ its value $\frac{bx}{\beta}$, we obtain

$$x + y = \frac{\beta + b}{\beta} . x = \frac{x}{f}.$$

(*i*) If this equation be solved we obtain

$$x = \left(\frac{lf + a + c}{2} \pm \sqrt{\left(\frac{lf + a + c}{4}\right)^2 - clf},\right.$$

this gives the actual ascent of the water; in most treatises it is assumed that the reservoir does not subside by any sensible quantity, when the water ascends by $x$, in which case $f = 1$.

(*k*) Let v, v′, be the respective volumes of the air at the temperature zero and 100, we have the following proportion:

$$v' : v :: 1{,}375 : 1, \text{ and } v' - v : v :: {,}375 : 1,$$

$\because$ v′ − v the total dilatation for $100^\circ =$ v.,375, consequently the dilatation for each degree of the centrigrade thermometer is v,00375. When we know the dilatation for each degree of this thermometer, it is easy to obtain the corresponding dilatations in the case of Fahrenheit's and Reaumer's thermometers, by multiplying the preceding fraction in the first case by $\frac{212}{100}$, and in the second by $\frac{80}{100}$, or by $\frac{4}{5}$; for these fractions express the ratio of the degrees in each case to a degree of the centrigrade. It is necessary here to observe, that according to recent experiments made with great accuracy by the late Professor Rudberg, an account of which is given in the second volume of Taylor's Scientific Memoirs, it appears that the value ,375 for the dilatation of gas, is greater than the true value, which is q. p. ,364.

($l$) For since $\alpha$V is the increase of volume for each degree of the centigrade thermometer, $\alpha$ being $=$ ,00375, $\alpha$V$\theta$ will be the increase for $\theta$ degrees, and as $D' : D :: V : V(1+\alpha\theta)$, $D' = \frac{D}{1+\alpha\theta}$.

($m$) $p : \Pi :: \rho : D'$, $\therefore p = \frac{\rho\Pi}{D'} = \rho\Pi . \frac{(1+\alpha\theta)}{D} = \rho k . (1+\alpha\theta)$.

($n$) $k = \frac{\Pi}{D} = m\text{G} . 0,^{m}76 \times \frac{10462}{m} = (7951,12)\ \text{G}$.

($o$) $p$ being considered as a function of $z$ we have $dp = \frac{dp}{dz} dz$.

($p$) $c = \log \Pi - \frac{gr}{k(1+\alpha\theta)}$, $\therefore \log p = \frac{gr}{k(1+\alpha\theta)} . \left(\frac{r}{r+z} - 1\right)$ $+ \log \Pi$, $\therefore \log \frac{P}{\Pi} = - \frac{grz}{k(1+\alpha\theta)(r+z)}$.

($q$) By making these substitutions, equation (3) becomes

$$\frac{dp}{p} = \frac{-gr^2}{k(1+\alpha\theta)} . \frac{dz}{r+z)^2};$$

the integral of which is

$$\log p = \frac{gr^2}{k(1+\alpha\theta)} . \frac{1}{(r+z)} + c,$$

now when $z = 0$, this is the value of log $\Pi$, and when $z = 0$, $c = 0$.

($r$) The differential of $e^{\frac{-grz}{k.(1+\alpha\theta)(r+z)}}$ is

$$\left(\frac{-grdz}{k.(1+\alpha\theta)(r+z)} + \frac{grzdz}{k.(1+\alpha\theta)(r+z)^2}\right) e^{\frac{-grz}{k.(1+\alpha\theta)(r+z)}},$$

which is evidently reducible to $\frac{-gr^2 . dz}{k(1+\alpha\theta)(r+z)^2} . e^{\frac{-grz}{k.(1+\alpha\theta)(r+z)}}$,

therefore the integral of

$$\frac{\Pi gr^2 dz}{k.(1+\alpha\theta)(r+z)^2} . e^{\frac{-grz}{k.(1+\alpha\theta)(r+z)}} \text{ is } -\Pi . e^{\frac{-grz}{k.(1+\alpha\theta)(r+z)}}.$$

($s$) By substituting its value for $c$ in the expression for $\frac{dz^2}{dt^2}$, and dividing by $c$, we obtain

$$\frac{dz^2}{dt^2} = \frac{2\Pi .}{c}\left(1 - e^{\frac{-grz}{k.(1+\alpha\theta)(r+z)}}\right) - 2.g. \left(r - \frac{r^2}{r+z}\right).$$

($t$) In the value of $c\frac{d^2z}{dt^2} = 0$, there will be a common factor,

namely, $\frac{gr^2}{(r+z)^2}$, therefore, by suppressing this factor, it will be reduced to $\frac{\Pi}{k(1+\alpha\theta)} \cdot e^{\frac{-grz}{k.(1+\alpha\theta)(r+z)}} - c = 0$, and, consequently,

$$\frac{-gr}{k.(1+\alpha\theta)\left(\frac{r}{z}+1\right)} = \log\left(c\,\frac{k.(1+\alpha\theta)}{\Pi}\right),$$

from which it appears, that the value of $z$ can be determined by means of a logarithm; in the case of $\frac{dz^2}{dt^2} = 0$, we have evidently,

$$\frac{2\Pi}{c}\left(1 - e^{\frac{-gr}{k(1+\alpha\theta)\left(1+\frac{r}{z}\right)}}\right) = \frac{2gr}{1+\frac{r}{z}},$$

in which transcendental equation the value of $z$ can only be computed by approximation.

($u$) From the values of $\Pi$ and $p$ given here, we obtain $\log\frac{P}{\Pi}\left(= \text{by equation (4)}, \frac{-grz}{k(1+\alpha\theta)(r+z)}\right) = \log . mgh' . \frac{r^2}{(r+z)^2} \div \log mgh$, now

$$\log mgh' . \frac{r^2}{(r+z)^2} = \log mgh' + 2\log . \frac{r}{r+z},$$

consequently, there results

$$\log mgh - \log mgh' = \log\frac{h}{h'} + 2\log\frac{r+z}{r} = \frac{grz}{k.(1+\alpha\theta)(r+z)}.$$

($v$) $\theta = \frac{t+t'}{2}$, $\alpha = 0{,}004 = \frac{4}{1000}$, $\therefore \alpha\theta = 2\left(\frac{t+t'}{1000}\right)$.

($x$) See No. 193.

($y$) By substituting in the expressions for $k$ given in No. 625, the mean between 7951,12G, and 7971,09G, which is nearly 7962$^m$,10G, we shall have $k = 7961{,}10.$G, and, therefore, by putting its value for M, we have

$$\frac{k}{M}[1 - 0{,}002588\cos 2\,(48^\circ\,50'\,14'')] = (18337^m, 46)\,G.$$

($z$) By substituting for $g$ its value given above, and $2\frac{(t+t')}{1000}$ for $\alpha\theta$, equation (5) becomes equal to $\left(\text{as } \frac{rz}{r+z} = \frac{z}{1+\frac{z}{r}}\right) \log\frac{h}{h'} + 2\log\frac{r+z}{r}$

$$= \frac{(1-0{,}002588\cos 2\psi)\,G.}{(1-0{,}002588\cos 2\,(48^\circ\,50'\,14'')} \cdot \frac{1}{k\left(1+\frac{2(t+t')}{1000}\right)} \cdot \frac{z}{1+\frac{z}{r}};$$

hence in order to convert $\log \frac{h}{h'}$ and $\log \frac{r+z}{r}$ into vulgar logarithms, by substituting

$$(1837^m,46)\,\text{G for } \frac{k}{\text{M}}(1-0{,}002588\cos 2\,(48^\circ\,50'\,14'')$$

we obtain

$$\left(\log\frac{h}{h'}+2\log\frac{r+z}{r}\right)=\left(\frac{(1-0{,}002588\cos 2\psi.)\,\text{G}\,1}{\text{M}\,.\,(18337^m,46)\,(1+2\,(\frac{t'+t}{1000})}\right)\cdot\frac{z}{1+\frac{z}{r}};$$

therefore it is evident, that in the value of $z$ which may be deduced from this equation, the logarithms are those in the vulgar system, or of that in which the base is 10, as is observed in the text.

($a'$) By No. 255, when $\rho'$, the density of the stratum, is equal to half $\rho$ the mean density of the earth, $g'=\frac{gr^2}{(r+z)^2}+\frac{3gz}{4r}$, and when $z^2$ is neglected, $\frac{gr^2}{(r+z)^2}=g\cdot\left(1-\frac{2z}{r}\right)$, $\therefore g'=g\cdot\left(1-\frac{2z}{r}+\frac{3z}{4r}\right)=g\cdot\left(1-\frac{5z}{4r}\right)$, hence $p=mg'h'=mgh'\left(1-\frac{5z}{4r}\right)$, $\therefore \log\frac{\Pi}{p}=\log\,.\left(\frac{h}{h'}\cdot\frac{1}{1-\frac{5z}{4}}\right)=\log\frac{h}{h'}+\log\frac{1}{1-\frac{5z}{4}}$, now when $z^2$ is neglected $\frac{1}{1-\frac{5z}{4}}=1+\frac{5z}{4r}$, and also $\left(1+\frac{5z}{8r}\right)^2=1+\frac{5z}{4r}$, on the same supposition, $\therefore \log\left(1+\frac{5z}{4r}\right)=\log\left(1+\frac{5z}{8r}\right)^2=2\log\,.\left(1+\frac{5z}{8r}\right)$, in like manner, in the equation $\frac{dp}{p}=-\frac{g'dz}{k.(1+\alpha\theta)}$, from which equations (4) and (5) are derived, if for $g'$ its value be substituted, the second member becomes

$$\frac{-g}{k.(1+\alpha\theta)}dz\,.\left(1-\frac{5z}{4r}\right)\text{ and }\therefore \log\frac{\text{P}}{\Pi}=\frac{-g\,.}{k.(1+\alpha\theta)}z\,.\left(1-\frac{5z}{8r}\right),$$

and $\log\frac{\Pi}{p}=\frac{g}{k\,(1+\alpha\theta)}z\,.\left(1-\frac{5z}{8r}\right)$, therefore, when the action of the stratum, the height of which is equal to $z$, is taken into account, this is the expression which must be substituted for the second member of equation (5), and when by means of this equation, the value of $z$ is deduced, it is evident that in this value $1-\frac{5z}{8r}$ will be changed into $1+\frac{5z}{8r}$ for $\left(1-\frac{5z}{8r}\right)^{-1}=1+\frac{5z}{8r}$.

($b'$) When the tension of the maximum quantity of vapour of water is $0{,}76^m$, then $D'$, its density is to $D_{,}$ that of perfectly dry air :: 10 : 16, or $D' = \frac{10}{16} D_{,}$, now $D_{,}$, the density when pressure is 0,016, is to $D$ the density when pressure is $0{,}76^m$ as $0{,}016 : 0{,}76^m$, $\therefore D' = \frac{10}{16} \cdot \frac{,016}{,0{,}76} D$, but $\frac{0{,}016}{0{,}76} = \frac{1{,}6}{76}$, $\therefore \frac{10}{16} \cdot \frac{1{,}6}{76} = \frac{1}{76}$, and $\therefore D' = \frac{D}{76}$.

($c'$) At the temperature zero, $P = \frac{1000}{0{,}76}$, $\because$ as the increase of volume is 0,00375 for each degree of temperature, this increase for $18°{,}75$, is (18,75(00375).

($d'$) Since $a$ is the tension of the vapour and $h$ the pressure produced by the mixture, if $d$ be the density of dry air whose elastic force or tension is $h-a$, we have $d : \Delta :: h-a : h \therefore d = \Delta \frac{h-a}{h}$, in like manner by what precedes, $\frac{10}{16} \Delta$ may be considered as the density under the pressure $h$, and $\therefore \frac{10}{16} \cdot \frac{a}{h} \Delta$ is the density due to the pressure or tension $a$.

($e'$) The elastic force varies as the density and temperature, No. 633, therefore, when the density is a maximum, it must depend on the temperature.

($f'$) Since, as stated in No. 633, the temperature varies as density, it follows that as the density of the vapour is much less than that of ordinary air, the temperature must be less also. And as it appears from the computation made in the text, that the weight of a column of aqueous vapour, on the supposition of there being no atmosphere, is less than the weight of all the vapour contained in our atmosphere, it follows that our atmosphere does not prevent water from vaporizing.

## CHAPTER VI.

($a$) By differentiating equation (1), we obtain

$$dp = k(1 + \alpha\theta)\, d\rho + \alpha k \rho . d\theta,$$

consequently, if $p$ be considered as constant, and $\because dp = 0$, we have

$$\frac{d\rho}{d\theta} = \frac{-\alpha\rho}{1 + \alpha\theta},$$

and when $\rho$ is supposed to be constant, we have

$$\frac{dp}{d\theta}=\alpha k\rho=\left(\text{by substituting for }\rho\text{ its value }\frac{p}{k.(1+\alpha\theta)}\right)\frac{\alpha p}{1+\alpha\theta}.$$

(*b*) In the first case, $c=\frac{dq}{d\theta}=\frac{dq}{d\rho}\cdot\frac{d\rho}{d\theta}=-\frac{dq}{d\rho}\frac{\alpha\rho}{1+\alpha\theta}$, in the second case, $c'=\frac{dq}{d\theta}=\frac{dq}{dp}\frac{dp}{d\theta}=\frac{dq}{dp}\frac{\alpha p}{1+\alpha\theta}$; consequently, we obtain

$$1+\alpha\theta=-\frac{\alpha\rho}{c}\cdot\frac{dq}{d\rho}=\frac{dq}{dp}\cdot\frac{\alpha p}{c'},\ \because\ \frac{dq}{d\rho}\rho+\frac{c}{c'}p\cdot\frac{dq}{dp}.$$

(*c*) See Lubbock on Heat of Vapours and Astronomical Refraction, and also London and Edinburgh Philosophical Journal, No. 128.

(*d*) Since $p=k\rho''[1+\alpha(\theta+\omega)]$, $p''=k\rho''(1+\alpha\theta)$, we have

$$\frac{p}{p''}=\frac{1+\alpha(\theta+\omega)}{1+\alpha\theta},\text{ consequently, }\frac{p-p''}{p''}=\frac{\alpha\omega}{1+\alpha\theta},$$

from this equation $\omega$ may be obtained.

(*e*) $1^\circ,3173:1^\circ::0,0133$: to the condensation requisite to produce an increase of temperature equal to one degree, therefore, this condensation $=\frac{0,0133.1^\circ}{1,3173}=0,00101$.

(*f*) In fact when the pressure remains the same, we have by equation (1), $p=k\rho(1+\alpha\theta)=k\rho(1+\delta)[1+\alpha(\theta-\varepsilon)$, $\because 1+\alpha\theta=1+\alpha(\theta-\varepsilon)+\delta(1+\alpha\theta)$, $\delta\varepsilon$ being neglected as infinitely small, consequently there results

$$\alpha\varepsilon=\delta(1+\alpha\theta),\ \because\ \delta=\frac{\alpha\varepsilon}{1+\alpha\theta}.$$

(*g*) In equation (3), if we suppose $\frac{dq}{d\rho}=\text{Q}$ and $\frac{dq}{dp}=\text{P}$, it will become $\rho\text{Q}+\gamma p\text{P}=0$, now $dq=\frac{dq}{d\rho}d\rho+\frac{dq}{dp}dp=\text{Q}d\rho+\text{P}dp$, $\because \text{Q}=\left(\frac{dq-\text{P}dp}{d\rho}\right)$, substituting this value of Q in the given equation, we obtain, by concinnating, $\rho dq-\rho\text{P}dp=-\gamma p\text{P}d\rho$, consequently, $dq=$

$$\text{P}.\left(\frac{\rho dp-\gamma p.d\rho}{\rho}\right)=\frac{\gamma\text{P}\rho}{p^{\frac{1}{\gamma}-1}}\left(\frac{\rho.d.p^{\frac{1}{\gamma}}-p^{\frac{1}{\gamma}}.d\rho}{\rho^2}\right)=\frac{\gamma\text{P}\rho}{p^{\frac{1}{\gamma}-1}}d.\left(\frac{p^{\frac{1}{\gamma}}}{\rho}\right);\ \because$$

$\frac{\gamma\text{P}\rho}{p^{\frac{1}{\gamma}-1}}$ must be of the form $f'\frac{p^{\frac{1}{\gamma}}}{\rho}$, and $\because dq=f'\frac{p^{\frac{1}{\gamma}}}{\rho}d.\frac{p^{\frac{1}{\gamma}}}{\rho}$ and $q=$

$f\left(\frac{p^{\frac{1}{\gamma}}}{\rho}\right)$; and $\frac{p^{\frac{1}{\gamma}}}{\rho} = \phi' q$, and $\because p = k\rho\,(1 + \alpha\theta) = \phi q \cdot \rho^{\gamma}$, consequently, $\theta = \frac{1}{\alpha k}\rho^{\gamma-1} - \frac{1}{\alpha}$.

(*h*) $\phi q = \alpha k\theta \frac{1}{\rho^{\gamma-1}} + \frac{'k}{\rho^{\gamma-1}}$, and by substituting this value in the expression for $\theta'$, we obtain

$$\theta' = \theta\left(\frac{\rho'}{\rho}\right)^{\gamma-1} + \frac{1}{\alpha}\cdot\left(\frac{\rho'}{\rho}\right)^{\gamma-1} - \frac{1}{\alpha} =$$

$\left(\text{when } 266°67 \text{ is substituted for } \frac{1}{\alpha}\right)$ the value of $\theta$ given in text.

(*i*) In the value of $q$, as the increment of $q$ considered as a function of $\theta$ is constant, $q$ must be a linear function of $\theta$.

(*k*) $0{,}2669 = \text{B}\Pi^{\frac{1}{\gamma}-1}$, $c = \text{B}p^{\frac{1}{\gamma}-1} = \text{B}\Pi^{\frac{1}{\gamma}-1}\left(\frac{h}{0{,}76}\right)^{\frac{1}{\gamma}-1} = 0{,}2669.$ $\left(\frac{0{,}76}{h}\right)^{1-\frac{1}{\gamma}}$.

(*l*) Since $m = nc$, by substituting for $c$ we obtain $m =$

$$n.(0{,}2669)\left(\frac{0{,}76}{h}\right)^{1-\frac{1}{\gamma}}.$$

(*m*) By substituting 0,76 for $p$, and making $\theta = 100°$, we have

$$c = \frac{dg}{d\theta} = \text{B}.0{,}76^{\frac{1}{\gamma}-1}, \;\because \text{B} = c.0{,}76^{\frac{\gamma-1}{\gamma}},$$

consequently, we must have

$$\text{C} = \text{A} + \left(c.0{,}76^{\frac{\gamma-1}{\gamma}}\right)(266°{,}67 + 100)(0{,}76)^{\frac{1-\gamma}{\gamma}} = \text{A} + c.(366{,}67),$$

$\because \text{A} = \text{C} - c.(366°67)$, and if this value of A be substituted in the general expression of $q$ there results equation (8), for $h^{\frac{1}{\gamma}-1} = \left(\frac{1}{h}\right)^{\frac{\gamma-1}{\gamma}}$.

(*n*) By equation (1) we have $0{,}76 = k\text{D}.\left(1 + \frac{100}{266{,}67}\right) = k\text{D}.$ $\left(\frac{266{,}67 + 100}{266{,}67}\right) = k.\frac{366{,}67}{266{,}67}$; $h : 0{,}76 :: k\frac{\rho(266{,}67 + \theta)}{266{,}67} : k\text{D}.\frac{366{,}67}{266{,}67}$, $\because h. = \rho.\frac{0{,}76°,(266{,}66 + \theta)}{\text{D}.366{,}67}$, $\because \rho = \frac{\text{D}h}{0{,}76°}.\frac{366{,}67}{(266{,}67 + \theta)}$.

($o$) As the volume of a gas increases by 0,00375 for each additional degree of heat, the increase for 100° is ,0,375, consequently, the weight of the litre at the temperature $100^\circ = \frac{1^{gr},21433}{1,375}$.

($p$) $w$ the weight of $v : 0,55$ the weight of a litre of vapour $:: v\rho : \text{D} \times 1$, for a litre is the unit of capacity, consequently, $w = \frac{0^{gr},55 . v\rho}{\text{D}\times 1} =$ (by substituting for $\rho$ its value) $\frac{vh}{0,76} \, \frac{0^{gr},55 . 366^\circ,67}{266,67 + \theta}$, which is the value of $w$ given in the text, since $0^{gr},55 \times 366,67 = 201^{gr},6685$, of which the two last places are neglected in the expression in text.

($q$) Since the density is supposed to have attained its maximum, and $\theta$ is invariable, $p$, which depends on $\rho$ and $\theta$, must be also invariable.

($r$) We have $v : \frac{v.}{p}(p + p') :: p : p + p'$.

# BOOK VI.

## CHAPTER I.

(a) $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, are functions of $x, y, z, t$, and as $x, y, z$, are functions of $x', y', z'$, $\frac{dx}{dt}, \frac{dy}{dt}$, &c., are functions of $x', y', z', t$, &c.

(b) $dp = 0$ is more general than $p = 0$, for at the surface the pressure should rather be considered as constant than as cipher.

(c) As the value of M′C′ is equal to the square root of the sums of the squares of the differences of the corresponding coordinates of M′ and C′, it evidently must be equal to the expression given in the text; now if this value of M′C′ be extracted by means of the binomial theorem, it is evident that there will result by considering $\left(dz + \frac{dw}{dz}dzdt\right)^2$ as the first term

$$\text{M}'\text{C}' = dz + \frac{dw}{dz}dzdt + \tfrac{1}{2}\left(dz + \frac{dw}{dz}dzdt\right)^{\frac{3}{2}}\left[\left(\frac{du}{dz}\right)^2 dz^2dt^2 + \left(\frac{dv}{dz}\right)^2 dz^2dt^2\right] - \&c.;$$

from which it is evident that, when infinitely small quantities of the third and higher orders are neglected, the expression for M′C′ is reduced to its first term; in like manner, in order to obtain the value of D′G′, we obtain (by substituting $x + dx$, $y + dy$, for $x$ and $y$) the coordinates of D′ =

$$x + dx + udt + \frac{du}{dx}dx\,dt + \frac{du}{dy}dy\,dt,$$

$$y + dy + vdt + \frac{dv}{dx}dx\,dt + \frac{dv}{dy}dy\,dt,$$

$$z + wdt + \frac{dw}{dx}dx\,dt + \frac{dw}{dy}dy\,dt;$$

in the same manner the coordinates of G′ will be

$$x + dx + udt + \frac{du}{dx}dx\,dt + \frac{du}{dy}dy\,dt + \frac{d^2u}{dx\,dz}dx\,dz\,.\,dt + \frac{d^2u}{dy\,dz}dy\,dz\,dt,$$

$$y + dy + vdt + \frac{dv}{dx}dx\,dt + \frac{dv}{dy}dy\,dt + \frac{d^2v}{dx\,dz}\,.\,dx\,dz\,dt + \frac{d^2v}{dy\,dz}\,.\,dy\,dz\,dt,$$

$$z+dz+wdt+\frac{dw}{dx}.dx\,dt+\frac{dw}{dy}dy\,dt+\frac{d^2w}{dx\,dt}.dx\,dz\,dt+\frac{d^2w}{dy\,dz}dy\,dz\,dt+\frac{dw}{dz}dz\,dt;$$

and the differences of the corresponding coordinates are respectively

$$\frac{d^2u}{dx\,dz}dx\,dz\,dt+\frac{d^2u}{dy\,dz}dy\,dz\,dt,$$

$$\frac{d^2v}{dx\,dz}dx\,dz\,dt+\frac{d^2v}{dy\,dz}dy\,dz\,dt,$$

$$dz+\frac{dw}{dz}dz\,dt+\frac{d^2w}{dx\,dz}.dx\,dz\,dt+\frac{d^2w}{dy\,dz}.dy\,dz\,dt,$$

from which it is evident at once, that when terms of the third and higher orders are neglected, the expression for D′G′ will be reduced to its first term.

(*d*) If the infinitely small quantity by which the angle A′M′B′ differs from a right angle be denoted by $dx$, we shall have sin A′M′B′ $=\sin(90^\circ\pm dx)=1-\frac{dx^2}{1.2}$, therefore, as the same is true for sin C′M′P′, if they be taken into account in the expression M′A′.M′B′.M′C′. sin A′M′B′. sin C′M′P′, it is evident that as M′A′, M′B′, M′C′, are respectively of the first order, there will result terms of the fifth and higher orders in the resulting expressions.

(*e*) When their values are substituted for these factors, their product is

$$dx.\left(1+\frac{du}{dx}dt\right).dy.\left(1+\frac{dv}{dy}dt\right).dz.\left(1+\frac{dw}{dz}dt\right),$$

which, when infinitely small quantities of the fifth order are neglected, becomes the expression in the text.

(*g*) In this case, each stratum heated from above must expand, and as no lateral expansion can take place, every stratum must rise vertically, and be replaced by the stratum immediately beneath; this ascent commences with the uppermost stratum, in this way it will be evident, that the length of the fluid column will increase; however, as the lighter strata are always the uppermost ones, it is evident that there will be no interchange between the strata, and, consequently, that the continuity of the fluid will not be interrupted. *f*

(*g*) This is the complete differential with respect to $t$, for as

$u=\frac{dx}{dt}, u\frac{df}{dx}=\frac{df}{dx}\frac{dx}{dt}=\frac{df}{dt}$, and as in the case of a *fixed* division, it is evident that the time $t$ is not explicitly contained in $f$, the first term of equation (4) is cipher, and it consequently becomes equation (9).

($h$) This must be the case, because infinitely small quantities of the second or higher order are neglected.

($i$) If the law of Mairiotte obtains in the case of motion, then we have $p \propto \rho$, consequently, $\int\frac{dp}{\rho}$ will be expressed by a logarithm, when the variations of temperature are taken into account, $\rho$ will not be $\div l$ to $p$, consequently, $\int\frac{dp}{\rho}$ cannot be expressed by a logarithm, neither will it be thus expressed, when $\rho$ is constant as in the case of a homogeneous liquid, for then $\int\frac{dp}{\rho}=\frac{1}{\rho}\int dp=\frac{p}{\rho}$.

($k$) In this case, by substituting for $dp$ its value, there results

$$\int\frac{1}{\rho}\left[\frac{dp}{dx}dx+\frac{dp}{dy}dy+\frac{dp}{dz}dz\right]=\text{P},$$

consequently,

$$\frac{1}{\rho}\frac{dp}{dx}=\frac{d\text{P}}{dx}; \text{ \&c.}$$

($l$) If the equation $udx+vdy+wdz=d\phi$ be differenced with respect to $t, x, y, z$, respectively, we obtain,

$$\frac{du}{dt}dx+\frac{dv}{dt}dy+\frac{dw}{dt}dz=d.\frac{d\phi}{dt},$$

$$\frac{du}{dx}dx+\frac{dv}{dx}dy+\frac{dw}{dx}dz=d.\frac{d\phi}{dx},$$

$$\frac{du}{dy}dx+\frac{dv}{dy}dy+\frac{dw}{dy}dz=d.\frac{d\phi}{dy},$$

$$\frac{du}{dz}dx+\frac{dv}{dz}dy+\frac{dw}{dz}dz=d.\frac{d\phi}{dz};$$

therefore, if in equations (3) we substitute $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$, for $u, v, w$, we obtain, as $\frac{1}{\rho}\frac{dp}{dx}=\frac{d\text{P}}{dx}$, $\text{X}=\frac{d\text{V}}{dx}$, &c. the values of $\frac{d\text{P}}{dx}, \frac{d\text{P}}{dy}, \frac{d\text{P}}{dz}$ given in the text.

Now if these equations be multiplied by $dx, dy, dz$, respectively,

and if they be then added together, there will result by substituting $d\text{P}$ for $\frac{d\text{P}}{dx}dx + \frac{d\text{P}}{dy}dy + \frac{d\text{P}}{dz}dz$, $d\text{V}$ for $\text{X}dx + \text{Y}dy + \text{Z}dz$, and by observing that

$$\tfrac{1}{2}d.\left(\frac{d\phi}{dx}\right)^2 = \frac{d\phi}{dx}.\frac{d^2\phi}{dx^2}dx + \frac{d\phi}{dx}.\frac{d^2\phi}{dx.dy}dy + \frac{d\phi}{dz}.\frac{d^2\phi}{dx.dz}dz,$$

$$\tfrac{1}{2}.d.\left(\frac{d\phi}{dy}\right)^2 = \frac{d\phi}{dy}.\frac{d^2\phi}{dx.dy}dx + \frac{d\phi}{dy}.\frac{d^2\phi}{dy^2}dy + \frac{d\phi}{dy}.\frac{d^2\phi}{dy.dz}dz,$$

$$\tfrac{1}{2}d.\left(\frac{d\phi}{dz}\right)^2 = \frac{d\phi}{dz}.\frac{d^2\phi}{dz\,dx}dx + \frac{d\phi}{dz}.\frac{d^2\phi}{dy.dz}dy + \frac{d\phi}{dz}.\frac{d^2\phi}{dz^2}dz,$$

the value of $d\text{P}$ given in text.

(*m*) Since $\varepsilon$ is infinitely small, we may neglect all terms after the second in the expression of $u$; $\frac{d\phi_{\prime}}{dx}$ is evidently the value of $u$ when $\varepsilon = 0$.

(*n*) By multiplying these equations by $dx$, $dy$, $dz$, respectively, adding them together and observing that $\text{X}dx + \text{Y}dy + \text{Z}dz = d\text{V}$, $\frac{1}{\rho}\left(\frac{dp}{dx}dx + \frac{dp}{dy}dy + \frac{dp}{dz}dz\right) = d\text{P}$, we obtain the expression in text, for in differentiating $\frac{d\phi^2}{dx^2}, \frac{d\phi^2}{dy}, \frac{d\phi^2}{dz^2}$, each of them must be considered as functions of $x, y, z$; see Note (*l*).

(*o*) In this case $\frac{du}{dx}, \frac{du}{dz}$, are respectively equal to cipher, consequently, we shall have $\left(\text{as } \frac{du}{dy} = -\omega\right)$ the first equation (3) equal to

$$\frac{1}{\rho}.\frac{dp}{dx} = \text{X} + \omega^2 x, \text{ and } \frac{1}{\rho}\frac{dp}{dy} = \text{Y} + \omega^2 y, \frac{1}{\rho}\frac{dp}{dz} = \text{Z},$$

from which we may obtain the value of $\frac{1}{\rho}dp$, by multiplying these equations by $dx$, $dy$, $dz$, respectively, and then adding them together.

## CHAPTER II.

(*a*) For we have by this law, when the temperature is invariable,

$$p : mgh :: \text{D}.(1+s) : \text{D} \therefore p = mgh\,(1+s);$$

but, when in consequence of the sudden increase of temperature,

$p$ varies in a greater ratio than $\rho$, or D $(1+s)$, we have $p = gmh.(1+s+\beta s)$, and $\because dp = gmh(1+\beta)\,ds$, hence $\frac{dp}{\rho} = \frac{gmh(1+\beta)ds}{\text{D}.(1+s)}$, and $\int \frac{dp}{\rho} = \frac{gmh.(1+\beta)}{\text{D}}.\log.(1+s)$, now when the square and higher powers of $s$ are neglected $\log(1+s) = s$, $\therefore \int \frac{dp}{\rho} = \text{P} = a^2 s$; consequently when this value is substituted for P in equation (b) and V is suppressed, and $\left(\frac{d\phi}{dx}\right)^2, \left(\frac{d\phi}{dy}\right)^2$, &c. are neglected, we obtain $-a^2 s = \frac{d\phi}{dt}$.

(*b*) In integrating the values of $udt$, $vdt$, $wdt$, at *any instant*, as the displacements are very small, the error which is committed when in this operation, $x', y', z'$, is substituted for $x, y, z$, is inconsiderable, so that it may be neglected, and $\because$ $x, y, z$, may be regarded as constant.

(*c*) Equation (c) becomes when D$(1+s)$ is substituted for $\rho$,

$$d.\frac{\text{D}.(1+s)}{dt} + d.\frac{\text{D}.(1+s).\frac{d\phi}{dx}}{dx} + d.\frac{\text{D}.(1+s).\frac{d\phi}{dy}}{dy} + d.\frac{\text{D}.(1+s).\frac{d\phi}{dz}}{dz}$$

$$= \left(\text{when } s.\left(\frac{d\phi}{dx} + \frac{d\phi}{dy} + \frac{d\phi}{dz}\right) \text{is neglected}\right) \text{D}.\left(\frac{ds}{dt} + \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right),$$

which when $-\frac{1}{a^2}.\frac{d^2\phi}{dt^2}$ is substituted for $\frac{ds}{dt}$ is the equation (3).

(*d*) By substituting for $u$, $v$, $w$, their respective values there results

$$udx + vdy + wdz = \frac{\zeta(xdx + ydy + zdz)}{r} = \zeta dr,$$

which, as $\zeta$ is a function of $r$ and $t$, will be an exact differential of a function of $r$ and $t$.

$$(e)\ \frac{d\phi}{dx} = \frac{d\phi}{dr}.\frac{dr}{dx} = \frac{d\phi}{dr}.\frac{x}{r}, \ \therefore \frac{d^2\phi}{dx^2} = \frac{d^2\phi}{dr^2}.\frac{dx}{dr}.\frac{x}{r}\left(= \frac{d^2\phi}{dr^2}.\frac{x^2}{r^2}\right) + \frac{d\phi}{dr}\left(\frac{1}{r} - \frac{x^2}{r^3}\right) = \frac{d\phi}{dr}\left(\frac{y^2+z^2}{r^3}\right);$$

in like manner

$$\frac{d^2\phi}{dy^2} = \frac{d^2\phi}{dr^2}.\frac{y^2}{r^2} + \frac{d\phi}{dr}.\frac{x^2+z^2}{r^3}, \quad \frac{d^2\phi}{dz^2} = \frac{d^2\phi}{dr}.\frac{z^2}{r^2} + \frac{d\phi}{dr}.\frac{x^2+y^2}{r^2},$$

consequently we shall have

$$\frac{d^2\phi}{dt^2} = a^2.\frac{d^2\phi}{dr^2}\left(\frac{x^2+y^2+z^2}{r^2}\right) + 2\left(\frac{x^2+y^2+z^2}{r^3}\right).\frac{d\phi}{dr} = a^2.\frac{d^2\phi}{dr^2} + \frac{2}{r}.\frac{d\phi}{dr}.$$

(*f*) Now since $r$ is the independent variable, we have

$$\frac{d^2.r\phi}{dr^2}=r.\frac{d^2\phi}{dr^2}+2\frac{dr}{dr}\frac{d\phi}{.dr}\text{ and }a^2.\frac{d^2.r\phi}{dr^2}=a^2r.\frac{d^2\phi}{dr^2}+2a^2.\frac{d\phi}{dr},$$

therefore, if both members of the equation

$$\frac{d^2\phi}{dt^2}=a^2.\left(\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr}\right),$$

be multiplied by $r$, it will coincide with equation (4) for $r\frac{d^2\phi}{dt^2}=\frac{d^2r\phi}{dt^2}$.

(*g*) $\phi=\frac{1}{r}.(f(r+at)+\text{F}.(r-at))$, $\because\zeta=\frac{d\phi}{dr}=\frac{1}{r}(f'(r+at)+$ $\text{F}'(r-at))-\frac{1}{r^2}[f(r+at)+\text{F}(r-at)]$ and $\frac{d\phi}{dt}=\frac{a}{r}.[f'(r+at)-$ $\text{F}'(r-at)]$, $\because s=-\frac{1}{a^2}\frac{d\phi}{dt}=$ the second equation (5), and from this second equation it is evident, that the velocity is always proportional to the condensation.

(*h*) Since the first formula vanishes at the same time as $r$, when it is expressed in a series proceeding according to the powers of $r$, each term must have $r$, or some power of $r$, as a factor, and therefore when $r$ is infinitely small, all powers after the first may be neglected; now when equations (5) are applied to the case of the propagation of sound in our atmosphere, we are warranted in assuming that $r$ is infinitely small, because the agitation of each particle is so inconsiderable, that it does not appear to move sensibly from its state of rest; for when sound is transmitted through a mass of air, in which motes are observed to float, they do not appear to be actuated by any sensible motion. See article on Sound in the Encyclopædia Metropolitana, No. 54.

(*i*) $\Psi r$ denotes a certain velocity as well as $\psi r$, because $\zeta$ is to $s$ in a given ratio.

(*k*) $\psi r=\frac{1}{r}.\frac{d.fr}{dr}-\frac{1}{r^2}.\frac{fr}{dr}+\frac{1}{r}.\frac{d.\text{F}r}{dr}-\frac{1}{r^2}.\frac{d.\text{F}r}{dr},=\frac{d.\frac{1}{r}.fr}{dr}+\frac{d.\frac{1}{r}.\text{F}r}{dr}.$

(*l*) In fact, by making this substitution, we get

$$\zeta=\frac{1}{r}.(\tfrac{1}{2}b+\tfrac{1}{2}b)-\frac{1}{r^2}.[\tfrac{1}{2}b(r+at)-\tfrac{1}{2}c+\tfrac{1}{2}b(r-at)+\tfrac{1}{2}c]=0,$$

in like manner it may be shown, that when corresponding substitutions are made in the second equation (5), the expression for $s$ be-

comes cipher, it thus appears that when $\psi_1 r$ and $\Psi_1 r$ are supposed to be cipher for a certain value of $r$, which is the supposition that it is always made in order to determine the values of the arbitrary constants $b$ and $c$, then the values of $\zeta$ and $s$ become cipher.

(*m*) By adding and subtracting equations (7), and substituting $z$ for $r$, we obtain

$$fz = \tfrac{1}{2} z\psi_1 z - \tfrac{1}{2}\Psi_1 z, \quad \text{F}z = \tfrac{1}{2} z\psi_1 z + \tfrac{1}{2}\Psi_1 z,$$

now

$$d\psi_1 z = \psi z dz, \quad d\Psi_1 z = z\Psi z dz,$$

consequently,

$$f'z \,.\, dz = \tfrac{1}{2} dz\psi_1 z + \tfrac{1}{2} z \,.\, \psi z dz - \tfrac{1}{2}\Psi z dz,$$

hence then there results

$$f'z = \tfrac{1}{2}\psi_1 z + \tfrac{1}{2} \,.\, z \,.\, (\psi z - \Psi z).$$

(*n*) If $\dfrac{\varepsilon - r}{a}$ be substituted for $t$ in the functions $f(r + at)$, $f'(r + at)$, the results are $f\varepsilon$, $f'\varepsilon$, consequently, as long as $t$ is less than $\dfrac{\varepsilon - r}{a}$ the values of $f(r + at)$, $f'(r + at)$, will be less than $f\varepsilon$, $f'\varepsilon$, and therefore will not be cipher; and the same is true for $\text{F}(r - at)$, $\text{F}'(r - at)$, when $t$ is $\angle$ than $\dfrac{r}{a}$, $r$ being by hypothesis less than $\varepsilon$; from the equation $t = \dfrac{\varepsilon - r}{a}$ it is evident that the greater $r$ is, or, the nearer to the point that is agitated is to the surface of the sphere, the less will be the duration.

(*o*) See No. 497, notes; since the motion commences when $at = r - \varepsilon$, and ceases when $at' = r + \varepsilon$, the duration $= t' - t = \dfrac{2\varepsilon}{a}$, and the breadth of the sonorous wave $= 2\varepsilon$.

(*p*) At a great distance from the centre of the agitation, the functions $f(r + at), f'(r + at)$, vanish, so that the second term of second member of the value of $\zeta$ given in page 584 vanishes; consequently $\zeta =$ evidently $as$; if, on the contrary, $r$ was very small, then by expanding the values of $f'(r' + at)$, $\text{F}'(r - at)$, $f(r + at)$, $\text{F}(r - at)$, into series proceeding according to powers of $r$, we obtain

$$\frac{1}{r} f'(r + at) = \frac{f'at}{r} + \frac{f''at \,.\, r}{r} + \frac{f'''at \,.\, r^2}{1.2.r} + \&c.$$

$$\frac{1}{r}\text{F}'(r - at) = \frac{\text{F}'(-at)}{r} + \frac{\text{F}''(-at)\,r}{r} + \frac{\text{F}'''(-at)\,.\,r^2}{1.2r} + \&c.,$$

$$\frac{1}{r^2}f(r+at)=\frac{1}{r^2}.f(at)+\frac{1}{r^2}.f'at.r+\frac{1}{r^2}.f''\frac{at.r^2}{1.2}+\&c.,$$

$$\frac{1}{r^2}\text{F}(r-at)=\frac{\text{F}(-at)}{r^2}+\frac{\text{F}'(-at)}{r^2}.r+\frac{\text{F}''(-at)r^2}{r^2.1.2}+\&c.;$$

consequently, the expression for $\zeta$ becomes, by concinnating and obliterating, $=-\frac{1}{r^2}.[(fat)+\text{F}(-at)]$ i. e. $=\frac{\psi t}{r^2}$ nearly.

($q$) The number of functions $f, f_{\prime}, f_{\prime\prime}$, &c. of equation (9) which the most general expression for $\phi$ contains, will depend on the number of distinct agitations at the commencement of the motion, the agitation resulting from the several distinct ones will be the algebraic sum of these functions; this remarkable conclusion results from the forms of the original differential equations. See No. 551.

($r$) $u$ is equal in general to $\zeta$ resolved in the direction of the axis of $x$, but as the agitation does not extend as far as the plane AB, $u$ must be cipher; now this condition is satisfied in this case, for the expression for $u=\frac{d\phi}{dr}\frac{x}{r}+\frac{d\phi}{dr_{\prime}}\frac{x_1}{r^1}$ becomes when $r=r_1$, $x=h$, and $x_1=-h$, equal to cipher.

($s$) $p=\frac{gmh}{\text{D}}.\rho(1+s+\beta s).(1+s)^{-1}=$ (neglecting $s^2$) $\frac{g.mh}{\text{D}}.\rho$ $(1+s-s+\beta s)=\frac{gmh}{\text{D}}\rho(1+\beta s)$.

($t$) By substituting in the equation $p=k\rho.(1+\alpha(\theta+n))$ for $k$, we obtain $p=\frac{gmh\rho}{\text{D}.(1+\alpha\theta)}.(1+\alpha\theta)+\alpha n)=\frac{gmh\rho}{\text{D}}.\left(1+\frac{\alpha n}{1+\alpha\theta}\right)$.

($u$) If the density in the first case be $\rho$, and in the second $\rho\left(1+\frac{1}{125}\right)$, then the increment $=\rho\sqrt{1+\frac{1}{125}}-1=\frac{\rho}{250}q.p.$

($v$) If in the value of $p=\frac{gmh}{\text{D}}\rho.(1+\beta s)$, we substitute $\text{D}(1+s)$ for $\rho$, it becomes $gmh.(1+s+\beta s)$, $s^2$ being always neglected, but on the same supposition $1+s+\beta s=(1+s)^{1+\beta}$, therefore $p=gmh.(1+s)^{1+\beta}$.

($x$) See *Comptes Rendus* for August, 1841, where an account is given of some experiments made in Geneva on the intensity and distance to which sound can be propagated under water.

## CHAPTER III.

(*a*) By substituting for $y'$ its value, we have

$$(p'-p)\,y' = (p'-p)\left(y+\frac{dy}{dx}dx\right), = \left(\text{as } p'-p = \frac{dp}{dx}dx\right)\frac{dp}{dx}dx.$$
$$\left(y+\frac{dy}{dx}dx\right),$$

and when $dx^2$ is neglected it becomes $\frac{dp}{dx}.ydx.$

(*b*) When the two members are multiplied by $dx$, the equation becomes

$$\left(\text{as } \frac{dy}{dx}dx = dy, \text{ and } \frac{dp}{dx}dx = dp\right)$$
$$dp = g\rho dx - \alpha\rho\frac{du}{dt}\frac{dx}{y} + \rho\alpha^2u^2.\frac{dy}{y^3};$$

the integral of which is evidently the value of $p$ given in the text.

(*c*) At the orifice

$$p = \Pi - g\rho c = \Pi + g\rho\,(l-\theta) - \alpha\rho\frac{du}{dt}.\frac{1}{\lambda} - \frac{\rho u^2}{2}.\left(1-\frac{\alpha^2}{\omega^2}\right),$$
$$\because\ g\rho\,(h+c) - \rho\alpha\frac{1}{\lambda}\frac{du}{dt} - \rho\frac{u^2}{2}.\left(1-\frac{\alpha^2}{\omega^2}\right) = 0;$$

when $\alpha$ is $>\omega$ and therefore $\beta$ is negative, the value of $p$ at the lower orifice is then greater than $\Pi$, therefore the flowing out must take place at the upper orifice of which section is minimum.

(*d*) From equation (4) we obtain $\lambda dt = \frac{2\alpha du}{2g.(h+c)-\beta^2u^2}$, when $g\rho c$ is inconsiderable with respect to $\Pi$, we may neglect $c$ in the expression $2g.(h+c)$, now if we make $x = \frac{\beta u}{\sqrt{2g.h}}$ this differential equation becomes $\lambda dt = \frac{\alpha.\sqrt{2}.dx}{\beta.\sqrt{gh}.(1-x^2)}$ of which the integral is $\frac{\alpha}{\beta\sqrt{2gh}}.$ $\log.\frac{1+x}{1-x}$, and by substituting its value for $x$ we obtain the expression in the text.

(*e*) Since $t=0$ when $u=0$, the value of the constant which

might occur in the integral of the value of $\lambda . dt$ must be cipher, for it is equal to $\frac{a}{\beta\sqrt{2gh}}.\log.1$.

($f$) From equation (6) we obtain

$$\beta u.\left(1+e^{\frac{\beta\lambda t\sqrt{2gh}}{a}}\right)=2gh.\left(e^{\frac{\beta\lambda t\sqrt{2gh}}{a}}-1\right)\therefore u=\frac{\sqrt{2gh}}{\beta}\left(\frac{e^{\frac{\beta\lambda t\sqrt{2gh}}{a}}-1}{e^{\frac{\beta\lambda t\sqrt{2gh}}{a}}+1}\right),$$

which, by multiplying numerator and denominator by $e^{\frac{-\beta\lambda t\sqrt{2gh}}{2a}}$, becomes equal to the expression in the text.

($g$) When the value of $u$ is multiplied by $dt$, the differential is evidently that of a logarithm, for the numerator is the differential of the denominator multiplied by $\frac{2a}{\beta.\lambda\sqrt{2gh}}$, therefore,

$$a\int udt=\frac{2a^2}{\beta^2\lambda}\log\left[e^{\frac{\beta\lambda t\sqrt{2gh}}{2a}}+e^{\frac{-\beta\lambda t\sqrt{2gh}}{a}}\right]+c;$$

now as $q=0$, when $t=0$, we have $c=\frac{-2a^2}{\beta^2\lambda}.\log 2$, hencethen there results the expression in the text.

($h$) When equation (4) is multiplied by $dt$, we obtain

$$(g.(h+c)-\tfrac{1}{2}\beta^2u^2)\,dt-\frac{a}{\lambda}du=0,$$

and equation (5) becomes in the same case

$$dh+\frac{au}{\omega}dt=0,$$

therefore, by eliminating $dt$ between these equations we obtain the expression in text.

($i$) See examples of the integral and differential calculus, page 329.

($k$) $\beta^2=1-\frac{a^2}{\omega^2}$, $\omega=a$, and $a=\frac{a}{n}$, $\therefore\ \beta^2=1-\frac{1}{n^2}=\frac{n^2-1}{n^2}$.

($l$) In this case $\lambda\omega=\frac{a^2}{h}$, $a^2=\frac{a^2}{n^2}$, $\therefore\ \frac{\beta^2\lambda\omega}{a^2}=\frac{n^2-1}{n^2}.\frac{a^2}{h}.\frac{n^2}{a^2}=\frac{n^2-1}{h}$ and $\frac{\lambda\omega h}{a^2}=\frac{a^2}{h}.\frac{h}{a^2}n^2=n^2$, $\therefore$ equation (7) becomes $\frac{dz}{dh}-\frac{(n^2-1)}{h}z+n^2=0$.

(*m*) By multiplying all the terms of this equation by $h$ and $dh$, we obtain

$$dzh + (1 - n^2)\, zdh + n^2 hdh = 0,$$

and, in order to reduce this to a perfect integral, by multiplying it by $h^{-n^2}$, there results

$$dzh^{1-n^2} + (1-n^2)\, zh^{-n^2}.\, dh + n^2 h^{1-n^2}\, dh = d.zh^{1-n^2} + n^2 dh\, h^{1-n^2} = 0,$$

$$\therefore zh^{1-n^2} + \frac{n^2}{2-n^2}.h^{2-n^2} = c, \text{ and } z = ch^{n^2-1} - \frac{n^2}{2-n^2}.h.$$

(*n*) In virtue of equation (5) $dt = \dfrac{\omega}{\alpha u} dh = \dfrac{ndh}{u}$.

(*o*) Equation (9) may be written as follows:

$$dt = \frac{dh}{2gh}\left(\frac{(2-n^2)\, H^{n^2-2}}{h^{n^2-2} - H^{n^2-2}}\right)^{\frac{1}{2}}.$$

If we suppose $H + h' = h$, then the expression is

$$\left(\frac{(2-n^2)\, H^{n^2-2}}{(H+h')^{n^2-2} - H^{n^2-2}}\right)^{\frac{1}{2}} =$$

$$\left((2-n^2)\, H^{n^2-2}\right)^{2}\left[\frac{1}{H^{n^2-2}\left[1-1+\frac{n^2-2}{1}H^{-1}h' + \left(\frac{n^2-2}{1}\right)\left(\frac{n^2-3}{2}\right)H^{-2}h'^2 + \left(\frac{n^2-2}{1}\right)\left(\frac{n^2-3}{2}\right)\left(\frac{n^2-4}{3}\right)H^{-3}h'^3 + \&c.\right]}\right]^{\frac{1}{2}}$$

$=$ by dividing by $(2-n^2).\, H^{n^2-2}$, and supposing $n^2 = 2$,

$$\frac{1}{+\frac{h'}{H} + \frac{1}{2}.\frac{h'^2}{H}. + \frac{1}{3}\frac{h'^3}{H}} \&c.$$

equal by substituting $h - H$ for $h'$,

$$\frac{1}{\left(\frac{h}{H}-1\right) + \frac{1}{2}\left(\frac{h}{H}-1\right)^2 + \frac{1}{3}\left(\frac{h}{H}-1\right)^3} \tfrac{1}{2}\,\&c. = \log\left(\frac{H}{h}\right)^{-\frac{1}{2}}$$

Now by substituting their values for $h$ and $dh$, we obtain, as

$$\log\left(\frac{H}{h}\right)^{-\frac{1}{2}} = \frac{1}{x},\ dt = \frac{-4He^{-2x^2}xdx}{\sqrt{2gHe^{-2x^2}}}\frac{1}{x} = \text{expression in text.}$$

(*p*) By equation (1) we have, when $g$ is neglected, as it is in this case,

$\dfrac{dp}{dx} = -\rho\dfrac{dv}{dt}$, consequently we shall have $\dfrac{1}{\rho}\dfrac{dp}{dx}.dt = -dv =$

$$-\frac{dv}{dt}dt-\frac{dv}{dx}v.dt\ \therefore\ \frac{1}{\rho}\frac{dp}{dx}=-\frac{dv}{dt}-v.\frac{dv}{dx}.$$

$$(q)\ \frac{d.\rho yv}{dt}=\rho v\frac{dy}{dt}+\rho v\frac{dy}{dx}.\frac{dx}{dt}+y.\frac{d.\rho v}{dt}+y.\frac{d.\rho v}{dx}.\frac{dx}{dt}$$

$$=\rho v\frac{dy}{dt}+\rho v^2\frac{dy}{dx}+y.\frac{d.\rho v}{dt}+yv.\frac{d.\rho v}{dx}.$$

(*r*) When the capacity is great the pressure on the surface of the vessel for a short space of time may be considered as constant.

(*s*) Multiplying the second $=n$ by $dx$, we obtain $pvdy+yd.pv=0$, $\therefore pvy=c$,

(*t*) If both sides be multiplied by $dx$ we shall obtain

$$k.\frac{dp}{p}+\alpha^2\Pi^2.u^2.\frac{1}{py}.d.\frac{1}{py}=0,\ \therefore k.\log p+\alpha^2\Pi^2u^2.\frac{1}{2p^2y^2}=c'.$$

# NOTES TO ADDITION.

(*a*) When a machine is in motion, if the moving forces do not coincide with the directions along which the points of the machine move, they must at least make acute angles with those directions, for if they made right angles they would not produce any effect at all in the directions in which the points move, and if they made obtuse angles their tendency would be to retard these motions.

(*b*) If after multiplying the first members of these respective equations together, and also their second members, they be added together, we obtain

$$\text{M}.(\text{X}dx + \text{Y}dy + \text{Z}dz) = \text{P}.ds.(\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu)$$
$$= \text{P}ds.\cos\sigma = \text{P}dp.$$

(*c*) It would be useless for the purposes of machinery that the motion of the machine should be continually accelerated; if the moving forces acted always with the same intensity this would be the case; in this case, in order to render the motion uniform, the resistance is made to increase, or if this cannot be conveniently done, then the action of the motive force is either intermitted, or made to become a resisting power, by which means, though we cannot render the motion of the machine uniform, we confine its variations within certain limits.

(*d*) When the body sets out from a state of rest, as there is no initial velocity, $\Sigma mk^2$ must vanish; likewise as the effect of the motive forces must be greater than that of the resisting forces, when the motive forces are of that description that they act more forcibly on bodies which are at rest than on those which are in motion, the acceleration produced by these forces must continually diminish with the increased acceleration of the machine until $\text{P}dp = \text{Q}dq$; and it is evident from these considerations that the successive differences between $\text{P}dp$ and $\text{Q}dq$ constitute a decreasing series, so that the process by which the machine is brought to a constant uniform state, i. e. to one in which $\Sigma\text{P}dq = \Sigma\text{Q}dq$, is most rapid at the commencement of the motion, and gets continually slower; indeed, strictly speaking, this uniformity is not attained until after the lapse of an indefinite time.

(*e*) The diminution estimated in the direction of the surfaces comes under the expression for the effect produced by friction.

(*f*) It appears from the expression $\text{U}(\varepsilon+\alpha)$ that when a man walks on a horizontal plane without a load there are two effects produced, he raises or depresses his centre of gravity, and he also impresses on this point a horizontal velocity, each of these requires a distinct effort, the last is evidently much less than the first; but there are no experiments made by means of which their relative intensity can be determined; as it is evident the less $\text{U}(\varepsilon + a)$ is, every thing else being the same, the greater will be the power of the man; the skill in walking so as to economize this power, consists in going as closely as possible to the ground, so that $\varepsilon$ may be the least possible; the expression for the work done when a man ascends a height, and carries no load, is $\text{U}(\varepsilon+\alpha+h)+\text{F}l$; now when a man carries only his own weight, the height he is able to ascend in a given time multiplied into $\text{U}$ is $>$ than when he carries an additional load; i. e. $\text{U}h$ is greater than $\text{K}h'$, $h'$ being the height he can ascend to in the same time when loaded, and $\text{K} = \text{U} + \text{L}$, $\text{L}$ being the additional load; this appears from some experiments made by Coulomb, which led him to infer that this diminution of action or difference between $\text{U}h$ and $\text{K}h'$ is proportional to the additional load carried; and, as it also appeared from his experiments, that when the load carried was equal to $\text{U}$ the weight of the man, the diminution was one-half, or equal to $\frac{\text{U}h}{2}$, when the load is $\text{L}$, it is equal to $\frac{h\text{L}}{2}$, therefore we must have

$$(\text{U}+\text{L})h' = \text{U}h - \frac{h\text{L}}{2} \text{ and } h' = \text{U}h\left(1 - \frac{\text{L}}{2\text{U}}\right)$$

divided by $\text{U}+\text{L}$. This formula should be considered only as an approximation, for it would appear from it that when $\text{L} = 2\text{U}$, $h' = 0$, i. e. if a man carries a load equal to twice his weight, he could not ascend. —See *Hachette Traite des Machines, Chapitre Premier*.

(*g*) If $\text{L} = 0$, is a function $\text{F}$ of these variables, or $\text{F}(txyz) = 0$, then when they become $t+dt$, $x+dx$, $y+dy$, $z+dz$, we shall have $\text{F}(t+dt,\ x+dx,\ y+dy,\ z+dz) = 0$, and when infinitely small quantities of the second and higher orders are neglected,

$$\text{L} + \frac{d\text{L}}{dt}dt + \frac{d\text{L}}{dx}dx + \frac{d\text{L}}{dy}dy + \frac{d\text{L}}{dz}dz = 0.$$

(*h*) Since $\frac{d^2x}{dt^2}d_{,}x + \frac{d^2y}{dt^2}d_{,}y + \frac{d^2z}{dt^2}d_{,}z = \frac{1}{2}d.v_{,}^2 + \frac{du'}{dt}d_{,}x + \frac{dv'}{dt}d_{,}y + \frac{dw'}{dt}d_{,}z$, by substituting for the first member of this equation, and concinnating, we obtain the value of $\frac{1}{2}d.mv_{,}^2$ given in the text.

(*i*) Since in this case the forces P act in the direction of gravity, $\int\Sigma\text{P}(dp - d_{,}p)$ must be equal to $\Sigma\text{P}$, or $\Pi$ multiplied into the vertical space described by the centre of gravity.

(*k*) Since the motion is uniform, the space $\zeta'$ described by the centre of gravity is $\div l$ to the time, consequently as $a\cos\alpha$ is the value of the velocity resolved in the direction of the vertical, we shall have $\zeta' = at\cos\alpha$.

THE END.

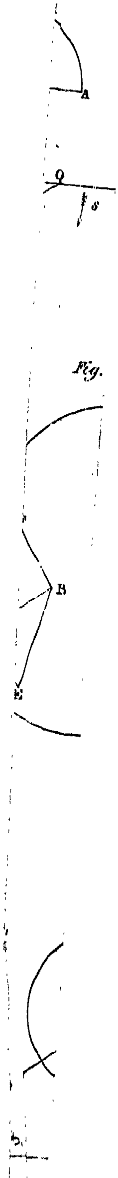



*Fig.*

Fig. 1

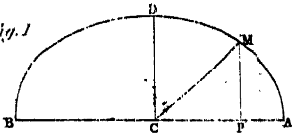



Fig. 2

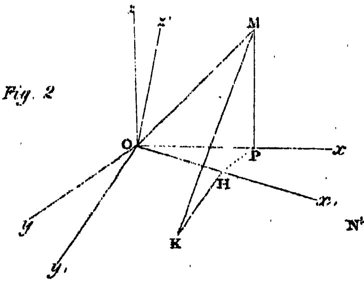



Fig. 5

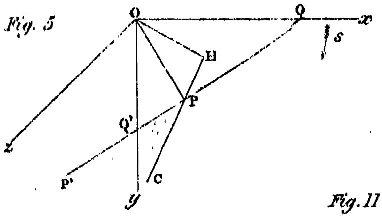



Fig. 6

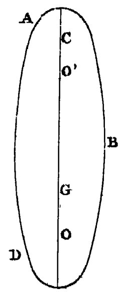



Fig. 7

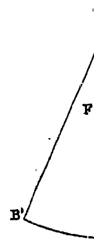



Fig. 11

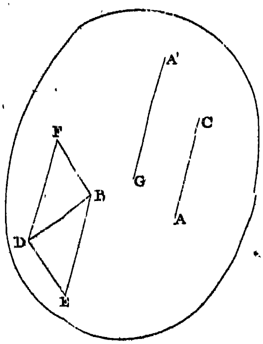
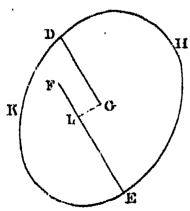



Fig. 10

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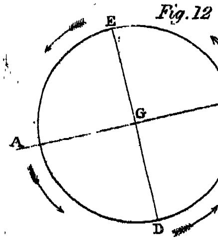


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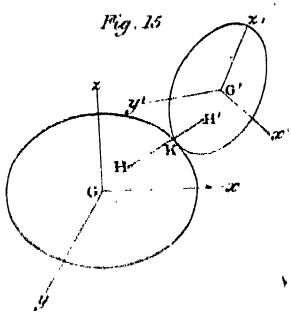
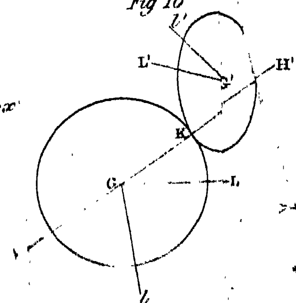



Fig 16




Fig

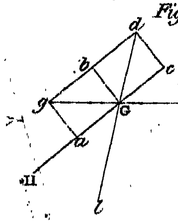


Fig. 3

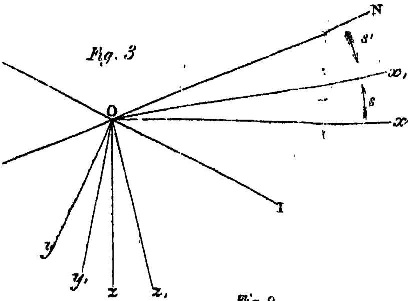



Fig 4.

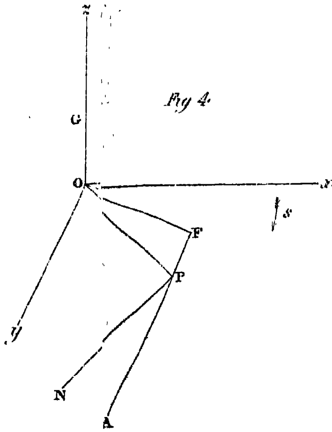



Fig. 8

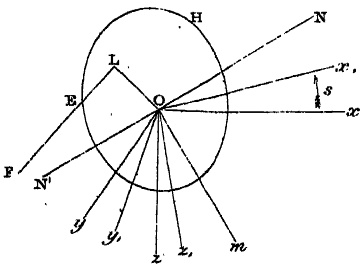



Fig. 9

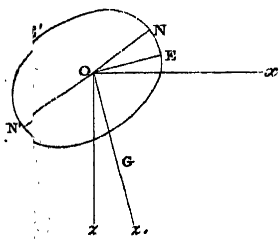



Fig. 13

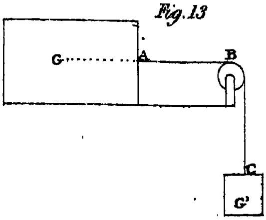



Fig 14

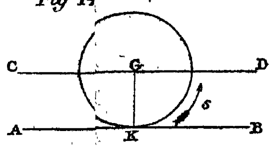



Fig 18

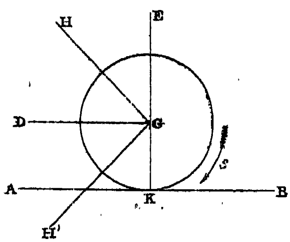



Fig. 19